Kirchhoff Rod

Constrained & Unconstrained

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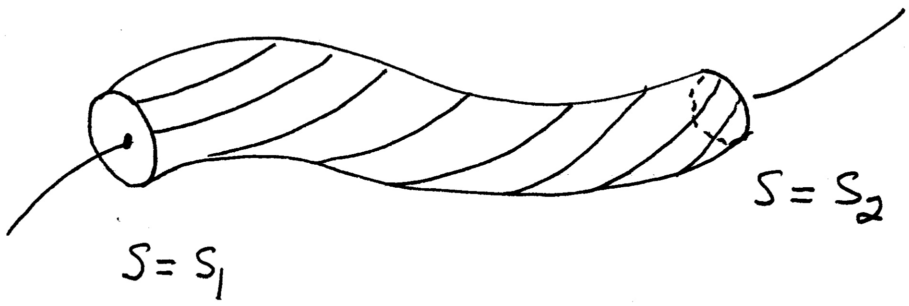
The purpose of these notes is to provide an expository introduction to Kirchhoff rod theory as background for the paper:

Dynamics of a Closed Rod with Bend and Twist in Fluid, by Sookkyng Lim, Anca Ferent, X. Sheldon Wang, and Charles S. Peskin SIAM J SCI COMPUT 31(1): 273-302, 2008

Equilibrium of a bent twisted rod with applied forces and torques



Let the surface of a rod be

(1) $$\underline{x} = \underline{X}(s, \theta)$$

with

(2) $$\underline{X}(s, \theta + 2\pi) = \underline{X}(s, \theta)$$

and with area element

(3) $$a(s, \theta)\, ds\, d\theta$$

Let $\underline{g}(s, \theta)$ be a force per unit area that is applied to the surface of the rod.

The total force and total torque applied to the part of the surface with $s \in (s_1, s_2)$ are then given by

$$(4) \qquad \underline{F}(s_1, s_2) = \int_{s_1}^{s_2} \int_0^{2\pi} \underline{g}(s,\theta)\, a(s,\theta)\, d\theta\, ds$$

$$(5) \qquad \underline{\tau}(s_1, s_2) = \int_{s_1}^{s_2} \int_0^{2\pi} \left(\underline{X}(s,\theta) \times \underline{g}(s,\theta) \right) a(s,\theta)\, d\theta\, ds$$

Let $\underline{X}^0(s)$ and $\underline{g}^0(s)$ be defined by

$$(6) \qquad \int_0^{2\pi} \left(\underline{X}(s,\theta) - \underline{X}^0(s) \right) a(s,\theta)\, d\theta = 0$$

$$(7) \qquad \int_0^{2\pi} \left(\underline{g}(s,\theta) - \underline{g}^0(s) \right) a(s,\theta)\, d\theta = 0$$

Also let

(8) $$\underline{f}(s) = \underline{g}^0(s) \int_0^{2\pi} a(s,\theta)\, d\theta$$

Then

(9) $$\underline{F}(s_1, s_2) = \int_{s_1}^{s_2} \underline{f}(s)\, ds$$

and the torque $\underline{\tau}(s_1, s_2)$ can be rewritten as follows

(10) $$\underline{\tau}(s_1, s_2) =$$

$$\int_{s_1}^{s_2} \int_0^{2\pi} \left(\underline{X}^0(s) + \left(\underline{X}(s,\theta) - \underline{X}^0(s) \right) \right) \times$$

$$\left(\underline{g}^0(s) + \left(\underline{g}(s,\theta) - \underline{g}^0(s) \right) \right) a(s,\theta)\, d\theta\, ds$$

$$= \int_{s_1}^{s_2} \left(\underline{X}^0(s) \times \underline{f}(s) + \underline{n}(s) \right) ds$$

where

(11) $$\underline{n}(s) =$$

$$\int_0^{2\pi} \left(\underline{X}(s,\theta) - \underline{X}^0(s)\right) \times \left(\underline{g}(s,\theta) - \underline{g}^0(s)\right) a(s,\theta)\, d\theta$$

In the above rewrite of the formula for $\underline{\tau}(s_1, s_2)$ we got only two terms in the end instead of four because the other two terms are equal to zero. This follows from the definitions of $\underline{X}^0(s)$ and $\underline{g}^0(s)$, equations (6-7).

We shall refer to the curve $\underline{x} = \underline{X}^0(s)$ as the <u>centerline</u> of the rod.

The functions $\underline{f}(s)$ and $\underline{n}(s)$ summarize what we need to know about the applied force per unit area $\underline{g}(s,\theta)$.

They are the applied <u>force density</u> and <u>couple density</u> with respect to the measure ds.

Note that the couple density $\underline{n}(s)$ as given by (11) is zero if $\underline{g}(s,\theta)$ is independent of θ, since in that case $\underline{g}(s,\theta) = \underline{g}^0(s)$. Also,

the torque $\underline{n}(s)\,ds$ is evaluated with respect to the nearby point $\underline{X}^0(s)$ even though $\underline{\tau}(s_1, s_2)$, to which $\underline{n}(s)\,ds$ contributes, is defined as torque about the origin. A change of origin changes $\underline{\tau}(s_1, s_2)$ but does not change $\underline{n}(s)\,ds$.

Since the rod is elastic and deformed, there is some force per unit area (also known as stress) that acts across each section of the rod $s = \text{constant}$.

In the same manner as above, we can derive from this stress both a force $\underline{F}(s)$ and a couple $\underline{N}(s)$ that are applied by the part of the rod with $s' > s$ onto the part of the rod with $s' < s$.

Of course it then follows that $-\underline{F}(s)$ and $-\underline{N}(s)$ are applied in the opposite direction across the section of the rod at s.

The torque transmitted across the section of the rod at s with the same sign convention as above is then given by

$$(12) \quad \underline{\tau}(s) = \left(\underline{X}^0(s) \times \underline{F}(s)\right) + \underline{N}(s)$$

Note that there is an approximation involved in equation (12), since $\underline{X}^0(s)$ was defined at the centroid of the area-weighted circumferential curve at s, but now we are using it as the centroid of the entire cross-section of the rod at s.

Equilibrium of the segment of rod for which $s \in (s_1, s_2)$ requires that the force and torque on that segment should be zero. This gives the following equation

(13)
$$0 = \underline{F}(s_1, s_2) + \underline{F}(s_2) - F(s_1)$$

$$= \int_{s_1}^{s_2} \left(\underline{f}(s) + \frac{d\underline{F}}{ds}(s) \right) ds$$

(14)
$$0 = \underline{\tau}(s_1, s_2) + \underline{\tau}(s_2) - \underline{\tau}(s_1)$$

$$= \int_{s_1}^{s_2} \Big(\left(\underline{X}^0(s) \times \underline{f}(s) \right) + \underline{n}(s)$$
$$+ \frac{d}{ds} \left(\left(\underline{X}^0(s) \times \underline{F}(s) \right) + \underline{N}(s) \right) \Big) ds$$

Since s_1 and s_2 are arbitrary, the integrands in (13-14) must be zero. From (13), we get

(15) $$0 = \underline{f}(s) + \frac{d\underline{F}}{ds}(s)$$

and this can be used to simplify (14), which then gives

(16) $$0 = \underline{n}(s) + \frac{d\underline{N}}{ds}(s) + \frac{d\underline{X}^0}{ds}(s) \times \underline{F}(s)$$

These are the fundamental equilibrium equations of a thin* elastic rod.

