Collision of Billiard Balls in 3D with Spin and Friction

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Abstract

The collision of billiard balls is influenced by their spin. This influence is a consequence of friction that acts during the very brief time that the balls are in contact. Here we use the Coulomb friction law, which is especially convenient because it does not require knowledge of the area of contact. In center-of-mass coordinates, we solve the general problem of the collision of identical billiard balls in 3D. When sliding friction is the only dissipative mechanism, we are able to express the outgoing state of the balls unambiguously in terms of their incoming state. When there is additional dissipation associated with compression of the balls during the collision, there is a one-parameter family of solutions depending on the extent of this additional dissipation.

Formulation in Center-of-Mass Coordinates

We consider two identical billiard balls, denoted A and B, each with mass m, radius r, and scalar moment of inertia $\alpha m r^2$, where α is a dimensionless constant that depends on the radial mass distribution. The coefficient of sliding friction between the two balls will be denoted μ.

At the moment of collision, the surfaces of the two balls are tangent to each other, and the point of contact is the center of mass of the system, since the balls are identical. We choose our frame of reference and our coordinate system within that frame so that the center of mass of the system is fixed at the origin. We also choose the x axis so that it goes through the centers of both balls at the moment of collision, with the center of ball A at $x = -r$ and the center of ball B at $x = +r$. The common tangent plane to the spherical surfaces of the two balls at the point of contact at the moment of collision is the plane $x = 0$.

Since the center of mass of the system is at rest and the balls are identical, the velocity of ball B is minus the velocity of ball A, and this is true at all times, before, during, and after the collision.

We denote the velocity of ball A by $\mathbf{U} = (U, V, W)$, and then the velocity of ball B is $-\mathbf{U}$. The angular velocities of the two balls are denoted $\boldsymbol{\Omega}^{\mathrm{A}} = (\Omega_{\mathrm{x}}^{\mathrm{A}}, \Omega_{\mathrm{y}}^{\mathrm{A}}, \Omega_{\mathrm{z}}^{\mathrm{A}})$ and $\boldsymbol{\Omega}^{\mathrm{B}} = (\Omega_{\mathrm{x}}^{\mathrm{B}}, \Omega_{\mathrm{y}}^{\mathrm{B}}, \Omega_{\mathrm{z}}^{\mathrm{B}})$.

Equations of Motion During the Collision

We envision the collision as taking place during a short time interval $(0, \tau)$. This time interval is so brief, and the balls are so nearly rigid, that the geometric arrangement of the balls (two spherical balls that touch at the origin with their centers at $\pm r$ on the x axis) can be thought of as remaining constant during the collision. The velocities and angular velocities of the two balls are changing rapidly, however, during $(0, \tau)$, and we shall derive differential equations that govern how they change.

Let $f(t) > 0, t \in (0, \tau)$ be the magnitude of the normal force of one ball on the other during the collision. This force is applied at the point of contact (i.e., at the origin) in the negative x direction to ball A and in the positive x direction to ball B. Note that this force does not result in any torque about the center of either ball.

According to Coulomb's law of sliding friction, the magnitude of the frictional force of either ball on the other is $\mu f(t)$. This force acts at the origin in a direction that is tangent to the plane $x = 0$. To find the specific dirction within this plane in which the frictional force acts, we need to evaluate the relative velocity of the material surfaces of the two spherical balls at the point of contact.

The velocity of a material point on the surface of ball B that happens to be at the origin at any time during the collision is given by

$$-\mathbf{U} + \left(\boldsymbol{\Omega}^{\mathrm{B}} \times (-r\mathbf{e}_{\mathrm{x}})\right), \tag{1}$$

where $\mathbf{e}_{\mathrm{x}}$ is a unit vector in the positive x direction, since $(-r\mathbf{e}_{\mathrm{x}})$ is the vector from the center of ball B to the origin. Similarly, the velocity of a material point on the surface of ball A that happens to be at the origin at any time during the collision is given by

$$\mathbf{U} + \left(\boldsymbol{\Omega}^{\mathrm{A}} \times (r\mathbf{e}_{\mathrm{x}})\right). \tag{2}$$

Subtracting (2) from (1) and projecting the result onto the plane $x = 0$ we get the tangential velocity of the surface of ball B relative to that of the surface of ball A as follows:

$$-2(0,V,W)-r\,(\mathbf{\Omega}\times\mathbf{e}_{\mathrm{x}})=-(0,2V+r\Omega_{\mathrm{z}},2W-r\Omega_{\mathrm{y}}),\tag{3}$$

where we have made the definition

$$\mathbf{\Omega}=\mathbf{\Omega}^{A}+\mathbf{\Omega}^{B},\tag{4}$$

and of course the components of $\mathbf{\Omega}$ are $\Omega_{\mathrm{x}},\Omega_{\mathrm{y}},\Omega_{\mathrm{z}}$.

