Parallel Transport and Gaussian Curvature

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1 Introduction

On a smooth curved surface S, parallel transport, suitably defined, of a vector around a smooth, simple closed curve Γ does not, in general, result in a the same vector at the end of the parallel-transport process as at the beginning. Although the initial and final vectors have the same magnitude, there is in general a nonzero angle between them. From the initial and final vectors themselves, this angle is only determined modulo 2π.

To remove this ambiguity, it is necessary to keep track of angular changes thoughout the parallel-transport process, and for this purpose, a comparison or reference vector is needed. A common choice of reference vector is the tangent vector to the curve, and this leads to the local Gauss-Bonnet theorem, which states that

$$\int_\Gamma k_{\mathrm{g}} \mathrm{d}s + \int_\Omega K \mathrm{d}A = 2\pi. \tag{1}$$

Here Ω is the part of the surface S that is enclosed by Γ, $\mathrm{d}s$ is the element of arclength on Γ, $\mathrm{d}A$ is the element of area on S, k_{g} is the geodesic curvature of Γ on S, and K is the Gaussian curvature of the surface S.

The role of parallel transport in equation (1) is that k_{g} is the rate of change with respect to arclength of the angle between any vector undergoing parallel transport and the unit tangent vector to the curve Γ. The term 2π on the right-hand side of equation (1) actually has nothing to do with parallel transport or with Gaussian curvature. It is simply the angle though which the unit tangent vector turns during one passage around a closed curve. This is most easily seen in the case of a plane curve, for which $K = 0$. Parallel transport of a vector along a curve in the plane is the same as parallel transport in the Euclidean sense, i.e., the vector does not change at all. Thus there is no angular discrepancy involved in parallel transport around a closed curve in the plane. Nevertheless the integral of the geodesic curvature is 2π because the tangent vector to the curve has turned through an angle of 2π.

What these considerations suggest is that we could dispense with the term 2π and obtain a more fundamental result by using a different reference vector, other than the unit tangent to the curve Γ, to keep track of the changes in angle that occur during parallel transport. This can indeed be done. What we need is simply a smooth field of unit vectors tangent to the surface S that is defined throughout Ω.[1] For a surface S described in parametric form $\mathbf{x}(u, v)$, such a field is easily obtained by differentiation with respect to either coordinate, followed by normalization, e.g., $\dfrac{\partial \mathbf{x}}{\partial u}/\|\dfrac{\partial \mathbf{x}}{\partial u}\|$, but there is no need for the reference unit vector field to be related to the coordinates in any particular way, and we will not assume any such relationship in the following.

An objection to the above idea might be that the result could depend on the choice of the reference field, but in fact we will see in the following that this is not the case. An intuitive explanation runs as follows: First, we are interested only in the change of angle that occurs during one complete passage by parallel transport around a closed curve. This is the angular difference between two vectors at the same point, so the reference angle at that point obviously cancels out. This leaves only an ambiguity of $2\pi n$, where n is any integer, and it is precisely this ambiguity that we need a reference field to resolve. Note, however, that any smooth unit vector field defined on Ω that is tangential to S can be continuously deformed into any other such field (recall that the region Ω is topologically equivalent to a disc in $\mathbb{R}^2$), and the integer n in the ambiguity of $2\pi n$ cannot change continuously, so it cannot depend on the choice of reference field. The conclusion of this argument will be confirmed in the following by the observation that the results we derive do not depend at all on the choice of the reference field.

The use of a smooth unit vector field to keep track of angular changes during parallel transport has additional advantages besides elimination of the somewhat spurious term 2π. One of these is that the concept of geodesic curvature is not

[1]Note that there is *not* any such field that reduces to the unit tangent to the curve Γ when restricted to Γ. This is because any continuous vector field defined througout $\Omega \cup \Gamma$, tangent to S, and equal to the unit tangent vector to Γ when evaluated on Γ must be zero somewhere in Ω, and thus cannot be a field of *unit* vectors. To prove the claimed existence of a zero of a vector field with the above properties, note that $\Gamma \cup \Omega$ together with the associated vector field can be mapped bicontinuously onto a sphere in such a way that the image of Γ is the equator and the image of Ω is an open hemisphere with the equator as its boundary.. In fact we can consider two such mappings that are reflections of each other in the equatorial plane of the sphere. This produces a vector field that is continuous on the whole sphere and has mirror symmetry in the equatorial plane. By the hairy ball theorem, there is at least one point on the sphere where this vector field is equal to zero. This point cannot be on the equator, since the vector field there is the image of a unit vector field under a bicontinuous mapping. Thus, by symmetry there must be at least two points, one in each hemisphere, where the vector field is zero, and at the corresponding point in Ω the corresponding vector field must be zero as well.

needed – instead we can work directly with the angular discrepancy generated by parallel transport. Related to this, there is no need for the use of arclength in the parametric description of the curve Γ. An important property of parallel transport is that it does not depend on the choice of parameterization, so it is unnatural that the local Gauss-Bonnet theorem singles out arclength as a preferred parameterization.