To proceed further, we need additional hypotheses.

* the rod needs to be thin to justify the approximation concerning the centerline that was mentioned above.

The configuration of a Kirchhoff rod is defined by

(17) $$\underline{X}^0(s),\ \underline{D}^1(s),\ \underline{D}^2(s),\ \underline{D}^3(s)$$

in which $\underline{D}^1(s), \underline{D}^2(s), \underline{D}^3(s)$ is a right-handed orthonormal triad at each value of s. The vector $\underline{D}^3(s)$ is constrained to be aligned with the tangent vector to the centerline $d\underline{X}^0/ds$. Another constraint is that the centerline is inextensible, so the parameter s may be chosen such that s = arclength along the centerline. These constraints are summarized by

(18) $$\frac{d\underline{X}^0}{ds}(s) = \underline{D}^3(s)$$

Since $\|\underline{D}^3(s)\| = 1$, this implies not only that $D^3(s)$ is aligned with $d\underline{X}^0/ds$ but also that s = arclength.

The intuitive idea is that the triad $\underline{D}^1(s)$, $\underline{D}^2(s)$, $\underline{D}^3(s)$ is embedded in the material of the rod and can be therefore used to describe the local orientation of the material. Accordingly, the triad is called a <u>material frame</u>. Although it resembles the Frenet frame of differential geometry, the two are different and should not be confused. In particular, the Frenet frame of a space curve is determined by the curve, but the triad of Kirchhoff rod theory is not so determined (except for the vector $\underline{D}^3(s)$, which is determined by (18)). Two configurations of a rod with the same centerline may be twisted differently, and then their triads will not be the same.

Since the triad rotates as a rigid body as s varies, there is a vector $\underline{K}(s)$ such that

(19) $$\frac{d\underline{D}^i}{ds}(s) = \underline{K}(s) \times \underline{D}^i(s) \quad , \quad i=1,2,3$$

Note the analogy between $\underline{K}(s)$ and the angular velocity of a rigid body. In fact, if we move along the centerline at speed 1, the $\underline{K}(s)$ is the angular velocity with which the triad rotates.

We can find the components of $\underline{K}(s)$ in the basis $\{\underline{D}^1(s), \underline{D}^2(s), \underline{D}^3(s)\}$ in the following way.

Let (i,j,k) be a cyclic permutation of $(1,2,3)$, and apply $\cdot\,\underline{D}^j(s)$ to both sides of (19):

$$(20) \qquad \frac{d\underline{D}^i}{ds}(s)\cdot\underline{D}^j(s) = \left(\underline{K}(s)\times\underline{D}^i(s)\right)\cdot\underline{D}^j(s)$$

$$= \underline{K}(s)\cdot\left(\underline{D}^i(s)\times\underline{D}^j(s)\right)$$

$$= \underline{K}(s)\cdot\underline{D}^k(s) \quad = K_k(s)$$

Two conventions here that we shall use extensively in the following are that (i,j,k) is a cyclic permutation of $(1,2,3)$ and that a component of a vector refers to its component in the basis $\{\underline{D}^1(s), \underline{D}^2(s), \underline{D}^3(s)\}$.

We now have all of the apparatus needed to state the key hypothesis of Kirchhoff rod theory, which is that the components of the couple $\underline{N}(s)$ are related to the components of $\underline{K}(s)$ in the following simple way

$$N_i(s) = a_i \, K_i(s) \quad , \quad i=1,2,3 \tag{21}$$

where a_1, a_2, a_3 are given constants.

The constants a_1, a_2 are bending moduli and a_3 is the twist modulus of the rod. These moduli are independent of s because we are assuming that the rod is homogeneous. The relationship (21) is diagonal because $\underline{D}^1(s)$ and $\underline{D}^2(s)$ are assumed to be aligned in what may be called the principal directions within each cross section. In the special case that the cross sections are isotropic, $a_1 = a_2$.

Our task now is to write the equilibrium equations (15-16) in components with the triad as our basis, so that we can make use of (21). The interesting terms are $d\underline{F}/ds$ and $d\underline{N}/ds$. Since

$$\underline{F}(s) = \sum_{l=1}^{3} F_l(s)\, \underline{D}^l(s) \tag{22}$$

we have

$$\frac{d\underline{F}}{ds} = \sum_{l=1}^{3}\left(\frac{dF_l}{ds}\underline{D}^l + F_l \frac{d\underline{D}^l}{ds}\right) \tag{23}$$

$$\left(\frac{d\underline{F}}{ds}\right)_i = \frac{dF_i}{ds} + \sum_{l=i,j,k} F_l \left(\underline{K}\times\underline{D}^l\right)\cdot\underline{D}^i \tag{24}$$

$$= \frac{dF_i}{ds} + \sum_{l=i,j,k} F_l\, \underline{K}\cdot\left(\underline{D}^l\times\underline{D}^i\right)$$

$$= \frac{dF_i}{ds} - F_j K_k + F_k K_j$$

Thus,

$$\left(\frac{d\underline{F}}{ds}\right)_i = \frac{dF_i}{ds} + K_j F_k - K_k F_j \tag{25}$$

and similarly

$$\left(\frac{d\underline{N}}{ds}\right)_i = \frac{dN_i}{ds} + K_j N_k - K_k N_j \tag{26}$$

$$= a_i \frac{dK_i}{ds} + (a_k - a_j) K_j K_k$$

In the last step, we used the Kirchhoff hypothesis, (21).

Another term that we need to evaluate is

$$\frac{d\underline{X}^0}{ds} \times \underline{F} = \underline{D}^3 \times \sum_{l=1}^{3} F_l \underline{D}^l = F_1 \underline{D}^2 - F_2 \underline{D}^1 \tag{27}$$

which gives

$$\left(\frac{d\underline{X}^0}{ds} \times \underline{F}\right)_i = F_1 \delta_{i2} - F_2 \delta_{i1} \tag{28}$$

Therefore, the equilibrium equations (15–16) can be written out in components as follows:

(29) $$0 = f_i + \frac{dF_i}{ds} + K_j F_k - K_k F_j$$

(30) $$0 = n_i + a_i \frac{dK_i}{ds} + (a_k - a_j) K_j K_k + F_1 \delta_{i2} - F_2 \delta_{i1}$$

These equations hold for $i = 1, 2, 3$, and for each i, j and k are determined by the condition that (i, j, k) is a cyclic permutation of $(1, 2, 3)$.

