Consider an incompressible elastic material filling all of space. Let its motion be described by

$$\underline{x} = \underline{X}(\xi, t) \tag{1}$$

where $\xi = \xi_1, \xi_2, \xi_3$ are material coordinates

This is called a Lagrangian description of the motion. The incompressibility of the material is expressed by

$$\frac{\partial}{\partial t} \det\left(\frac{\partial \underline{X}}{\partial \xi}\right) = 0 \tag{2}$$

where $\partial \underline{X}/\partial \xi$ is the 3×3 matrix with

$$\left(\frac{\partial \underline{X}}{\partial \xi}\right)_{ij} = \frac{\partial X_i}{\partial \xi_j} \tag{3}$$

We assume that the elastic material has a mass density $M(\xi)$ with respect to the measure

$$d\xi = d\xi_1 \, d\xi_2 \, d\xi_3 \tag{4}$$

Of course, $M(q)$ is independent of time, since q_1, q_2, q_3 are material coordinates and since the mass of any material element is conserved.

We assume that the elastic energy is given by a functional of the form

$$E\left[\underline{X}(\ ,t)\right] \tag{5}$$

The variation (denoted δ_{var} to distinguish it from the Dirac delta function which will appear later) of this elastic energy defines a force density $\underline{F}(q,t)$ with respect to the measure dq in the following way

$$\delta_{var} E = -\int \underline{F}(q,t) \cdot \delta_{var} \underline{X}(q,t)\, dq \tag{6}$$

This defines $-\underline{F}$ as the variational derivative of E. The physical significance of $\underline{F}$ will emerge as we proceed.

The action over a time interval $(0,T)$ is defined as the integral over that time interval of the difference between the kinetic energy and the potential energy. Thus

$$S=\int_0^T\left(\frac{1}{2}\int M(\underline{\xi})\left|\frac{\partial \underline{X}}{\partial t}\right|^2 d\xi - E[\underline{X}(\,,t)]\right)dt \qquad (7)$$

We seek the motion $\underline{X}(\xi,t)$ that minimizes S subject to the following constraints

$$\underline{X}(\xi,0)=\underline{X}_0(\xi) \qquad (8)$$

$$\underline{X}(\xi,T)=\underline{X}_T(\xi) \qquad (9)$$

$$\det\left(\frac{\partial \underline{X}}{\partial \xi}\right)=J(\xi) \qquad (10)$$

where $\underline{X}_0(\xi)$, $\underline{X}_T(\xi)$, and $J(\xi)$ are given functions. Of course, the consistency of Eqs. (8–10) implies a relationship between these given functions, namely

$$\det\left(\frac{\partial \underline{X}_0}{\partial \xi}\right)=\det\left(\frac{\partial \underline{X}_T}{\partial \xi}\right)=J(\xi) \qquad (11)$$

Now consider a motion of the form

$$\underline{x} = \underline{X}(\xi,t) = \underline{X}^0(\xi,t) + \delta_{var}\underline{X}(\xi,t) \qquad (12)$$

where $\underline{X}^0(\xi,t)$ is the motion that minimizes S subject to the constraints stated above, and where $\delta_{var}\underline{X}(\xi,t)$ is a small perturbation of this motion.

In the following, we treat δ_{var} as a differential operator: When applied to any variable it produces the change in that variable, <u>to first order</u>, that is induced by the perturbation $\delta_{var}\underline{X}(\xi,t)$. Thus, δ_{var} is analogous to the operator d of ordinary calculus, as in "$d\phi = \phi'(x)\,dx$."

Applying δ_{var} to the constraint equations, we get

$$\delta_{var}\underline{X}(\xi,0) = 0 \qquad (12)$$

$$\delta_{var}\underline{X}(\xi,T) = 0 \qquad (13)$$

$$\delta_{var}\left(\det\left(\frac{\partial \underline{X}}{\partial \xi}\right)\right) = 0 \qquad (14)$$

The differentiation of the determinant in Eq. 14 will be discussed below.

We now apply δ_{var} to the action S, and set the result equal to zero, in order to minimize S. In the kinetic energy term we integrate by parts with respect to time, and in the potential energy term we use the definition of $\underline{F}$ as given by Eq. 6:

$$0=\delta_{var}S=\int_0^T\int\left(M(q)\frac{\partial\underline{X}^0}{\partial t}\cdot\frac{\partial\delta_{var}\underline{X}}{\partial t}+\underline{F}(q,t)\cdot\delta_{var}\underline{X}(q,t)\right)dq\,dt$$

$$=\int_0^T\int\left(-M(q)\frac{\partial^2\underline{X}^0}{\partial t^2}(q,t)+\underline{F}(q,t)\right)\cdot\delta_{var}\underline{X}(q,t)\,dq\,dt \qquad (15)$$

In the above integration by parts with respect to time, there are no boundary terms because of Eqs. 12-13.