Finally, there is the issue of a closed curve that is only piecewise smooth, i.e., a closed curve with smooth arcs that meet at corners. Each corner makes a finite contribution to the integral of the geodesic curvature equal to the angle through which the tangent vector turns in turning the corner. But in parallel transport there is no change at all at a corner of the vector that is undergoing parallel transport. This difference shows up especialy strongly when a region bounded by a closed curve is cut into subregions, and new corners are thereby introduced. The sum of the integrals of the geodesic curvatures around the subregions, including the contributions from corners, is now greater than what it was for the parent region. All of the increase is due to the introduction of new corners, since smooth arcs make contributions of equal magnitude but opposite sign to the regions on their two sides. No complication invoving the new corners appears, however, in the case of parallel transport. If a region is cut into subregions, the angular discrepancy produced by parallel transport around the original region is the sum of the angular discrepancies produced by parallel transport around the subregions.

2 Setting

We are given a smooth surface S in parametric form $\mathbf{x}(u,v)$ with unit normal

$$\mathbf{n}(u,v) = \frac{\dfrac{\partial \mathbf{x}}{\partial u} \times \dfrac{\partial \mathbf{x}}{\partial v}}{\left\| \dfrac{\partial \mathbf{x}}{\partial u} \times \dfrac{\partial \mathbf{x}}{\partial v} \right\|}. \tag{2}$$

In the (u,v) parameter plane, we are given a smooth Jordan curve γ, which is specified in parametric form in terms of two smooth functions $u(s), v(s)$. These functions are defined for all real s, and they are periodic with period 1:

$$u(s+1) = u(s), \tag{3}$$
$$v(s+1) = v(s). \tag{4}$$

There is no value of s at which the derivatives of u and v are both zero. The functions $u(s), v(s)$ must also be such that the curve they describe is a Jordan curve. To avoid self-intersetion, we must impose the condition that if $u(s_1) = u(s_2)$ and $v(s_1) = v(s_2)$, then $s_1 - s_2$ is required to be an integer (since u and v are periodic with period 1). Let ω denote the part of the (u, v) parameter plane that is enclosed by γ. We assume that s increases (by 1) during each *counterclockwise* passage along γ around ω.

Let

$$\begin{aligned} \Gamma &= \mathbf{x}(\gamma), && (5) \\ \Omega &= \mathbf{x}(\omega). && (6) \end{aligned}$$

Then

$$\mathbf{X}(s) = \mathbf{x}(u(s), v(s)) \quad (7)$$

is a parametric description of the curve Γ, which is a smooth simple closed curve on S, and the part of S enclosed by Γ is Ω.

It will be useful in the following to have a right-handed orthonormal triad

$$\left(\mathbf{a}(u, v), \mathbf{b}(u, v), \mathbf{n}(u, v)\right), \quad (8)$$

in which $\mathbf{n}$ is the surface normal defined by equation (2), and in which all three members of the triad are smooth functions of (u, v) defined on $\omega \cup \gamma$.

To construct such a triad, all we need is a nonzero vector field tangential to S, i.e., $\mathbf{w}(u, v)$ such that $\mathbf{w} \cdot \mathbf{n} = 0$ and $\mathbf{w} \neq 0$. For example, we could choose $\mathbf{w} = \dfrac{\partial \mathbf{x}}{\partial u}$ or $\mathbf{w} = \dfrac{\partial \mathbf{x}}{\partial v}$. Then, we need only set

$$\begin{aligned} \mathbf{a}(u, v) &= \frac{\mathbf{w}(u, v)}{\|\mathbf{w}(u, v)\|}, && (9) \\ \mathbf{b}(u, v) &= \mathbf{n}(u, v) \times \mathbf{a}(u, v). && (10) \end{aligned}$$

3 Parallel Transport

In the above setting, a vector field $\mathbf{P}(s)$ is said to be parallel-transported along the curve Γ on the surface S if it satisfies the following two conditions:

$$\begin{aligned} \mathbf{N}(s) \cdot \mathbf{P}(s) &= 0, && (11) \\ \mathbf{N}(s) \times \frac{\mathrm{d}\mathbf{P}}{\mathrm{d}s} &= 0. && (12) \end{aligned}$$

Here

$$\mathbf{N}(s) = \mathbf{n}(u(s), v(s)), \tag{13}$$

so that $\mathbf{N}(s)$ is the surface normal evaluated at the point on the curve Γ for which the parameter has the value s. Equation (12) is equivalent to the existence of a scalar field $\lambda(s)$ such that

$$\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}s} = \lambda(s)\mathbf{N}(s). \tag{14}$$

From (11) and (14) it is obvious that the length of any vector undergoing parallel transport is constant, and from now on we consider the transport of unit vectors only. In that case, we may write

$$\mathbf{P}(s) = \mathbf{a}(u(s), v(s)) \cos\theta(s) + \mathbf{b}(u(s), v(s)) \sin\theta(s). \tag{15}$$

It will also be useful to introduce

$$\mathbf{Q}(s) = -\mathbf{a}(u(s), v(s)) \sin\theta(s) + \mathbf{b}(u(s), v(s)) \cos\theta(s) \tag{16}$$