Part I

Homework: Solve for the circular equilibrium of a bent, twisted Kirchhoff rod in the absence of applied forces and couples, so $\underline{f}(s)$ and $\underline{n}(s)$ are both zero. The rod has radius R and length $2\pi R$ so its centerline is given by

$$\underline{X}^0(s) = R\left(\cos\left(\frac{s}{R}\right), \sin\left(\frac{s}{R}\right), 0\right) \tag{31}$$

Assume that $a_1 = a_2$. Impose the condition that the triad makes an integer number of turns in traveling once around the rod, so that the functions $\underline{D}^i(s)$ are periodic with period $2\pi R$. (This condition is not essential. Since $a_1 = a_2$ there is actually a continuum of smooth solutions, but their description requires in general a multivalued triad.)

Evaluate all variables. For $\underline{K}, \underline{F}, \underline{N}$, find their components in the triad frame and also in the lab frame. In what lab-frame direction does the vector $\underline{F}$ point?

In Kirchhoff rod theory, $\underline{N}(s)$ is determined by equation (21) when the configuration of the rod is given, but there is no similar equation for $\underline{F}(s)$. This is because $\underline{F}(s)$ is a constraint force. It takes on whatever values may be needed to enforce inextensibility of the centerline and alignment of $\underline{D}^3(s)$ with the tangent vector to the centerline.

To make use of Kirchhoff rod theory in the context of a rod immersed in fluid, we would therefore have to formulate the inextensibility and alignment constraints as conditions that have to be satisfied by the fluid velocity field in the neighborhood of the rod, and then we would have to figure out how to solve for $\underline{F}(s)$ such that those constraints continue to be satisfied as time evolves. These are daunting tasks!

Alternatively, we can generalize Kirchhoff rod theory, by allowing (18) to be violated, but at the same time by providing a physical mechanism that <u>tends</u> to maintain both inextensibility and alignment, without enforcing those conditions strictly.

Accordingly, we postulate the following elastic energy functional

(32) $$E = \frac{1}{2}\int_0^L \sum_{i=1}^{3} \left(a_i K_i^2(s) + b_i \left(\underline{D}^i(s) \cdot \frac{d\underline{X}^0}{ds}(s) - \delta_{3i} \right)^2 \right) ds$$

where

(33) $$K_i(s) = \frac{d\underline{D}^j}{ds}(s) \cdot \underline{D}^k(s)$$

Here s is a material coordinate that is equal to arclength in the unstressed configuration of the rod, but not in general. The constant L is the unstressed length of the rod.

Note that E is a functional of the four vector-valued functions

(34) $$\underline{X}^0(s),\ \underline{D}^1(s),\ \underline{D}^2(s),\ \underline{D}^3(s)$$

which we refer to collectively as the configuration of the rod. Although we have dropped the constraints of inextensibility and alignment, we retain the constraint that

(35) $$\underline{D}^1(s),\ \underline{D}^2(s),\ \underline{D}^3(s)$$

is a right-handed orthonormal triad at every s. This constraint will play an important role in the variational argument that we use below to obtain equations of equilibrium.

The constants a_i have the same meaning as before, and the terms involving these constants are the elastic energy of a standard Kirchoff rod.

The new terms involving the constants b_i tend to enforce both alignment and inextensibility. The terms involving b_1 and b_2 are trying to keep $\underline{D}^1$ and $\underline{D}^2$ orthogonal to $d\underline{X}^0/ds$. To the extent that they succeed, $\underline{D}^3$ will be aligned* with $d\underline{X}^0/ds$, and then the term involving b_3 will tend to drive $\|d\underline{X}^0/ds\|$ towards 1. Thus the various terms cooperate in the enforcement of alignment and inextensibility.

* or anti-aligned, but the term involving b_3 has a strong preference for alignment over anti-alignment.

The applied forces and couples appear in our formulation in terms of the work that they do on an arbitrary perturbation of the configuration of the rod. Such a perturbation is given by

(36) $$\delta \underline{X}(s),\ \delta \underline{D}^1(s),\ \delta \underline{D}^2(s),\ \delta \underline{D}^3(s)$$

but the most general perturbation of the triad that we may consider is a rotation

(37) $$\delta \underline{D}^i(s) = \delta \underline{\Omega}(s) \times \underline{D}^i(s)$$

Here $\delta \underline{\Omega}(s)$ is a small arbitrary vector-valued function of s that generates the rotation. Thus, an arbitrary perturbation is characterized by arbitrary $\delta \underline{X}(s)$ and $\delta \underline{\Omega}(s)$, and the work done during such a perturbation by the applied forces and couples is given by

$$(38) \quad \delta W = \int_0^L \left(\underline{f}(s) \cdot \delta \underline{X}^0(s) + \underline{n}(s) \cdot \delta \underline{\Omega}(s) \right) ds$$

Our task now is to evaluate δE for the same perturbation so that we can apply the principle of virtual work to obtain the equation of equilibrium

From (32)

$$(39) \quad \delta E = \int_0^L \sum_{i=1}^{3} \Big(a_i K_i(s) \delta K_i(s)$$

$$+ b_i \left(\underline{D}^i(s) \cdot \frac{d\underline{X}^0}{ds}(s) - \delta_{3i} \right) \Big(\delta \underline{D}^i(s) \cdot \frac{d\underline{X}^0}{ds}(s)$$

$$+ \underline{D}^i(s) \cdot \frac{d\delta \underline{X}^0}{ds} \Big) \Big) ds$$

We can simplify the notation by anticipating a key result and letting

(40) $$F_i(s) = b_i \left(\underline{D}^i(s) \cdot \frac{d\underline{X}^0}{ds} - \delta_{3i} \right)$$

so that

(41) $$\underline{F}(s) = F_i(s)\, \underline{D}^i(s)$$

Then

(42) $$\delta E = \int_0^L \sum_{i=1}^{3} \left(a_i K_i(s) \delta K_i(s) + F_i(s) \delta \underline{D}^i(s) \cdot \frac{d\underline{X}^0}{ds}(s) \right) ds$$

$$+ \int_0^L \underline{F}(s) \cdot \frac{d\,\delta \underline{X}^0}{ds}(s)\, ds$$

Note that we have now separated δE into two parts. On the right-hand side of (42), the first line only involves $\delta\Omega$ (through δK_i and $\delta \underline{D}^i$), and the second line only involves $\delta \underline{X}^0$. We consider these two kinds of perturbations separately.