It will be useful to rewrite the vector (3) in polar form. To do this, we let $S>0$ and θ be implicitly defined by

$$2V+r\Omega_{\mathrm{z}} = S\cos\theta,\tag{5}$$
$$2W-r\Omega_{\mathrm{y}} = S\sin\theta.\tag{6}$$

and then let the unit vector $\mathbf{e}$ be defined by

$$\mathbf{e}=(0,\cos\theta,\sin\theta)\tag{7}$$

Then the relative velocity defined by (3) becomes $-S\mathbf{e}$, in which S is the sliding speed and $-\mathbf{e}$ is a unit vector in the direction of sliding of the surface of ball B relative to that of ball A at the point of contact. Therefore, the frictional force of ball B on ball A during the collision is $\mu f(t)(-\mathbf{e})$. Combining this with the normal force, we get the following equation of motion for $\mathbf{U}(t)$ during the collision:

$$m\frac{d\mathbf{U}}{dt}=-f(t)\mathbf{e}_{\mathrm{x}}-\mu f(t)\mathbf{e}.\tag{8}$$

We shall see later that the unit vector $\mathbf{e}$, and the angle θ on which it depends, are constants of the motion, but for now we allow for the possibility that they depend on time. The sliding speed S of one ball on the other is not constant, and its time dependence will be determined below.

We now turn our attention to the changes in the angular velocities of the two balls during the collision. The torque applied by ball B to ball A about the center of ball A is

$$(+r\mathbf{e}_{\mathrm{x}})\times(-\mathbf{e}\mu f(t)),\tag{9}$$

and the torque applied by ball A to ball B about the center of ball B is

$$(-r\mathbf{e}_{\mathrm{x}})\times(+\mathbf{e}\mu f(t)).\tag{10}$$

Note the important fact that these torques are *equal* (not opposite). The forces have opposite sign, but this is canceled by the opposite sign of the two vectors from the centers of the balls to the point of contact, where the forces are applied.

It may seem worrisome, if these torques do not cancel each other, how can angular momentum be conserved? Note, however, that the total angular momentum of the system involves not only the spin angular momenta of the two balls, but also their orbital angular momenta with respect to the center of mass of the whole system. A formula for the total angular momentum will be given later, and it will be left to the reader to verify that the incoming angular momentum and the outgoing angular momentum are equal.

From the two equal torque expressions (9) and (10) we get the equations of motion for $\mathbf{\Omega}^{\mathrm{A}}$ and $\mathbf{\Omega}^{\mathrm{B}}$:

$$\alpha m r^2 \frac{d\mathbf{\Omega}^{\mathrm{A}}}{dt} = \alpha m r^2 \frac{d\mathbf{\Omega}^{\mathrm{B}}}{dt} = -\mu r f(t) \left(\mathbf{e}_{\mathrm{x}} \times \mathbf{e} \right). \tag{11}$$

From equation (11), we can see that the following are all constants of the motion during the collision:

$$\mathbf{\Omega}^{\mathrm{A}} - \mathbf{\Omega}^{\mathrm{B}}, \;\; \Omega_{\mathrm{x}}^{A}, \;\; \Omega_{\mathrm{x}}^{\mathrm{B}}. \tag{12}$$

By adding the two forms of equation (11) and then dividing both sides by $\alpha m r^2$, we get

$$\frac{d\mathbf{\Omega}}{dt} = -\frac{2\mu f(t)}{\alpha m r} \left(\mathbf{e}_{\mathrm{x}} \times \mathbf{e}(t) \right). \tag{13}$$

We now have equation (8) for $d\mathrm{U}/dt$ and (13) for $d\mathbf{\Omega}/dt$. Since $\mathbf{\Omega}^{\mathrm{A}} - \mathbf{\Omega}^{\mathrm{B}}$ is a constant of the motion during the collision, and since $\mathbf{\Omega} = \mathbf{\Omega}^{\mathrm{A}} + \mathbf{\Omega}^{\mathrm{B}}$ by definition, we can always solve for $\mathbf{\Omega}^{\mathrm{A}}$ and $\mathbf{\Omega}^{\mathrm{B}}$ separately if we know $\mathbf{\Omega}$.