Then $(\mathbf{P}(s), \mathbf{Q}(s))$ is an orthonormal basis for the plane that is tangent to the surface S at the point $\mathbf{X}(s)$. Differentiation with respect to s in (15) gives

$$\begin{aligned} \frac{\mathrm{d}\mathbf{P}}{\mathrm{d}s} &= \mathbf{Q}\frac{\mathrm{d}\theta}{\mathrm{d}s} \\ &+ (\cos\theta)\left(\frac{\partial \mathbf{a}}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}s} + \frac{\partial \mathbf{a}}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}s}\right) \\ &+ (\sin\theta)\left(\frac{\partial \mathbf{b}}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}s} + \frac{\partial \mathbf{b}}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}s}\right). \end{aligned} \tag{17}$$

Now we need to enforce the condtion that $\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}s}$ is normal to the surface S. Since $\mathbf{P}$ is a unit vector, it is already guaranteed that $\mathbf{P} \cdot \frac{\mathrm{d}\mathbf{P}}{\mathrm{d}s} = 0$, so we just need to impose

$$\begin{aligned} 0 = \mathbf{Q} \cdot \frac{\mathrm{d}\mathbf{P}}{\mathrm{d}s} &= \frac{\mathrm{d}\theta}{\mathrm{d}s} \\ &+ \left(\cos^2\theta\right)\left(\mathbf{b} \cdot \frac{\partial \mathbf{a}}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}s} + \mathbf{b} \cdot \frac{\partial \mathbf{a}}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}s}\right) \\ &- \left(\sin^2\theta\right)\left(\mathbf{a} \cdot \frac{\partial \mathbf{b}}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}s} + \mathbf{a} \cdot \frac{\partial \mathbf{b}}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}s}\right). \end{aligned} \tag{18}$$

Since $\mathbf{a}$ and $\mathbf{b}$ are unit vectors, the derivatives of $\mathbf{a}$ are orthogonal to $\mathbf{a}$, and the derivatives of $\mathbf{b}$ are orthogonal to $\mathbf{b}$. That is why we do not have any terms involving $\sin\theta\cos\theta$ in the foregoing. Moreover, since $\mathbf{a}\cdot\mathbf{b} = 0$ we also have the identities

$$\mathbf{b}\cdot\frac{\partial\mathbf{a}}{\partial u} = -\mathbf{a}\cdot\frac{\partial\mathbf{b}}{\partial u}, \tag{19}$$

$$\mathbf{b}\cdot\frac{\partial\mathbf{a}}{\partial v} = -\mathbf{a}\cdot\frac{\partial\mathbf{b}}{\partial v}. \tag{20}$$

Making use of these identities, we see that the coefficient of $\cos^2\theta$ and the coefficient of $\sin^2\theta$ in equation (18) are the same, and that equation (18) is therefore equivalent to

$$\frac{\mathrm{d}\theta}{\mathrm{d}s} = \mathbf{a}\cdot\frac{\partial\mathbf{b}}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}s} + \mathbf{a}\cdot\frac{\partial\mathbf{b}}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}s}. \tag{21}$$

Note that the right-hand side of equation (21) does not involve θ. This shows that parallel transport actually involves a rigid rotation of the whole tangent plane (viewed as a collection of vectors undergoing parallel transport) as the point of tangency migrates along the curve.

4 Gaussian Curvature

We are now ready to integrate on both sides of equation (21) with respect to s over the interval $(0, 1)$. On the left-hand side we get the angular discrepancy produced by parallel transport once around Γ in the direction of increasing s. On the right-hand side we evaluate the integral by using Stokes' theorem in the (u, v) parameter plane:

$$\begin{aligned}
\Delta_{\Gamma,S} &= \theta(1) - \theta(0) \\
&= \int_0^1 \left(\mathbf{a}\cdot\frac{\partial\mathbf{b}}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}s} + \mathbf{a}\cdot\frac{\partial\mathbf{b}}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}s}\right)\mathrm{d}s \\
&= \oint_\gamma \left(\mathbf{a}\cdot\frac{\partial\mathbf{b}}{\partial u}\mathrm{d}u + \mathbf{a}\cdot\frac{\partial\mathbf{b}}{\partial v}\mathrm{d}v\right) \\
&= \iint_\omega \left(\frac{\partial}{\partial u}\left(\mathbf{a}\cdot\frac{\partial\mathbf{b}}{\partial v}\right) - \frac{\partial}{\partial v}\left(\mathbf{a}\cdot\frac{\partial\mathbf{b}}{\partial u}\right)\right)\mathrm{d}u\mathrm{d}v \\
&= \iint_\omega \left(\frac{\partial\mathbf{a}}{\partial u}\cdot\frac{\partial\mathbf{b}}{\partial v} - \frac{\partial\mathbf{a}}{\partial v}\cdot\frac{\partial\mathbf{b}}{\partial u}\right)\mathrm{d}u\mathrm{d}v.
\end{aligned} \tag{22}$$