First, let $\delta \underline{\Omega} = 0$. Then

(43) $$\delta W = \int_0^L \underline{f}(s) \cdot \delta \underline{X}^0(s)\, ds$$

and

(44) $$\delta E = \underline{F}(s) \cdot \delta \underline{X}^0(s) \Big|_0^L - \int_0^L \frac{d\underline{F}}{ds}(s) \cdot \delta \underline{X}^0(s)\, ds$$

By the principle of virtual work, $\delta W = \delta E$ for arbitrary $\delta \underline{X}^0$, and this implies

(45) $$0 = \underline{f}(s) + \frac{d\underline{F}}{ds}(s)$$

and also the boundary condition

(46) $$\underline{F}(0) = \underline{F}(L) = 0 \quad .$$

The agreement of (45) with (15) justifies the notation introduced in (40-41).

Next, let $\delta \underline{X}^0 = 0$. Then

(47) $$\delta W = \int_0^L \underline{n}(s) \cdot \delta \underline{\Omega}(s)\, ds$$

In the equation for δE, we can (slightly) simplify the notation by anticipating the Kirchhoff hypothesis and setting

(48) $$N_i = a_i K_i, \qquad i = 1, 2, 3$$

Then

(49) $\delta E =$

$$\int_0^L \sum_{i=1}^{3} \left(N_i(s)\, \delta K_i(s) + F_i(s)\, \delta \underline{D}^i(s) \cdot \frac{d\underline{X}^0}{ds}(s) \right) ds$$

From (33),

(50) $$\delta K_i(s) = \frac{d\delta \underline{D}^j}{ds}(s) \cdot \underline{D}^k(s) + \frac{d\underline{D}^j}{ds}(s) \cdot \delta \underline{D}^k(s)$$

Therefore, after integration by parts

(51) $$\delta E = \sum_{i=1}^{3} N_i(s)\, \delta \underline{D}^j(s) \cdot \underline{D}^k(s) \Big|_0^L$$

$$+ \int_0^L \sum_{i=1}^{3} \Big(-\frac{dN_i}{ds}(s)\, \delta \underline{D}^j(s) \cdot \underline{D}^k(s)$$

$$- N_i(s)\, \delta \underline{D}^j(s) \cdot \frac{d\underline{D}^k}{ds}(s)$$

$$+ N_i(s) \frac{d\underline{D}^j}{ds}(s) \cdot \delta \underline{D}^k(s)$$

$$+ F_i(s)\, \delta \underline{D}^i(s) \cdot \frac{d\underline{X}^0}{ds} \Big)\, ds$$

In equation (51), the boundary term and the first term in the integrand both involve

(52) $$\delta\underline{D}^j \cdot \underline{D}^k = (\delta\underline{\Omega} \times \underline{D}^j) \cdot \underline{D}^k$$
$$= \delta\underline{\Omega} \cdot (\underline{D}^j \times \underline{D}^k)$$
$$= \delta\underline{\Omega} \cdot \underline{D}^i$$

since this term has (i,j,k) cyclic.

The next two terms combine nicely, since

(53) $$-\delta\underline{D}^j \cdot \frac{d\underline{D}^k}{ds} + \frac{d\underline{D}^j}{ds} \cdot \delta\underline{D}^k$$
$$= -(\delta\underline{\Omega} \times \underline{D}^j) \cdot \frac{d\underline{D}^k}{ds} + \frac{d\underline{D}^j}{ds} \cdot (\delta\underline{\Omega} \times \underline{D}^k)$$
$$= -\delta\underline{\Omega} \cdot \left(\underline{D}^j \times \frac{d\underline{D}^k}{ds}\right) + \delta\underline{\Omega} \cdot \left(\underline{D}^k \times \frac{d\underline{D}^j}{ds}\right)$$
$$= -\delta\underline{\Omega} \cdot \frac{d}{ds}\left(\underline{D}^j \times \underline{D}^k\right) = -\delta\underline{\Omega} \cdot \frac{d}{ds}\underline{D}^i$$

again, we are using i,j,k cyclic.

Finally, in the last term

$$(54) \quad \sum_{i=1}^{3} F_i(s)\, \delta \underline{D}^i(s) \cdot \frac{d\underline{X}^0}{ds}$$

$$= \sum_{i=1}^{3} F_i(s) \left(\delta \underline{\Omega} \times \underline{D}^i(s) \right) \cdot \frac{d\underline{X}^0}{ds}$$

$$= \sum_{i=1}^{3} \delta \underline{\Omega} \cdot \quad F_i(s)\, \underline{D}^i(s) \times \frac{d\underline{X}^0}{ds}$$

$$= \delta \underline{\Omega} \cdot \left(\underline{F}(s) \times \frac{d\underline{X}^0}{ds} \right)$$

Then, putting everything together, and again using the principle of virtual work and the arbitrariness of $\delta \underline{\Omega}$, we get

(55) $$0 = \underset{\sim}{n}(s) + \frac{d\underset{\sim}{N}}{ds}(s) + \frac{d\underset{\sim}{X}^0}{ds}(s) \times \underset{\sim}{F}(s)$$

together with the boundary condition

(56) $$\underset{\sim}{N}(0) = \underset{\sim}{N}(L) = 0$$

Note that the boundary conditions (46) & (56) simply say that no force or couple can be transmitted across the ends of the rod. This is because we have not applied any force or couple at the ends. In other words, these are <u>free-end</u> boundary conditions.

Equation (55) is exactly the same as (16), and this justifies the Kirchoff hypothesis (and shows that it is still applicable to the generalized Kirchoff rod).

What has been gained here, even though the equilibrium equations are the same, is that we now have an equation for $\underset{\sim}{F}(s)$, so that $F(s)$, like $\underset{\sim}{N}(s)$ is determined when the configuration of the rod is known.

Then, from equations (45) & (55), we immediately know what the applied force distribution $\underline{f}(s)$ and the applied couple distribution $\underline{n}(s)$ have to be for the configuration of the rod to be in equilibrium, so that we can apply $-\underline{f}(s)$ and $-\underline{n}(s)$ to the surrounding fluid. The motion of the fluid, moreover, is <u>unconstrained</u>.

Even though equations (45) & (55) look the same as in standard Kirchhoff rod theory, they are not really the same because s is not arclength, $d\underline{X}^0/ds$ is not in general a unit vector and moreover it is not in general aligned with $\underline{D}^3$.