In components, equations (8) and (13) read as follows:

$$
\begin{aligned}
m\frac{dU}{dt} &= -f(t), && (14)\\
m\frac{dV}{dt} &= -\mu f(t)\cos\theta, && (15)\\
m\frac{dW}{dt} &= -\mu f(t)\sin\theta, && (16)\\
\frac{d\Omega_{\mathrm{x}}}{dt} &= 0, && (17)\\
\frac{d\Omega_{\mathrm{y}}}{dt} &= \frac{2\mu f(t)}{\alpha m r}\sin\theta, && (18)\\
\frac{d\Omega_{\mathrm{z}}}{dt} &= -\frac{2\mu f(t)}{\alpha m r}\cos\theta. && (19)
\end{aligned}
$$

We can use these equations, together with equations (5-6), which implicitly define $S(t)$ and $\theta(t)$ to derive equations for dS/dt and $d\theta/dt$. To do so, we differentiate with respect to t in (5-6) to obtain

$$
\begin{aligned}
\frac{d}{dt}\left(S\cos\theta\right) &= 2\frac{dV}{dt} + r\frac{d\Omega_{\mathrm{z}}}{dt}\\
&= -2\frac{\mu f(t)}{m}\left(1+\frac{1}{\alpha}\right)\cos\theta, && (20)\\
\frac{d}{dt}\left(S\sin\theta\right) &= 2\frac{dW}{dt} - r\frac{d\Omega_{\mathrm{y}}}{dt}\\
&= -2\frac{\mu f(t)}{m}\left(1+\frac{1}{\alpha}\right)\sin\theta. && (21)
\end{aligned}
$$

To obtain the last line of (20), we used equations (15) and (19), and to obtain the last line of (21) we used equations (16) and (18). By writing out $\frac{d}{dt}\left(S\cos\theta\right)$ and $\frac{d}{dt}\left(S\sin\theta\right)$ in the foregoing, we get a pair of equations that can be solved for dS/dt and $d\theta/dt$. The results are

$$
\begin{aligned}
\frac{dS}{dt} &= -2\frac{\mu f(t)}{m}\left(1+\frac{1}{\alpha}\right), && (22)\\
\frac{d\theta}{dt} &= 0. && (23)
\end{aligned}
$$

Outgoing State in Terms of the Incoming State

With θ constant, it is straightforward to integrate equations (14-19) over $(0, \tau)$ with the following results:

$$\begin{aligned} U(\tau) - U(0) &= -\frac{1}{m}P, & (24)\\ V(\tau) - V(0) &= -\frac{\mu}{m}P\cos\theta, & (25)\\ W(\tau) - W(0) &= -\frac{\mu}{m}P\sin\theta, & (26)\\ \Omega_{\mathrm{x}}(\tau) - \Omega_{\mathrm{x}}(0) &= 0, & (27)\\ \Omega_{\mathrm{y}}(\tau) - \Omega_{\mathrm{y}}(0) &= \frac{2\mu}{\alpha m r}P\sin\theta, & (28)\\ \Omega_{\mathrm{z}}(\tau) - \Omega_{\mathrm{z}}(0) &= -\frac{2\mu}{\alpha m r}P\cos\theta, & (29) \end{aligned}$$

where

$$P = \int_0^\tau f(t)\ \ dt \qquad (30)$$

The parameter P, with units of momentum, is called the *impulse* of the collision. We shall see later to what extent P can be determined.

Now recall that $\mathbf{\Omega} = \mathbf{\Omega}^{\mathrm{A}} + \mathbf{\Omega}^{\mathrm{B}}$ and also that $\mathbf{\Omega}^{\mathrm{A}} - \mathbf{\Omega}^{\mathrm{B}}$ is constant. Because of these relationships, equations (27-29) imply

$$\begin{aligned} \Omega_{\mathrm{x}}^{\mathrm{A,B}}(\tau) - \Omega_{\mathrm{x}}^{\mathrm{A,B}}(0) &= 0, & (31)\\ \Omega_{\mathrm{y}}^{\mathrm{A,B}}(\tau) - \Omega_{\mathrm{y}}^{\mathrm{A,B}}(0) &= \frac{\mu}{\alpha m r}P\sin\theta, & (32)\\ \Omega_{\mathrm{z}}^{\mathrm{A,B}}(\tau) - \Omega_{\mathrm{z}}^{\mathrm{A,B}}(0) &= -\frac{\mu}{\alpha m r}P\cos\theta, & (33) \end{aligned}$$