In the last step of the foregoing, note the cancelation of second-derivative terms leading to the above result. In the integrand on the last line of (22), the dot products can be evaluated in the basis $(\mathbf{a}, \mathbf{b}, \mathbf{n})$. Since the derivatives of $\mathbf{a}$ are orthogonal to $\mathbf{a}$ and the derivatives of $\mathbf{b}$ are orthogonal to $\mathbf{b}$, only the coefficients of $\mathbf{n}$ conribute to the dot products. Thus,

$$
\begin{aligned}
\frac{\partial \mathbf{a}}{\partial u} \cdot \frac{\partial \mathbf{b}}{\partial v} - \frac{\partial \mathbf{a}}{\partial v} \cdot \frac{\partial \mathbf{b}}{\partial u} &= \left(\mathbf{n} \cdot \frac{\partial \mathbf{a}}{\partial u}\right)\left(\mathbf{n} \cdot \frac{\partial \mathbf{b}}{\partial v}\right) - \left(\mathbf{n} \cdot \frac{\partial \mathbf{a}}{\partial v}\right)\left(\mathbf{n} \cdot \frac{\partial \mathbf{b}}{\partial u}\right) \\
&= \left(\frac{\partial \mathbf{n}}{\partial u} \cdot \mathbf{a}\right)\left(\frac{\partial \mathbf{n}}{\partial v} \cdot \mathbf{b}\right) - \left(\frac{\partial \mathbf{n}}{\partial v} \cdot \mathbf{a}\right)\left(\frac{\partial \mathbf{n}}{\partial u} \cdot \mathbf{b}\right) \\
&= \left(\frac{\partial \mathbf{n}}{\partial u} \times \frac{\partial \mathbf{n}}{\partial v}\right) \cdot (\mathbf{a} \times \mathbf{b}) \\
&= \left(\frac{\partial \mathbf{n}}{\partial u} \times \frac{\partial \mathbf{n}}{\partial v}\right) \cdot \mathbf{n}. \qquad (23)
\end{aligned}
$$

Equation (22) can therefore be rewritten as

$$
\begin{aligned}
\Delta_{\Gamma,S} &= \iint_{\omega} \mathbf{n} \cdot \left(\frac{\partial \mathbf{n}}{\partial u} \times \frac{\partial \mathbf{n}}{\partial v}\right) \mathrm{d}u \mathrm{d}v \\
&= \iint_{\Omega} K \mathrm{d}A, \qquad (24)
\end{aligned}
$$

where $K : \Omega \to \mathbb{R}$ is implicitly defined by

$$
K\left(\mathbf{x}(u, v)\right) = \frac{\mathbf{n} \cdot \left(\dfrac{\partial \mathbf{n}}{\partial u} \times \dfrac{\partial \mathbf{n}}{\partial v}\right)}{\mathbf{n} \cdot \left(\dfrac{\partial \mathbf{x}}{\partial u} \times \dfrac{\partial \mathbf{x}}{\partial v}\right)}, \qquad (25)
$$

and where

$$
\mathrm{d}A = \mathbf{n} \cdot \left(\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}\right) \mathrm{d}u \mathrm{d}v \qquad (26)
$$

is the area element on S. Equation (25) defines the Gaussian curvature of the surface S as the ratio of the area swept by $\mathbf{n}$ on the unit sphere to that swept by $\mathbf{x}$ on the surface S. This formula for the Gaussian curvature can be written in another way by making use of equation (2) for $\mathbf{n}$:

$$\begin{aligned}
K\left(\mathbf{x}(u,v)\right) &= \frac{\left(\dfrac{\partial \mathbf{x}}{\partial u} \times \dfrac{\partial \mathbf{x}}{\partial v}\right) \cdot \left(\dfrac{\partial \mathbf{n}}{\partial u} \times \dfrac{\partial \mathbf{n}}{\partial v}\right)}{\left(\dfrac{\partial \mathbf{x}}{\partial u} \times \dfrac{\partial \mathbf{x}}{\partial v}\right) \cdot \left(\dfrac{\partial \mathbf{x}}{\partial u} \times \dfrac{\partial \mathbf{x}}{\partial v}\right)} \\
&= \frac{\left(\dfrac{\partial \mathbf{x}}{\partial u} \cdot \dfrac{\partial \mathbf{n}}{\partial u}\right)\left(\dfrac{\partial \mathbf{x}}{\partial v} \cdot \dfrac{\partial \mathbf{n}}{\partial v}\right) - \left(\dfrac{\partial \mathbf{x}}{\partial u} \cdot \dfrac{\partial \mathbf{n}}{\partial v}\right)\left(\dfrac{\partial \mathbf{x}}{\partial v} \cdot \dfrac{\partial \mathbf{n}}{\partial u}\right)}{\left\|\dfrac{\partial \mathbf{x}}{\partial u}\right\|^2 \left\|\dfrac{\partial \mathbf{x}}{\partial v}\right\|^2 - \left(\dfrac{\partial \mathbf{x}}{\partial u} \cdot \dfrac{\partial \mathbf{x}}{\partial v}\right)^2} \\
&= \frac{\left(\mathbf{n} \cdot \dfrac{\partial^2 \mathbf{x}}{\partial u^2}\right)\left(\mathbf{n} \cdot \dfrac{\partial^2 \mathbf{x}}{\partial v^2}\right) - \left(\mathbf{n} \cdot \dfrac{\partial^2 \mathbf{x}}{\partial u \partial v}\right)^2}{\left\|\dfrac{\partial \mathbf{x}}{\partial u}\right\|^2 \left\|\dfrac{\partial \mathbf{x}}{\partial v}\right\|^2 - \left(\dfrac{\partial \mathbf{x}}{\partial u} \cdot \dfrac{\partial \mathbf{x}}{\partial v}\right)^2} \\
&= \frac{\det \begin{pmatrix} \left(\mathbf{n} \cdot \dfrac{\partial^2 \mathbf{x}}{\partial u^2}\right) & \left(\mathbf{n} \cdot \dfrac{\partial^2 \mathbf{x}}{\partial u \partial v}\right) \\ \left(\mathbf{n} \cdot \dfrac{\partial^2 \mathbf{x}}{\partial u \partial v}\right) & \left(\mathbf{n} \cdot \dfrac{\partial^2 \mathbf{x}}{\partial v^2}\right) \end{pmatrix}}{\det \begin{pmatrix} \left\|\dfrac{\partial \mathbf{x}}{\partial u}\right\|^2 & \left(\dfrac{\partial \mathbf{x}}{\partial u} \cdot \dfrac{\partial \mathbf{x}}{\partial v}\right) \\ \left(\dfrac{\partial \mathbf{x}}{\partial u} \cdot \dfrac{\partial \mathbf{x}}{\partial v}\right) & \left\|\dfrac{\partial \mathbf{x}}{\partial v}\right\|^2 \end{pmatrix}}. \qquad (27)
\end{aligned}$$