If $\delta_{var}\underline{X}$ were arbitrary on $(0,T)$, we would at this stage be able to conclude that $M\,\partial^2\underline{X}^0/\partial t^2=\underline{F}$, but this conclusion is incorrect because of the constraint of incompressibility, Eq. 14.

To put the constraint of incompressibility in a simpler form, we switch to Eulerian variables:

let $\underline{u}(\underline{x},t)$ be implicitly defined by

$$\frac{\partial \underline{X}^0}{\partial t}(\xi,t) = \underline{u}\left(\underline{X}^0(\xi,t),t\right) \qquad (16)$$

Differentiating with respect to time, we get

$$\frac{\partial^2 \underline{X}^0}{\partial t^2} = \frac{\partial \underline{u}}{\partial \underline{x}}\left(\underline{X}^0(\xi,t),t\right)\frac{\partial \underline{X}^0}{\partial t}(\xi,t) + \frac{\partial \underline{u}}{\partial t}\left(\underline{X}^0(\xi,t),t\right)$$

$$= \left(\frac{\partial \underline{u}}{\partial \underline{x}}\underline{u} + \frac{\partial \underline{u}}{\partial t}\right)\left(\underline{X}^0(\xi,t),t\right) = \frac{D\underline{u}}{Dt}\left(\underline{X}^0(\xi,t),t\right)$$

(17)

where

$$\left(\frac{\partial \underline{u}}{\partial \underline{x}}\right)$$

is the 3×3 matrix with components

$$\left(\frac{\partial \underline{u}}{\partial \underline{x}}\right)_{ij} = \frac{\partial u_i}{\partial x_j} \qquad (18)$$

and where we have defined:

$$\frac{D\underline{u}}{Dt} = \frac{\partial \underline{u}}{\partial t} + \frac{\partial \underline{u}}{\partial \underline{x}}\underline{u} = \frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u} \qquad (19)$$

As is well known, the incompressibility constraint expressed in terms of $\underline{u}(\underline{x},t)$ is $\nabla\cdot\underline{u}=0$. To derive this, we start from Eq. 2. The derivative of a determinant is evaluated as follows.

Let A be a square matrix with elements A_{ij}. Then

$$\frac{\partial}{\partial t}\det(A) = \sum_{ij} \frac{\partial A_{ij}}{\partial t}\frac{\partial \det(A)}{\partial A_{ij}} \qquad (20)$$

But

$$\frac{\partial \det(A)}{\partial A_{ij}} = \text{signed minor of } A_{ij} = (A^{-1})_{ji}\det(A) \qquad (21)$$

Therefore

$$\frac{\partial}{\partial t}\det(A) = \left(\sum_{ij}\frac{\partial A_{ij}}{\partial t}(A^{-1})_{ji}\right)\det(A)$$

$$= \operatorname{trace}\left(\frac{\partial A}{\partial t}A^{-1}\right)\det(A) \qquad (22)$$

In our case $A = \partial\underline{X}^0/\partial\underline{\xi}$, so

$$\frac{\partial A}{\partial t} = \frac{\partial}{\partial\underline{\xi}}\frac{\partial\underline{X}^0}{\partial t} = \frac{\partial}{\partial\underline{\xi}}\underline{u}\left(\underline{X}^0(\underline{\xi},t),t\right) = \frac{\partial\underline{u}}{\partial\underline{x}}\frac{\partial\underline{X}^0}{\partial\underline{\xi}} \qquad (23)$$

Thus Eq. 2 becomes

$$0 = \text{trace}\left(\frac{\partial \underline{u}}{\partial \underline{x}} \frac{\partial \underline{X}^0}{\partial \xi}\left(\frac{\partial \underline{X}^0}{\partial \xi}\right)^{-1}\right)$$

$$= \text{trace}\left(\frac{\partial \underline{u}}{\partial \underline{x}}\right) = \nabla \cdot \underline{u} \qquad (24)$$

Eqs. 16-24 are standard in the Eulerian description of the motion of an incompressible material

In a completely analogous way, we may also introduce a vector field $\underline{v}(\underline{x},t)$ associated with the perturbation $\delta_{var}\underline{X}(\xi,t)$ as follows:

$$\delta_{var}\underline{X}(\xi,t) = \underline{v}(\underline{X}^0(\xi,t),t) \qquad (25)$$

The constraints on $\delta_{var}\underline{X}$ then become

$$\underline{v}(\underline{x},0) = 0 \qquad (26)$$

$$\underline{v}(\underline{x},T) = 0 \qquad (27)$$

$$\nabla \cdot \underline{v} = 0 \qquad (28)$$

The derivation of Eq. 28 from Eq. 14 is completely analogous to the derivation of Eq. 24 from Eq. 2, with δ_{var} playing the role of $\partial/\partial t$.