Equations (24-26) and (31-33) determine the outgoing state of the system in terms of the incoming state provided that the two parameters θ and P are known. In fact, the parameter θ is itself determined by the incoming state. To see this, evaluate equations (5-6) at $t = 0$. This gives

$$\begin{aligned} 2V(0) + r\Omega_{\mathrm{z}}(0) &= S(0)\cos\theta, & (34)\\ 2W(0) - r\Omega_{\mathrm{y}}(0) &= S(0)\sin\theta, & (35) \end{aligned}$$

in which $S(0) > 0$. This uniquely defines $S(0)$ and θ (except of course that one can always add any multiple of 2π to θ) in terms of the incoming state of the system. Note that we could alternatively have obtained an equation for θ in terms of the outgoing state by evaluating equations (5-6) at $t = \tau$. We leave it as an exercise for the reader to show that the same result is obtained either way. The proof of this requires the use of equation (22) to relate $S(\tau)$ to $S(0)$.

Another good exercise for the reader is to verify that the incoming and outgoing angular momenta are the same. During the collision, the angular momentum of the whole system with respect to its center of mass is given by

$$\mathbf{L} = \alpha m r^2 \mathbf{\Omega}^{\mathrm{A}} + \alpha m r^2 \mathbf{\Omega}^{\mathrm{B}} + ((-r\mathbf{e}_{\mathrm{x}}) \times (m\mathbf{U})) + ((r\mathbf{e}_x) \times (m(-\mathbf{U}))) \,, \quad (36)$$

and this vector should be the same at $t = 0$ as at $t = \tau$.

Conservation of Energy and Evaluation of the Impulse of the Collision

Because we are working in center-of-mass coordinates and the balls are identical, their translational kinetic energies are the same as each other at all times, but their rotational kinetic energies may be different. Thus, the kinetic energy of the whole system is given by

$$\begin{aligned} E_{\mathrm{K}} &= 2\frac{1}{2}m\|\mathbf{U}\|^2 + \frac{1}{2}\alpha m r^2 \|\mathbf{\Omega}^{\mathrm{A}}\|^2 + \frac{1}{2}\alpha m r^2 \|\mathbf{\Omega}^{\mathrm{B}}\|^2 \\ &= 2\frac{1}{2}m\|\mathbf{U}\|^2 + \frac{1}{4}\alpha m r^2 \|\mathbf{\Omega}^{\mathrm{A}} + \mathbf{\Omega}^{B}\|^2 + \frac{1}{4}\alpha m r^2 \|\mathbf{\Omega}^{\mathrm{A}} - \mathbf{\Omega}^{B}\|^2. \quad (37) \end{aligned}$$

It is useful to express the kinetic energy in the second way because $\mathbf{\Omega}^{\mathrm{A}} - \mathbf{\Omega}^{\mathrm{B}}$ is a constant of the motion, so the term involving $\mathbf{\Omega}^{\mathrm{A}} - \mathbf{\Omega}^{\mathrm{B}}$ does not change during the collision. Also, we have already determined how the quantity $\mathbf{\Omega} = \mathbf{\Omega}^{\mathrm{A}} + \mathbf{\Omega}^{\mathrm{B}}$ varies during the collision, see equations (27-29).

The heat generated by the collision is given by

$$\begin{aligned} H &= E_{\mathrm{K}}(0) - E_{\mathrm{K}}(\tau) \\ &= m\left(\|\mathbf{U}(0)\|^2 - \|\mathbf{U}(\tau)\|^2\right) + \frac{1}{4}\alpha m r^2 \left(\|\mathbf{\Omega}(0)\|^2 - \|\mathbf{\Omega}(\tau)\|^2\right). \quad (38) \end{aligned}$$