It is instructive to rewrite equation (55) in components. The interesting term is

$$\frac{d\underline{X}^0}{ds} \times \underline{F} \tag{57}$$

and the ith component of this term is given by

(58) $$\left(\frac{d\underline{X}^0}{ds} \times \underline{F}\right) \cdot \underline{D}^i = \frac{d\underline{X}^0}{ds} \cdot \left(\underline{F} \times \underline{D}^i\right)$$

$$= \sum_{l=i,j,k} F_l \frac{d\underline{X}^0}{ds} \cdot \left(\underline{D}^l \times \underline{D}^i\right)$$

$$= -F_j \frac{d\underline{X}^0}{ds} \cdot \underline{D}^k + F_k \frac{d\underline{X}^0}{ds} \cdot \underline{D}^j$$

$$= -F_j \left(\frac{F_k}{b_k} + \delta_{3k}\right) + F_k \left(\frac{F_j}{b_j} + \delta_{3j}\right)$$

$$= F_j F_k \left(\frac{1}{b_j} - \frac{1}{b_k}\right) + \delta_{3j} F_k - \delta_{3k} F_j$$

$$= F_j F_k \left(\frac{1}{b_j} - \frac{1}{b_k}\right) + \delta_{i2} F_1 - \delta_{i1} F_2$$

In the last step of the foregoing, we used the fact that (i, j, k) is cyclic. Because of this,

(59) $$j=3 \Rightarrow (i=2 \ \& \ k=1)$$

(60) $$k=3 \Rightarrow (i=1 \ \& \ j=2)$$

The remaining terms in (55) are resolved into components in exactly the same way as in the derivation of equation (30), so the compontwise version of (55) is

(61) $$0 = n_i + a_i \frac{dK_i}{ds} + (a_k - a_j) K_j K_k + F_1 \delta_{i2} - F_2 \delta_{i1} + F_j F_k \left(\frac{1}{b_j} - \frac{1}{b_k} \right)$$

This is exactly the same as (30) except for the additional term $F_j F_k \left(\frac{1}{b_j} - \frac{1}{b_k} \right)$

This term approaches zero in the limit $b_i \to \infty$, $i=1,2,3$, as we would expect, but we can also make it zero without taking that limit by making the choice $b_1 = b_2 = b_3$.

Thus, in the special case $b_1 = b_2 = b_3$, we have the amazing result that even the componentwise equations of the unconstrained Kirchhoff rod are the same as those of the standard Kirchhoff rod. Nevertheless, for the reasons already stated, the two models are not the same. The best way to see this is to work out the details of an example:

Homework – Part 2

Solve for the circular equilibrium of a bent, twisted, unconstrained Kirchhoff rod. The rod has unstressed length $2\pi R$, but its equilibrium radius R_{eq} may be different from R, and R_{eq} may depend on the amount of twist. Thus the equilibrium centerline will be

$$\underline{X}^0(s) = R_{eq}\left(\cos\frac{s}{R},\ \sin\frac{s}{R},\ 0\right) \tag{62}$$

where R_{eq} remains to be determined. As in Part 1, assume that $a_1 = a_2$, and impose the restriction that the triad vectors are periodic functions of s with period $2\pi R$ (not $2\pi R_{eq}$, since the domain of s remains the same when the rod is stretched or compressed). As in Part 1, there are no applied forces or couples.

Consider the special case of the unconstrained rod in which

(63) $$b_1 = b_2 = b_3 = b$$

Evaluate all variables as in Part 1, and note their behavior as functions of b, especially in the limit $b \to \infty$.

(Since this is a closed rod, the free-end boundary conditions derived above are <u>not</u> applicable.)

Here are some hints for doing the homework:

- According to (45) with $\underline{f}(s)=0$, the lab-frame vector $\underline{F}(s)$ is independent of s, and by symmetry it can only have a z component:

(64) $$\underline{F} = (0, 0, F)$$

- Now use (62) and (64) to evaluate

$$\frac{d\underline{X}^0}{ds} \times \underline{F}$$ in the lab frame, and

substitute this into (55) with $\underline{n}(s)=0$.

Now you have a formula for $\frac{d\underline{N}}{ds}$ in the lab frame that you can integrate to find $\underline{N}(s)$. Be sure to allow for a constant of integration! By symmetry, that constant must be of the form $(0, 0, N^0)$

- The form of $\underline{D}^3(s)$ can be determined by taking advantage of the fact that all of the b_i are equal.
From (40-41),

(65) $$\underline{F} = b \sum_{i=1}^{3} \underline{D}^i(s)\left(\underline{D}^i(s)\cdot \frac{d\underline{X}^0}{ds}(s) - \delta_{3i}\right)$$

$$= b\left(\frac{d\underline{X}^0}{ds}(s) - \underline{D}^3(s)\right)$$

This determines $\underline{D}^3(s)$ in the lab frame up to the constants R_{eq}, F that remain to be determined.

- With $\underline{D}^3(s)$ and $\underline{N}(s)$ known (except for some constants), you can separate $\underline{N}(s)$ by projection into a part parallel to $\underline{D}^3$ and a part perpendicular to $\underline{D}^3$. This determines the lab-frame vector $\underline{K}(s)$, since

(66) $$\underline{K}(s) = \frac{1}{a_3}\,\underline{N}^{\parallel}(s) + \frac{1}{a}\,\underline{N}^{\perp}(s)$$

Here $a = a_1 = a_2$, and it is only because $a_1 = a_2$ that we can find $\underline{K}$ from $\underline{N}$ while only knowing $\underline{D}^3$.

At this point we should be able to find all three of the $\underline{D}^i(s)$ by solving the differential equations

(67) $$\frac{d\underline{D}^i}{ds} = \underline{K} \times \underline{D}^i$$

in the lab frame. In the case of $\underline{D}^3$ this should only confirm what we already know, but it might determine some constants. It should also be possible to determine some of the constants by imposing periodicity.

Discretization of the unconstrained Kirchhoff rod

We consider a closed rod of unstressed length L so that $\underline{X}^0(s)$ is periodic with period L. For simplicity, we assume that the rod has an integer number of twists, so that the triad vectors $\underline{D}^i(s)$ are also periodic with period L.

We choose Δs so that $L/\Delta s$ is an integer, and we locate the triads at integer multiples of Δs, and nodes of the centerline at half-integer multiples of Δs.

The discrete curvatures are discretized in the following way

$$K_i\left(s+\frac{\Delta s}{2}\right) = \frac{\underline{D}^j(s+\Delta s) - \underline{D}^j(s)}{\Delta s} \cdot \frac{\underline{D}^k(s+\Delta s) + \underline{D}^k(s)}{2} \tag{68}$$

$$= \frac{1}{2\Delta s}\left(\underline{D}^j(s+\Delta s)\cdot \underline{D}^k(s) - \underline{D}^j(s)\cdot \underline{D}^k(s+\Delta s)\right)$$

(i,j,k) cyclic

Note that the right-hand side of (68) is antisymmetric in (j,k). This reflects the continuous identity that

(69) $$\frac{d\underline{D}^j}{ds}\cdot\underline{D}^k = -\underline{D}^j\cdot\frac{d\underline{D}^k}{ds}$$

since $\underline{D}^j\cdot\underline{D}^k = 0$. We obtained (68) by discretizing the left-hand side of (69), but if we had discretized the right-hand side in the same way we would have obtained the exact same result.