Here we have used $\mathbf{n} \cdot \partial \mathbf{x}/\partial u = 0$ and $\mathbf{n} \cdot \partial \mathbf{x}/\partial v = 0$ to make the transition from the second line to the third line of equation (27). The symmetric matrices whose determinants appear on the last line of (27) have associated quadratic forms that are known as the first and second fundamental forms of the surface. The matrix in the denominator is the matrix of the first fundamental form, which is used in the evaluation of distance along a curve on the surface, and the matrix in the numerator is that of the second fundamental form, which is used in the evaluation of the curvatures of a curve on the surface.

5 Gauss' Theorema Egregium

Gauss' great discovery concerning the Gaussian curvature is that it is completely determined by the metric properties of the surface, independent of the embedding of the surface in $\mathbb{R}^3$. This will be proved if we can express the Gaussian curvature in terms of the first fundamental form and its derivatives. We will do this in the special case of orthogonal coordinates, in which the matrix of the first fundamental form is diagonal, that is,

$$\frac{\partial \mathbf{x}}{\partial u} \cdot \frac{\partial \mathbf{x}}{\partial v} = 0. \tag{28}$$

Then we may set

$$\mathbf{a} = \alpha \frac{\partial \mathbf{x}}{\partial u}, \quad \alpha = \left\| \frac{\partial \mathbf{x}}{\partial u} \right\|^{-1}, \tag{29}$$

$$\mathbf{b} = \beta \frac{\partial \mathbf{x}}{\partial v}, \quad \beta = \left\| \frac{\partial \mathbf{x}}{\partial v} \right\|^{-1}, \tag{30}$$

and it follows that

$$\frac{\partial \mathbf{b}}{\partial u} = \beta \frac{\partial^2 \mathbf{x}}{\partial u \partial v} + \frac{\mathrm{d}\beta}{\mathrm{d}u} \frac{\partial \mathbf{x}}{\partial v}, \tag{31}$$

$$\frac{\partial \mathbf{b}}{\partial v} = \beta \frac{\partial^2 \mathbf{x}}{\partial v^2} + \frac{\mathrm{d}\beta}{\mathrm{d}v} \frac{\partial \mathbf{x}}{\partial v}. \tag{32}$$

Because of (28), we then have

$$\begin{aligned} \mathbf{a} \cdot \frac{\partial \mathbf{b}}{\partial u} &= \alpha\beta \frac{\partial \mathbf{x}}{\partial u} \cdot \frac{\partial^2 \mathbf{x}}{\partial u \partial v} \\ &= \frac{\alpha\beta}{2} \frac{\partial}{\partial v} \left(\left\| \frac{\partial \mathbf{x}}{\partial u} \right\|^2 \right) \end{aligned} \tag{33}$$

$$\begin{aligned} \mathbf{a} \cdot \frac{\partial \mathbf{b}}{\partial v} &= \alpha\beta \frac{\partial \mathbf{x}}{\partial u} \cdot \frac{\partial^2 \mathbf{x}}{\partial v^2} \\ &= -\alpha\beta \frac{\partial^2 \mathbf{x}}{\partial u \partial v} \cdot \frac{\partial \mathbf{x}}{\partial v} \\ &= -\frac{\alpha\beta}{2} \frac{\partial}{\partial u} \left(\left\| \frac{\partial \mathbf{x}}{\partial v} \right\|^2 \right) \end{aligned} \tag{34}$$

Substitution of (33) and (34) into (21) then gives the result

$$\frac{\mathrm{d}\theta}{\mathrm{d}s} = \frac{\alpha\beta}{2}\frac{\partial}{\partial v}\left(\left\|\frac{\partial \mathbf{x}}{\partial u}\right\|^2\right)\frac{\mathrm{d}u}{\mathrm{d}s} - \frac{\alpha\beta}{2}\frac{\partial}{\partial u}\left(\left\|\frac{\partial \mathbf{x}}{\partial v}\right\|^2\right)\frac{\mathrm{d}v}{\mathrm{d}s}. \tag{35}$$