From equations (24-29) we have the following:

$$U^2(\tau) = U^2(0) - 2U(0)\frac{P}{m} + \left(\frac{P}{m}\right)^2, \tag{39}$$

$$V^2(\tau) = V^2(0) - 2V(0)\frac{\mu P\cos\theta}{m} + \mu^2\left(\frac{P}{m}\right)^2\cos^2\theta, \tag{40}$$

$$W^2(\tau) = W^2(0) - 2W(0)\frac{\mu P\sin\theta}{m} + \mu^2\left(\frac{P}{m}\right)^2\sin^2\theta, \tag{41}$$

$$\Omega_{\mathrm{x}}^2(\tau) = \Omega_{\mathrm{x}}^2(0), \tag{42}$$

$$\Omega_{\mathrm{y}}^2(\tau) = \Omega_{\mathrm{y}}^2(0) + 4\Omega_{\mathrm{y}}(0)\frac{\mu P\sin\theta}{\alpha m r} + \frac{4\mu^2}{\alpha^2 r^2}\left(\frac{P}{m}\right)^2\sin^2\theta, \tag{43}$$

$$\Omega_{\mathrm{z}}^2(\tau) = \Omega_{z}^2(0) - 4\Omega_{\mathrm{z}}(0)\frac{\mu P\cos\theta}{\alpha m r} + \frac{4\mu^2}{\alpha^2 r^2}\left(\frac{P}{m}\right)^2\cos^2\theta. \tag{44}$$

Substituting these results into equation (38) for H, we get

$$\begin{aligned} H = & \quad 2U(0)P \\ & + \mu\left(2V(0) + r\Omega_{\mathrm{z}}(0)\right)P\cos\theta \\ & + \mu\left(2W(0) - r\Omega_{\mathrm{y}}(0)\right)P\sin\theta \\ & - \left(1 + \mu^2\left(1 + \frac{1}{\alpha}\right)\right)\frac{P^2}{m}. \end{aligned} \tag{45}$$

This can be written more succinctly in terms of $S(0)$ by making use of equations (34-35) together with the identity that $\cos^2\theta + \sin^2\theta = 1$. The result is

$$H = \left(2U(0) + \mu S(0)\right)P - \left(1 + \mu^2\left(1 + \frac{1}{\alpha}\right)\right)\frac{P^2}{m} \tag{46}$$

Note that H is the total amount of heat that is generated by the collision. The amount of heat that is generated specifically by the sliding friction will be denoted H_{sfr} and is given by

$$H_{\text{sfr}} = \int_0^\tau \mu f(t)S(t)\,dt, \tag{47}$$

since $\mu f(t)$ is the magnitude of the frictional force, and $S(t)$ is the speed with which the surface of one ball is sliding over the surface of the other at the point of contact. To evaluate H_{sfr}, we use equation (22) for dS/dt, from which we obtain

$$S(t) = S(0) - \frac{2\mu}{m}\left(1 + \frac{1}{\alpha}\right)\int_0^t f(t')\,dt'. \tag{48}$$

Substution of this formula for the relative sliding speed $S(t)$ into equation (47) yields the following formula for H_{sfr}:

$$\begin{aligned} H_{\text{sfr}} &= \mu S(0) \int_0^\tau f(t)\,dt - \frac{2\mu^2}{m}\left(1+\frac{1}{\alpha}\right)\int_0^\tau f(t)\int_0^t f(t')\,dt'\,dt \\ &= \mu S(0) P - \frac{2\mu^2}{m}\left(1+\frac{1}{\alpha}\right)\frac{1}{2}\int_0^\tau \frac{d}{dt}\left(\left(\int_0^t f(t')\,dt'\right)^2\right)dt \\ &= \mu S(0) P - \mu^2\left(1+\frac{1}{\alpha}\right)\frac{P^2}{m}. \end{aligned} \tag{49}$$

Combining this with (46) we get the simple result that

$$H - H_{\text{sfr}} = 2U(0)P - \frac{P^2}{m}. \tag{50}$$

Note that all of the terms involving the friction coefficient μ have cancelled out.

Now we consider two different cases. In the first case, we assume that the *only* dissipative mechanism is sliding friction. Then $H = H_{\text{sfr}}$ and (50) becomes an equation for P:

$$0 = 2U(0)P - \frac{P^2}{m}. \tag{51}$$

The solution $P = 0$ would mean that there is no collision at all, so we reject this and then the only possibility is

$$P = 2mU(0). \tag{52}$$

This is the expected result for an elastic collision. What is interesting here is that the result is unchanged by the presence of sliding friction, even though kinetic energy is then not conserved. With P given by (52), we have completely determined the outgoing state of two billiard balls in terms of their incoming state, see equations (24-35).