With K_i given by (68), the discretized energy can be defined as follows

(70) $$E = \frac{1}{2}\sum_s\sum_{i=1}^{3} a_i K_i^2\left(s+\frac{\Delta s}{2}\right)\Delta s$$

$$+\frac{1}{2}\sum_s\sum_{i=1}^{3} b_i\left(\underline{D}^i(s)\cdot\frac{\underline{X}^0\left(s+\frac{\Delta s}{2}\right)-\underline{X}^0\left(s-\frac{\Delta s}{2}\right)}{\Delta s}-\delta_{3i}\right)^2\Delta s$$

in which the sum over s is over integer multiples of Δs.

In order to evaluate the forces and couples associated with this energy function we consider an arbitrary motion of the centerline and of the triads defined by

(71) $$\frac{d\underline{X}^0}{dt}\left(s+\frac{\Delta s}{2}\right) = \underline{U}^0\left(s+\frac{\Delta s}{2}\right)$$

(72) $$\frac{d\underline{D}^i}{dt}(s) = \underline{\Omega}(s) \times \underline{D}^i(s) \quad , \quad i=1,2,3$$

All variables are now functions of time, but we leave the time argument understood to simplify the notation.

Our task is now to express dE/dt in terms of $\underline{U}^0\left(s+\frac{\Delta s}{2}\right)$ and $\underline{\Omega}(s)$.

From equation (70), we immediately get

(73) $$\frac{dE}{dt} = \sum_s \sum_{i=1}^{3} N_i\left(s+\frac{\Delta s}{2}\right) \frac{dK_i}{dt}\left(s+\frac{\Delta s}{2}\right) \Delta s$$

$$+ \sum_s \sum_{i=1}^{3} F_i(s) \frac{d\underline{D}^i}{dt}(s) \cdot \frac{\underline{X}^0\left(s+\frac{\Delta s}{2}\right) - \underline{X}^0\left(s-\frac{\Delta s}{2}\right)}{\Delta s} \Delta s$$

$$+ \sum_s \sum_{i=1}^{3} F_i(s) \underline{D}^i(s) \cdot \frac{\underline{U}^0\left(s+\frac{\Delta s}{2}\right) - \underline{U}^0\left(s-\frac{\Delta s}{2}\right)}{\Delta s} \Delta s$$

where

(74) $$N_i\left(s+\frac{\Delta s}{2}\right) = a_i K_i\left(s+\frac{\Delta s}{2}\right)$$

(75) $$F_i(s) = b_i\left(\underline{D}^i(s) \cdot \frac{\underline{X}^0\left(s+\frac{\Delta s}{2}\right) - \underline{X}^0\left(s-\frac{\Delta s}{2}\right)}{\Delta s} - \delta_{3i}\right)$$

Also, for future reference let

(76) $$\underline{F}(s) = \sum_{i=1}^{3} F_i(s) \underline{D}^i(s)$$

By shifting indices (also known as summation by parts), we can rewrite the last line of (73) as

$$(77) \qquad -\sum_{s} \frac{\underline{F}(s+\Delta s)-\underline{F}(s)}{\Delta s} \cdot \underline{U}^{0}\left(s+\frac{\Delta s}{2}\right)$$

By making use of (72), we can rewrite the middle line of (73) in the following way

$$(78) \qquad \sum_{s}\sum_{i=1}^{3} F_i(s)\left(\underline{\Omega}(s)\times\underline{D}^{i}(s)\right)\cdot\frac{\underline{X}^{0}\left(s+\frac{\Delta s}{2}\right)-\underline{X}^{0}\left(s-\frac{\Delta s}{2}\right)}{\Delta s}\,\Delta s$$

$$= \sum_{s} \underline{\Omega}(s)\cdot\left(\underline{F}(s)\times\frac{\underline{X}^{0}\left(s+\frac{\Delta s}{2}\right)-\underline{X}^{0}\left(s-\frac{\Delta s}{2}\right)}{\Delta s}\right)\Delta s$$

To evaluate the first line of (73), we need a formula for dK_i/dt. This is obtained from (68) & (72):

(79) $$\frac{dK_i}{dt}\left(s+\frac{\Delta s}{2}\right) =$$

$$\frac{1}{2\Delta s}\left(\underline{\Omega}(s+\Delta s)\times \underline{D}^j(s+\Delta s)\right)\cdot \underline{D}^k(s)$$

$$+\frac{1}{2\Delta s}\,\underline{D}^j(s+\Delta s)\cdot\left(\underline{\Omega}(s)\times \underline{D}^k(s)\right)$$

$$-\frac{1}{2\Delta s}\left(\underline{\Omega}(s)\times \underline{D}^j(s)\right)\cdot \underline{D}^k(s+\Delta s)$$

$$-\frac{1}{2\Delta s}\,\underline{D}^j(s)\cdot\left(\underline{\Omega}(s+\Delta s)\times \underline{D}^k(s+\Delta s)\right)$$

$$=\frac{1}{2\Delta s}\left(\underline{\Omega}(s+\Delta s)-\underline{\Omega}(s)\right)\cdot\left(\underline{D}^j(s+\Delta s)\times \underline{D}^k(s)\right)$$

$$-\frac{1}{2\Delta s}\left(\underline{\Omega}(s)-\underline{\Omega}(s+\Delta s)\right)\cdot\left(\underline{D}^j(s)\times \underline{D}^k(s+\Delta s)\right)$$

$$= \frac{\underline{\Omega}(s+\Delta s) - \underline{\Omega}(s)}{\Delta s} \cdot$$

$$\frac{\underline{D}^j(s+\Delta s) \times \underline{D}^k(s) + \underline{D}^j(s) \times \underline{D}^k(s+\Delta s)}{2}$$

Now make the definition

(80) $$\underline{N}(s + \tfrac{\Delta s}{2}) = \sum_{i=1}^{3} N_i(s + \tfrac{\Delta s}{2})$$

$$\frac{\underline{D}^j(s+\Delta s) \times \underline{D}^k(s) + \underline{D}^j(s) \times \underline{D}^k(s+\Delta s)}{2}$$

Then the first line of (73) becomes

$$(81)\quad \sum_s \frac{\underline{\Omega}(s+\Delta s)-\underline{\Omega}(s)}{\Delta s}\cdot \underline{N}\left(s+\frac{\Delta s}{2}\right)\Delta s$$

$$= -\sum_s \underline{\Omega}(s)\cdot \frac{\underline{N}\left(s+\frac{\Delta s}{2}\right)-\underline{N}\left(s-\frac{\Delta s}{2}\right)}{\Delta s}\,\Delta s$$

Finally, putting everything together,

$$(82)\quad -\frac{dE}{dt} =$$

$$\sum_s \underline{\Omega}(s)\cdot\left(\frac{\underline{N}\left(s+\frac{\Delta s}{2}\right)-\underline{N}\left(s-\frac{\Delta s}{2}\right)}{\Delta s}\right.$$

$$\left. + \frac{\underline{X}^0\left(s+\frac{\Delta s}{2}\right)-\underline{X}^0\left(s-\frac{\Delta s}{2}\right)}{\Delta s}\times \underline{F}(s)\right)\Delta s$$

$$+\sum_s \underline{U}^0\left(s+\frac{\Delta s}{2}\right)\cdot\frac{\underline{F}(s+\Delta s)-\underline{F}(s)}{\Delta s}\,\Delta s$$