Then, by integration with respect to s over $(0,1)$, we get

$$\begin{aligned}
\Delta_{\Gamma,S} &= \int_0^1 \left(\frac{\alpha\beta}{2}\left(\frac{\partial}{\partial v}\left(\left\|\frac{\partial \mathbf{x}}{\partial u}\right\|^2\right)\right)\frac{\mathrm{d}u}{\mathrm{d}s} - \frac{\alpha\beta}{2}\left(\frac{\partial}{\partial u}\left(\left\|\frac{\partial \mathbf{x}}{\partial v}\right\|^2\right)\right)\frac{\mathrm{d}v}{\mathrm{d}s}\right)\mathrm{d}s \\
&= \oint_\gamma \left(\frac{\alpha\beta}{2}\left(\frac{\partial}{\partial v}\left(\left\|\frac{\partial \mathbf{x}}{\partial u}\right\|^2\right)\right)\mathrm{d}u - \frac{\alpha\beta}{2}\left(\frac{\partial}{\partial u}\left(\left\|\frac{\partial \mathbf{x}}{\partial v}\right\|^2\right)\right)\mathrm{d}v\right) \\
&= -\iint_\omega \left(\frac{\partial}{\partial u}\left(\frac{\alpha\beta}{2}\frac{\partial}{\partial u}\left(\left\|\frac{\partial \mathbf{x}}{\partial v}\right\|^2\right)\right) + \frac{\partial}{\partial v}\left(\frac{\alpha\beta}{2}\frac{\partial}{\partial v}\left(\left\|\frac{\partial \mathbf{x}}{\partial u}\right\|^2\right)\right)\right)\mathrm{d}u\mathrm{d}v.
\end{aligned} \tag{36}$$

Since our coordinates are orthogonal, the determinant of the matrix of the first fundamental form is given by

$$D = \left(\left\|\frac{\partial \mathbf{x}}{\partial u}\right\|\left\|\frac{\partial \mathbf{x}}{\partial v}\right\|\right)^2, \tag{37}$$

and the area element is given by

$$\mathrm{d}A = D^{1/2}\mathrm{d}u\mathrm{d}v = \frac{\mathrm{d}u\mathrm{d}v}{\alpha\beta} \tag{38}$$

Thus, we again have $\Delta_{\Gamma,S} = \iint_\Omega K\mathrm{d}A$ provided that we set

$$K = -\frac{D^{-1/2}}{2}\left[\frac{\partial}{\partial u}\left(D^{-1/2}\frac{\partial}{\partial u}\left(\left\|\frac{\partial \mathbf{x}}{\partial v}\right\|^2\right)\right) + \frac{\partial}{\partial v}\left(D^{-1/2}\frac{\partial}{\partial v}\left(\left\|\frac{\partial \mathbf{x}}{\partial u}\right\|^2\right)\right)\right]. \tag{39}$$

This formula expresses the Gaussian curvature entirely in terms of the elements of the matrix of the first fundamental form, and hence it proves Gauss' Theorema Egregium.

6 Principal Curvatures

We have managed to get this far without ever making use of the local definition of Gaussian curvature as the product of the principal curvatures of the surface. For completeness, we now evaluate the principal curvatures and show that their product is equal to the Gaussian curvature as previously defined.

Here we consider the family of smooth curves through a given point on the surface S. Each of the curves is parameterized by arclength, and is of the form

$$\mathbf{X}(s) = \mathbf{x}(u(s), v(s)). \tag{40}$$

Since s measures arclength we have

$$\left\| \frac{\mathrm{d}\mathbf{X}}{\mathrm{d}s} \right\|^2 = 1, \tag{41}$$

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}s} \cdot \frac{\mathrm{d}^2\mathbf{X}}{\mathrm{d}s^2} = 0. \tag{42}$$

The vector $\dfrac{\mathrm{d}\mathbf{X}}{\mathrm{d}s}$ is the unit tangent to the curve, and the vector $\dfrac{\mathrm{d}^2\mathbf{X}}{\mathrm{d}s^2}$ may be called the *curvature vector* of the curve. These two vectors can be expressed in terms of derivatives of $\mathbf{x}(u, v)$ as follows:

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}s} = \frac{\partial \mathbf{x}}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}s} + \frac{\partial \mathbf{x}}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}s}, \tag{43}$$

$$\begin{aligned} \frac{\mathrm{d}^2\mathbf{X}}{\mathrm{d}s^2} &= \frac{\partial^2 \mathbf{x}}{\partial u^2}\left(\frac{\mathrm{d}u}{\mathrm{d}s}\right)^2 + 2\frac{\partial^2 \mathbf{x}}{\partial u \partial v}\left(\frac{\mathrm{d}u}{\mathrm{d}s}\right)\left(\frac{\mathrm{d}v}{\mathrm{d}s}\right) + \frac{\partial^2 \mathbf{x}}{\partial v^2}\left(\frac{\mathrm{d}v}{\mathrm{d}s}\right)^2 \\ &+ \frac{\partial \mathbf{x}}{\partial u}\left(\frac{\mathrm{d}^2 u}{\mathrm{d}s^2}\right) + \frac{\partial \mathbf{x}}{\partial v}\left(\frac{\mathrm{d}^2 v}{\mathrm{d}s^2}\right). \end{aligned} \tag{44}$$