In the second case, we assume that there is also dissipation associated with the mechanism that generates the normal force $f(t)$, which physically comes from the slight compression of the balls when they are in contact. Without specifying that mechanism, the most we can say is that we have the inequality

$$H > H_{\text{sfr}}, \tag{53}$$

since the total heat generated is now the sum of two positive terms, one of which is H_{sfr}, and the other is the heat generated during the compression and expansion of the two balls. Applying this inequality to equation (50) we get

$$0 < 2U(0)P - \frac{P^2}{m}. \tag{54}$$

Since $P > 0$, this is equivalent to

$$P < 2mU(0). \tag{55}$$

On physical grounds, we also require $mU(0) \leq P$. Otherwise the outgoing value of U would have the same sign as the incoming value, see (24), and this would mean that the two balls go through each other, which is clearly impossible. Thus, we may conclude that

$$mU(0) \leq P < 2mU(0). \tag{56}$$

Within this range, we have a one-parameter family of possible outcomes. These are the allowed values of P when there is dissipation associated with the compression and expansion of the two balls during the collision. The borderline value $P = 2mU(0)$, which is not allowed by (56), is the value of P that was found above for the case in which the only dissipative mechanism is sliding friction.

An important restriction on the foregoing analysis is that $S(t)$ should not become negative during the collision. Recall that $dS/dt < 0$, see equation (42), and if $S(t) = 0$ at some time $t \in (0, \tau)$ then the surfaces of the two balls are momentarily at rest with respect to each other at that time, and a new phenomenon, namely static friction, comes into play. This makes our analysis inapplicable beyond such a time. Thus, we should impose the restriction that $S(\tau) \geq 0$, and from (48) this is equivalent to

$$0 \leq S(0) - \frac{2\mu}{m}\left(1 + \frac{1}{\alpha}\right)P. \tag{57}$$

This needs to be checked in any particular case to see whether the above results are applicable or not. Since P is not part of the given data but is restricted by (56), we can adopt a worst-case point of view and impose the condition

$$0 \leq S(0) - \frac{2\mu}{m}\left(1 + \frac{1}{\alpha}\right)2mU(0), \tag{58}$$

which implies (57) because of (56).

Special Case of a Rolling Ball Striking a Ball at Rest

Here we consider the special case in which a billiard ball (A) that is rolling without slipping on a table strikes another identical billiard ball (B) that is at rest on the table. Let the plane of the table be $z = -r$ so that when balls are on the table their centers are in the plane $z = 0$. Let the ball that is initially at rest have its center at the point $(r, 0, 0)$ prior to the collision. The ball that is initially rolling has the lab-frame velocity of its center of mass equal to $(u, v, 0)$. The rolling ball has been aimed in such a way that it will strike the other ball when the center of the rolling ball is at $(-r, 0, 0)$. Thus, the point of contact is the lab-frame origin. Since the moving ball is rolling without slipping its angular velocity vector prior to the collision is given by

$$\mathbf{\Omega}^{\mathrm{A}} = (-v, u, 0)/r, \tag{59}$$

and of course $\mathbf{\Omega}^{\mathrm{B}} = 0$ prior to the collision.

Now we change to center-of-mass coordinates with the two systems of coordinates having their corresponding axes parallel, and also having origins that coincide at the moment of collision. The incoming velocity of ball A in the center-of-mass system is $(u, v, 0)/2$ and the incoming velocity of ball B in the center-of-mass system is $-(u, v, 0)/2$. The angular velocities are the same in the center-of-mass system as in the laboratory frame.

We consider the case in which the only dissipative mechanism is sliding friction, so that

$$P = 2m(u/2) = mu. \tag{60}$$

The parameters $S(0)$ and θ are evaluated from equations (5-6) which in the present case become

$$v = S(0)\cos\theta \tag{61}$$

$$-u = S(0)\sin\theta \tag{62}$$

Thus, $S(0)$, which by definition is the speed with which the surface of one ball is sliding over the other as the collision is initiated, turns out to be equal to the lab-frame speed of the rolling ball. We denote this speed by s in the following, so that

$$S(0) = s = \sqrt{u^2 + v^2}. \tag{63}$$

Then

$$\cos\theta = \frac{v}{s} \tag{64}$$
$$\sin\theta = \frac{-u}{s} \tag{65}$$

We now have everything we need to evaluate the outgoing states of the two balls. The center-of-mass outgoing velocity of ball A (which is minus the center-of-mass outgoing velocity of ball B) is obtained from equations (24-26):

$$U(\tau) = \frac{u}{2} - u, \tag{66}$$
$$V(\tau) = \frac{v}{2} - \mu\frac{uv}{s}, \tag{67}$$
$$W(\tau) = \mu\frac{u^2}{s}. \tag{68}$$