Now let $\underline{n}(s)\,\Delta s$ be an external couple that is applied to the rod over the interval $(s - \frac{\Delta s}{2}, s + \frac{\Delta s}{2})$

and let $\underline{f}(s + \frac{\Delta s}{2})\,\Delta s$ be an external

force that is applied to the rod over the interval $(s, s+\Delta s)$. Then the rate at which external work is being done on the rod is given by

$$(83) \quad \dot{W} = \sum_s \left(\underline{\Omega}(s) \cdot \underline{n}(s) + \underline{U}^0(s + \tfrac{\Delta s}{2}) \cdot \underline{f}(s + \tfrac{\Delta s}{2}) \right) \Delta s$$

By conservation of energy

$$\dot{W} = \frac{dE}{dt}$$

and then, since $\underline{\Omega}$ and $\underline{U}^0$

are arbitrary

(84) $$0 = \underline{f}\left(s + \frac{\Delta s}{2}\right) + \frac{\underline{F}(s+\Delta s) - \underline{F}(s)}{\Delta s}$$

(85) $$0 = \underline{n}(s) + \frac{\underline{N}\left(s+\frac{\Delta s}{2}\right) - \underline{N}\left(s-\frac{\Delta s}{2}\right)}{\Delta s} + \frac{\underline{X}^0\left(s+\frac{\Delta s}{2}\right) - \underline{X}^0\left(s-\frac{\Delta s}{2}\right)}{\Delta s} \times \underline{F}(s)$$

where

(86) $$\underline{F}(s) = \sum_{i=1}^{3} b_i \left(\underline{D}^i(s) \cdot \frac{\underline{X}^0\left(s+\frac{\Delta s}{2}\right) - \underline{X}^0\left(s-\frac{\Delta s}{2}\right)}{\Delta s} - \delta_{3i} \right) \underline{D}^i(s)$$

(87) $$\underline{N}\left(s+\frac{\Delta s}{2}\right) = \sum_{i=1}^{3} a_i K_i\left(s+\frac{\Delta s}{2}\right) \underline{\tilde{D}}^i\left(s+\frac{\Delta s}{2}\right)$$

$$(88) \quad K_i\left(s+\frac{\Delta s}{2}\right) = \frac{\underline{D}^j(s+\Delta s)\cdot\underline{D}^k(s) - \underline{D}^j(s)\cdot\underline{D}^k(s+\Delta s)}{2\Delta s}$$

$$(89) \quad \underline{\tilde{D}}^i\left(s+\frac{\Delta s}{2}\right) = \frac{\underline{D}^j(s+\Delta s)\times\underline{D}^k(s) + \underline{D}^j(s)\times\underline{D}^k(s+\Delta s)}{2}$$

Of course, equations (84-85) could have been written down immediately by straightforward discretization of the equilibrium equations (15-16), and equation (86) is a straightforward discretization of equation (40).

Much less obvious, however, is how to discretize $\underline{N}(s+\frac{\Delta s}{2})$, especially since there is no triad at $(s+\frac{\Delta s}{2})$, and we now have the recipe given by equations (87-89).

In our use of the above equations for the simulation of the Kirchhoff rod in fluid, the configuration of the discretized rod

(90) $$\underline{X}^0(s+\tfrac{\Delta s}{2}),\ \underline{D}^1(s),\ \underline{D}^2(s),\ \underline{D}^3(s),$$

for s any integer multiple of Δs, is known at any particular time since it is part of the discretized state of the whole system. From these data we can derive

(91) $$\underline{F}(s),\ \underline{N}(s+\tfrac{\Delta s}{2})$$

from equations (86) & (87-89), and then we can evaluate

(92) $$\underline{f}(s+\tfrac{\Delta s}{2}),\ \underline{n}(s)$$

from equations (84) & (85), and finally we can apply $-\underline{f}(s+\frac{\Delta s}{2})$ and $-\underline{n}(s)$ to the surrounding fluid.

How to do this will be discussed next.

Interaction of the unconstrained Kirchhoff rod with fluid

The first issue to consider is that we must somehow model the thickness of the rod in order to capture its interaction with the fluid. In the limit of zero thickness, the rod has no interaction with the fluid at all. (This is a separate issue from the influence of thickness on the elastic parameters of the rod a_i and b_i.) To model the role of thickness in fluid-structure interaction, we use the width of the regularized delta function of the immersed boundary method. Thus, we refer here to $\delta_c(\underline{x})$ instead of $\delta_h(\underline{x})$, where c is a physical parameter that is roughly equal to the thickness of the rod. This approach has been validated and calibrated by Bringley*. Although one may set

* Bringley TT and Peskin CS: Validation of a simple method for representing spheres and slender bodies ... J. Computational Physics 227(11):5397-5425, 2008

$h = c$ in practice, the conceptually correct procedure is to let $h \to 0$ with c fixed. Note that in this case the support of δ_c as measured in meshwidths increases as the fluid mesh is refined. If we impose the restriction that c/h is an integer, then the identities that characterize the IB delta functions are still satisfied.

We are now ready to formulate the interaction equations of the IB method for an unconstrained Kirchhoff rod in fluid. We shall write these equations in their spatially discretized form, with time as a continuous variable. First we consider the equations of motion of the configuration of the rod, which are as follows:

(93) $$\frac{d\underline{X}^0}{dt}\left(s+\frac{\Delta s}{2},t\right) = \underline{U}^0\left(s+\frac{\Delta s}{2},t\right)$$

$$= \sum_{\underline{x}\in g_h} \underline{u}(\underline{x},t)\,\delta_c\left(\underline{x}-\underline{X}^0\left(s+\frac{\Delta s}{2},t\right)\right)h^3$$

(94) $$\frac{d\underline{D}^i}{dt}(s,t) = \underline{\Omega}(s,t)\times\underline{D}^i(s,t)$$

where

(95) $$\underline{\Omega}(s,t) = \sum_{\underline{x}\in g_h} \frac{1}{2}\left(\nabla_h\times\underline{u}\right)(\underline{x},t)\,\delta_c\left(\underline{x}-\underline{X}^0(s,t)\right)h^3$$

Equation (93) states that the centerline of the rod moves at the locally averaged fluid velocity, and equations (94–95) state that the triads rotate at the locally averaged angular velocity of the fluid.

Here we use the fact that the angular velocity of a small drop of fluid in a flow field $\underline{u}$ is equal to $\frac{1}{2}\nabla\times\underline{u}$.