Substitution of (43) into (41) then gives

$$\frac{\partial \mathbf{x}}{\partial u}\cdot\frac{\partial \mathbf{x}}{\partial u}\left(\frac{\mathrm{d}u}{\mathrm{d}s}\right)^2 + 2\frac{\partial \mathbf{x}}{\partial u}\cdot\frac{\partial \mathbf{x}}{\partial v}\left(\frac{\mathrm{d}u}{\mathrm{d}s}\right)\left(\frac{\mathrm{d}v}{\mathrm{d}s}\right) + \frac{\partial \mathbf{x}}{\partial v}\cdot\frac{\partial \mathbf{x}}{\partial v}\left(\frac{\mathrm{d}v}{\mathrm{d}s}\right)^2 = 1. \tag{45}$$

Also, if we define C_{n} as the component of the curvature vector that is normal to the surface, then

$$C_{\mathrm{n}} = \mathbf{n}\cdot\frac{\mathrm{d}^2\mathbf{X}}{\mathrm{d}s^2} = \mathbf{n}\cdot\frac{\partial^2 \mathbf{x}}{\partial u^2}\left(\frac{\mathrm{d}u}{\mathrm{d}s}\right)^2 + 2\mathbf{n}\cdot\frac{\partial^2 \mathbf{x}}{\partial u \partial v}\left(\frac{\mathrm{d}u}{\mathrm{d}s}\right)\left(\frac{\mathrm{d}v}{\mathrm{d}s}\right) + \mathbf{n}\cdot\frac{\partial^2 \mathbf{x}}{\partial v^2}\left(\frac{\mathrm{d}v}{\mathrm{d}s}\right)^2. \tag{46}$$

Note that the second derivatives of u and v do not appear here, since the terms in (44) that involve these second derivatives are tangential to the surface S. The quadratic forms that appear in equations (45) and (46) are known as the first and second fundamental forms, respectively.

Now we introduce the notation

$$\dot{u} = \frac{\mathrm{d}u}{\mathrm{d}s}, \quad \dot{v} = \frac{\mathrm{d}v}{\mathrm{d}s}. \tag{47}$$

$$E = \left\| \frac{\partial \mathbf{x}}{\partial u} \right\|^2, \quad F = \frac{\partial \mathbf{x}}{\partial u} \cdot \frac{\partial \mathbf{x}}{\partial v}, \quad G = \left\| \frac{\partial \mathbf{x}}{\partial v} \right\|^2. \tag{48}$$

$$L = \mathbf{n} \cdot \frac{\partial^2 \mathbf{x}}{\partial u^2}, \quad M = \mathbf{n} \cdot \frac{\partial^2 \mathbf{x}}{\partial u \partial v}, \quad N = \mathbf{n} \cdot \frac{\partial^2 \mathbf{x}}{\partial v^2}. \tag{49}$$

In matrix form, equations (45-46) read as follows:

$$1 = \begin{pmatrix} \dot{u} & \dot{v} \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix}, \tag{50}$$

$$C_{\mathrm{n}} = \begin{pmatrix} \dot{u} & \dot{v} \end{pmatrix} \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix}, \tag{51}$$

The matrix $\begin{pmatrix} E & F \\ F & G \end{pmatrix}$ is symmetric and positive definite. Non-negative definiteness follows from the Schwarz inequality, and positive definiteness then follows from the equality case of the Schwarz inequality, since the coordinate basis vectors $\frac{\partial \mathbf{x}}{\partial u}$ and $\frac{\partial \mathbf{x}}{\partial v}$ are not allowed to be aligned. It follows that this matrix has a unique positive definite square root, which we denote $\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{1/2}$. With the help of this matrix, we can ensure that the constraint (50) is satisfied by setting

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1/2} \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} \tag{52}$$

The normal curvature C_{n} then becomes a function of ϕ which is given by

$$C_{\mathrm{n}}(\phi) = \begin{pmatrix} \cos\phi & \sin\phi \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1/2} \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1/2} \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} \tag{53}$$

In equation (53), only ϕ is variable. This is because we are interested in a particular point on the surface S, and are considering all possible smooth curves through that point. At the point of interest, the coefficients of the first and second fundamental forms are specific numbers which do not depend on the orientation of the curve that is passing through that point. The orientation of the curve as it passes through the point of interest is encoded by ϕ.