To get the lab-frame outgoing velocities for ball A from these results, we simply add the vector $(u/2, v/2, 0)$. Thus, the lab-frame outgoing velocity of ball A is

$$(0, v - \mu uv/s, \mu u^2/s) \tag{69}$$

To get the outgoing velocity of ball B, we first change the sign of the results in (66-68) to get the outgoing velocity of ball B in center-of-mass coordinates, and then we add $(u/2, v/2, 0)$ to convert to lab-frame coordinates. The outgoing velocity of ball B in lab-frame coordinates is

$$(u, \mu uv/s, -\mu u^2/s) \tag{70}$$

Note that total linear momentum has been conserved, since it is equal to $m(u, v, 0)$ both before and after the collision. It is interesting that the outgoing velocities include vertical components. The consequences of this will be discussed below.

We still need to evaluate the outgoing angular velocities of the two balls. Here, there is no distinction between the lab frame and the center-of-mass frame. Also, the change in angular velociy is the same for both balls, see equations (31-33). In the present special case, these equations give the following change in angular velocity:

$$\left(0, -\frac{\mu u^2}{\alpha r s}, -\frac{\mu uv}{\alpha r s}\right). \tag{71}$$

Since the incoming angular velocity of ball B is zero, the vector (71) is equal to its outgoing velocity. For ball A, however, we must add the change (71) to the incoming angular velocity of ball A, which is $(-v, u, 0)/r$. Thus, the outgoing angular velocity of ball A is

$$\left(-\frac{v}{r}, \frac{u}{r} - \frac{\mu u^2}{\alpha r s}, -\frac{\mu u v}{\alpha r s}\right). \tag{72}$$

Complications Associated with the Table

In the foregoing, we did not consider the influence of the table at all, except in setting up the incoming conditions in which one of the balls is rolling without slipping. There are three separate influences of the table that should be considered, and we discuss them here.

The first question is whether friction with the table plays a role during the collision itself. The table applies a normal force to either ball, and thus there is also a frictional force whenever the contact point of a ball with the table is slipping. Prior to the collision, the normal force on either ball is mg, so the frictional force, if any slipping should occur, is $\mu_{\text{bt}} mg$, where μ_{bt} is the coefficient of friction between the ball and the table. Since the table is much softer than the ball, we assume that the collision will be over before this force has time to change appreciably, and therefore its consequences during the collision itself will be $\mathcal{O}(\tau)$ and therefore neglibible in comparison to the impulse of the collision. Thus, we are need only worry about the post-collisional influences of the table. This simplifies the matter in two ways: First, it means that we can use the outgoing velocities and angular velocities found above as initial conditions for the post-collision epoch; and second, it means that we can consider separately the influence of the table on each of the balls.

Recall that the outgoing velocities of both balls have a vertical component, with ball A having upward velocity and ball B having downward velocity. This means that ball B is immediately colliding with the table, and that ball A will leave the table briefly and collide with the table on its way down. In either case, we have the collision of a spinning ball with a table, and such a collision can be analyzed in much the same manner as the collision between two balls. Since the table is soft, it may be a realistic idealization to assume that the impulse of the collision is as small as possible, so that no bounce occurs. In that case, ball B loses its vertical velocity immediately, and ball A loses its vertical velocity upon its first return to the surface. These (inelastic) collisions with the surface also involve changes in angular velocity and in the horizontal components of velocity that can be calculated, but we omit the details.

Once the vertical velocity component has been eliminated, the situation for either ball is that its center of mass is moving horizontally, and the ball has some angular velocity as well. Unless the velocity and angular velocity are related in a very particular way, there is slip between the ball and the table at their point of contact, and by evaluating this slip and the associated frictional force we can obtain a differential equation for the evolution of the velocity and angular velocity of the ball. Note that both the velocity and the angular velocity are influenced by the frictional force, and the overall effect after some finite (!) time is to bring the ball into a state in which it is rolling without slipping.

The missing details of the foregoing are left, for now, as an exercise for the reader!