The operator ∇_h in (95) is the vector central difference operator corresponding to the vector differential operator ∇, also known as "del". (We have previously called this difference operator $\underline{D}$, but cannot do so here because we have used the notation $\underline{D}$ for the triad vectors.) Thus $\nabla_h\times\underline{u}$ has the same definition as $\nabla\times\underline{u}$, but with derivatives replaced by finite central differences.

Note that we apply the (discrete) curl operator <u>first</u> to the grid function $\underline{u}$, and <u>then</u> interpolate. One might think instead that we should interpolate first, and then apply the continuous curl operator to the interpolated velocity field. This can be done, but it involves differentiating δ_c, and important properties that were built into the definition of δ_c are thereby lost.

The next question is how to construct a force density $\underline{f}(\underline{x},t)$ that encodes the influence of the unconstrained Kirchhoff rod on the fluid. This influence occurs through the rod variables $\underline{f}$ and $\underline{n}$, which we now rename $\underline{f}^B$ and $\underline{n}^B$. These are Lagrangian variables, and we would normally denote them with upper case letters, but cannot do so here because we have already used $\underline{F}$ and $\underline{N}$ for the cross-sectional variables of the rod. Recall also the sign convention that

$\underline{f}^B$ and $\underline{n}^B$ are applied to the rod, so it is $-\underline{f}^B$ and $-\underline{n}^B$ that need to be applied to the fluid.

In the IB world, once we have decided how to interpolate, there is no longer any freedom in how to apply force to the fluid, since this is decided by duality, also known as conservation of energy. In the present context, we require that

(96) $$\sum_{\underline{x} \in g_h} \underline{u}(\underline{x},t) \cdot \underline{f}(\underline{x},t)\, h^3$$

$$= \sum_s \underline{U}^0(s+\tfrac{\Delta s}{2},t) \cdot \left(-\underline{f}^B(s+\tfrac{\Delta s}{2},t)\right) \Delta s$$

$$+ \sum_s \underline{\Omega}(s,t) \cdot \left(-\underline{n}^B(s,t)\right) \Delta s$$

$$= \sum_s \sum_{x \in g_h} \underline{u}(\underline{x}, t)\, \delta_c(\underline{x} - \underline{X}^0(s + \tfrac{\Delta s}{2}, t))\, h^3$$

$$\cdot \left(-\underline{f}^B(s + \tfrac{\Delta s}{2}, t)\right) \Delta s$$

$$+ \sum_s \sum_{\underline{x} \in g_h} \tfrac{1}{2}\left(\nabla_h \times \underline{u}\right)(\underline{x}, t)\, \delta_c(\underline{x} - \underline{X}^0(s, t))\, h^3$$

$$\cdot \left(-\underline{n}^B(s, t)\right) \Delta s$$

$$= \sum_{\underline{x} \in g_h} \underline{u}(\underline{x}, t) \cdot$$

$$\Big(\sum_s -\underline{f}^B(s + \tfrac{\Delta s}{2})\, \delta_c(\underline{x} - \underline{X}^0(s + \tfrac{\Delta s}{2}, t))\, \Delta s$$

$$+ \tfrac{1}{2} \nabla_h \times \sum_s -\underline{n}^B(s)\, \delta_c(\underline{x} - \underline{X}^0(s, t))\, \Delta s \Big) h^3$$

Here we have used the interpolation formulae (93) & (95) in order to express $\underline{U}^0(s+\frac{\Delta s}{2})$ and $\underline{\Omega}(s)$

in terms of $\underline{u}(\underline{x},t)$. Then, in the last step of the foregoing, we have interchanged the order of summation, and also we have made use of the identity

$$\sum_{\underline{x}\in g_h} (\nabla_h \times \underline{u})\cdot\underline{v}\, h^3 \tag{97}$$

$$= \sum_{\underline{x}\in g_h} \underline{u}\cdot(\nabla_h\times\underline{v})\, h^3$$

which holds on grids that are periodic in all directions and is easily proved by shifting indices, also known as summation by parts. Unlike most such identities, there is no change of sign in this case.

Since (96) is supposed to be an identity that holds for all $\underline{u}(\underline{x},t)$, we see by comparison of the first and last lines of (96) that

(98) $$\underline{f}(\underline{x},t) =$$

$$\sum_s -\underline{f}^B\left(s+\frac{\Delta s}{2},t\right)\delta_c\left(\underline{x}-\underline{X}^0\left(s+\frac{\Delta s}{2},t\right)\right)\Delta s$$

$$+\frac{1}{2}\nabla_h\times\sum_s -\underline{n}^B(s,t)\,\delta_c\left(\underline{x}-\underline{X}^0(s,t)\right)\Delta s$$

It should be noted that equation (95) and equation (98) can each be written in such a way that the operator ∇_h is applied as a gradient to δ_c instead of as a curl to a vector field. In the case of (95) the derivation involves summation by parts, and in the case of (98) we simply need to notice that $\underline{n}^B(s,t)$ is a constant vector insofar as the operator $\nabla_h\times$ is concerned. We omit the details.

For completeness, we recall here the Navier-Stokes equation in their spatially discretized form

(99) $$\rho\left(\frac{d\underline{u}}{dt} + S_h(\underline{u})\,\underline{u}\right) + \nabla_h p = \mu \Delta_h \underline{u} + \underline{f}$$

(100) $$\nabla_h \cdot \underline{u} = 0$$

where

(101) $$\left(S_h(\underline{u})\underline{u}\right)_i = \frac{1}{2}\left(\underline{u}\cdot\nabla_h u_i + \nabla_h(\underline{u}u_i)\right)$$

These equations are coupled to the unconstrained Kirchhoff rod through equations (93-95) and (98), and the rod itself is governed by equations 84-89, in which we should put a superscript B on $\underline{f}$ and $\underline{n}$ for consistency of notation. All of these equations together make a closed dynamical system in which the state of the fluid is its velocity field and the state of the rod is the configuration of its centerline and its triads.

Although we do not write it out here, the equations that we have just summarized have a sensible continuum limit in which $h, \Delta s \to 0$ with the parameter c held fixed. The limiting equations are a mathematical <u>model</u> of a thin elastic rod in fluid, in much the same spirit as the Kirchhoff rod model itself, which is not derived by solving any 3D elasticity problem but which aims to capture the essence of a thin rod's elastic behavior in a one-dimensional self-consistent continuum model. In much the same way, our formulation of the fluid-structure interaction equations for a thin rod in fluid does not exactly correspond to the solution of any 3D problem for the Navier-Stokes equations with an internal moving tubular boundary, but instead we work with the variables of the Kirchhoff rod model and couple them in a physically reasonable way to the fluid. Like the rod model itself, the resulting continuum model of a rod in fluid has an internal consistency that makes it seem worthy of mathematical study in its own right, besides being possibly useful in computer simulations in which twist is an important phenomenon.