Let

$$Q = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1/2} \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1/2}. \tag{54}$$

Then Q is symmetric, and

$$\begin{aligned} C_{\mathrm{n}}(\phi) &= Q_{11}\cos^2\phi + 2Q_{12}\cos\phi\sin\phi + Q_{22}\sin^2\phi \\ &= \frac{1}{2}(Q_{11}+Q_{22}) + \frac{1}{2}(Q_{11}-Q_{22})\cos(2\phi) + Q_{12}\sin(2\phi). \end{aligned} \tag{55}$$

Now let ψ be defined by

$$\cos(2\psi) = \frac{\frac{1}{2}(Q_{11}-Q_{22})}{D}, \tag{56}$$

$$\sin(2\psi) = \frac{Q_{12}}{D}, \tag{57}$$

where

$$D = \sqrt{\left(\frac{1}{2}(Q_{11}-Q_{22})\right)^2 + Q_{12}^2}. \tag{58}$$

Then

$$C_{\mathrm{n}}(\phi) = \frac{1}{2}(Q_{11}+Q_{22}) + D\cos(2(\phi-\psi)) \tag{59}$$

From this formula for C_{n} it is obvious that its maximum and minimum values are given by

$$\frac{1}{2}(Q_{11}+Q_{22}) \pm D. \tag{60}$$

These are called the principal curvatures, and their product is

$$\begin{aligned}
&\left(\tfrac{1}{2}\left(Q_{11}+Q_{22}\right)\right)^2 - D^2 \\
&= \left(\frac{1}{2}\left(Q_{11}+Q_{22}\right)\right)^2 - \left(\frac{1}{2}\left(Q_{11}-Q_{22}\right)\right)^2 - Q_{12}^2 \\
&= Q_{11}Q_{22} - Q_{12}^2 = \det(Q) \\
&= \det\left(\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1/2} \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1/2}\right) \\
&= \frac{\det\begin{pmatrix} L & M \\ M & N \end{pmatrix}}{\det\begin{pmatrix} E & F \\ F & G \end{pmatrix}} = K, \qquad (61)
\end{aligned}$$

see (27). A simpler but slightly more abstact way to obtain the above result starting from equations (53-54) is to note that the maximum and minimum of C_{n} are the larger and smaller, respectively, of the two eigenvalues of Q, and then that the product of the eigenvalues is the determinant of Q.

7 Geodesic Curvature

The setting here is exactly the same as in Section 2 except that here we assume that the parameter s measures arclength along the curve Γ on the surface S. Then $\boldsymbol{\tau} = \dfrac{\mathrm{d}\mathbf{X}}{\mathrm{d}s}$ is a unit vector, so it has a representation

$$\boldsymbol{\tau}(s) = \mathbf{a}(u(s), v(s)) \cos\alpha(s) + \mathbf{b}(u(s), v(s)) \sin\alpha(s). \qquad (62)$$

Here $\mathbf{a}(u, v)$ and $\mathbf{b}(u, v)$ are the same as in Section 2. It will also be useful to define

$$\boldsymbol{\sigma}(s) = -\mathbf{a}(u(s), v(s)) \sin\alpha(s) + \mathbf{b}(u(s), v(s)) \cos\alpha(s). \qquad (63)$$

Note that $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$ obey equations of the same form as those obeyed by $\mathbf{P}$ and $\mathbf{Q}$ in Section 3, with α here playing the role of θ. One very important difference, however, is that $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$ are *not* evolving according to parallel transport, and therefore

$\boldsymbol{\sigma}$ will not in general be orthogonal to $\dfrac{\mathrm{d}\boldsymbol{\tau}}{\mathrm{d}s}$. Indeed, the geodesic curvature of the curve Γ on the surface S is defined by

$$k_{\mathrm{g}} = \boldsymbol{\sigma} \cdot \frac{\mathrm{d}\boldsymbol{\tau}}{\mathrm{d}s} \tag{64}$$

Now following the same manipulations as were done in equations (15-21), we get

$$\begin{aligned} k_{\mathrm{g}} &= \frac{\mathrm{d}\alpha}{\mathrm{d}s} - \left(\mathbf{a} \cdot \frac{\partial \mathbf{b}}{\partial u} \frac{\mathrm{d}u}{\mathrm{d}s} + \mathbf{a} \cdot \frac{\partial \mathbf{b}}{\partial v} \frac{\mathrm{d}v}{\mathrm{d}s} \right) \\ &= \frac{\mathrm{d}\alpha}{\mathrm{d}s} - \frac{\mathrm{d}\theta}{\mathrm{d}s}, \end{aligned} \tag{65}$$

in which the last step makes use of equation (21).

Next, we integrate once around the closed curve Γ. The integral of $\dfrac{\mathrm{d}\theta}{\mathrm{d}s}$ has already been evaluated, see equation (24), and we claim that the integral of $\dfrac{\mathrm{d}\alpha}{\mathrm{d}s}$ is 2π. To prove this, note that the unit tangent vector $\boldsymbol{\tau}$ must return to the same vector value after one passage around the smooth closed curve Γ, and therefore the integral of $\dfrac{\mathrm{d}\alpha}{\mathrm{d}s}$ has to be an integer multiple of 2π. In fact, this integral is equal to 2π in our case, since the simple closed curve Γ can be continously deformed to a single point on the smooth surface S, and moreover this can be done in such a way that the curve is asymptotically circular in the tangent plane to S at the point towards which the curve is shrinking. For a planar circle, the integral of $\dfrac{\mathrm{d}\alpha}{\mathrm{d}s}$ is obviously equal to 2π, and the integer that multiplies 2π cannot change during a continuous deformation of the curve.

In this way equation (65) implies the local Gauss-Bonnet theorem. which we have stated at the outset, see equation (1).