We are now ready to transform Eq. 15 to Eulerian form. We do this in two stages. In the first stage, we use the Dirac delta function as a convenient device for changing variables. This introduces an additional integral $d\underline{x}$ besides the integrals $dq\,dt$ that are already present. In the second stage we make certain definitions that incorporate the integral dq (and the Dirac delta functions) so that only the integrals $d\underline{x}$, dt remain.

To carry out the first stage of the above program, we note that

$$\left(\frac{\partial^2 \underline{X}^0}{\partial t^2}\cdot\delta_{var}\underline{X}\right)(q,t)$$

$$= \int\left(\frac{D\underline{u}}{Dt}\cdot\underline{v}\right)(\underline{x},t)\,\delta\left(\underline{x}-\underline{X}^0(q,t)\right)d\underline{x} \qquad (29)$$

$$\delta_{var}\underline{X}\,(q,t) = \int \underline{v}(\underline{x},t)\,\delta\left(\underline{x}-\underline{X}^0(q,t)\right)d\underline{x} \qquad (30)$$

A slightly tricky point here is that we evaluate $\delta_{var}\underline{X}$ in two different ways: once in combination with $\partial^2\underline{X}^0/\partial t^2$ and once on its own.

Making use of Eqs. 29-30, we may rewrite Eq. 15 as follows:

$$0 = \int_0^T\iint\left(-M(q)\frac{Du}{Dt}(\underline{x},t) + \underline{F}(q,t)\right)\cdot\underline{v}(\underline{x},t)\,\delta\left(\underline{x}-\underline{X}(q,t)\right)d\underline{x}\,dq\,dt \tag{31}$$

Now we complete the change of variables by making the definitions

$$\rho(\underline{x},t) = \int M(q)\,\delta\left(\underline{x}-\underline{X}(q,t)\right)dq \tag{32}$$

$$\underline{f}(\underline{x},t) = \int \underline{F}(q,t)\,\delta\left(\underline{x}-\underline{X}(q,t)\right)dq \tag{33}$$

which define an Eulerian mass density and an Eulerian force density, respectively.

In terms of these densities, Eq. 31 becomes

$$0=\int_0^T\int\left(-\rho(\underline{x},t)\,\frac{D\underline{u}}{Dt}(\underline{x},t)+\underline{f}(\underline{x},t)\right)\cdot\underline{v}(\underline{x},t)\,d\underline{x}\,dt \tag{34}$$

This completes the transition to Eulerian variables, since all quantities are functions of fixed Cartesian coordinates $\underline{x}$ (and the time t).

Eq. 34 is supposed to hold for all $\underline{v}(\underline{x},t)$ that satisfy Eqs. 26-28. To see what this implies, we make use of the Hodge decomposition, that any vector field may be expressed as the sum of a divergence-free part and the gradient of a scalar. Thus, we may write

$$-\rho\,\frac{D\underline{u}}{Dt}+\underline{f}=\underline{W}+\nabla p \tag{35}$$

where

$$\nabla\cdot\underline{W}=0 \tag{36}$$

Then Eq. 34 becomes

$$0=\int_0^T\int\left(\underline{W}+\nabla p\right)\cdot\underline{v}\;d\underline{x}\,dt \tag{37}$$

Integration by parts shows that the term involving the pressure gradient is zero, since $\nabla \cdot \underline{v} = 0$:

$$\int (\nabla p \cdot \underline{v})\, d\underline{x} = -\int p(\nabla \cdot \underline{v})\, d\underline{x} = 0 \quad (38)$$

Thus, we are left with

$$0 = \int_0^T \int (\underline{w} \cdot \underline{v})\, d\underline{x}\, dt \quad (39)$$

for all $\underline{v}(\underline{x},t)$ that satisfy Eqs. (26-28). In particular, we may choose

$$\underline{v}(\underline{x},t) = a(t)\, \underline{w}(\underline{x},t) \quad (40)$$

where $a(t)$ is some continuous function with $a(0) = a(T) = 0$, and $a(t) > 0$ otherwise. Eq. 39 then becomes

$$0 = \int_0^T a(t) \int |\underline{w}(\underline{x},t)|^2\, d\underline{x}\, dt \quad (41)$$

from which we conclude that $\underline{w} = 0$.

It is now time to collect results. We have derived the following system of equations

$$\rho(\underline{x},t)\,\frac{D\underline{u}}{Dt} + \nabla p = \underline{f}(\underline{x},t) \tag{42}$$

$$\nabla\cdot\underline{u} = 0 \tag{43}$$

$$\rho(\underline{x},t) = \int M(q)\,\delta\big(\underline{x}-\underline{X}(q,t)\big)\,dq \tag{44}$$

$$\underline{f}(\underline{x},t) = \int \underline{F}(q,t)\,\delta\big(\underline{x}-\underline{X}(q,t)\big)\,dq \tag{45}$$

$$\frac{\partial\underline{X}}{\partial t}(q,t) = \underline{u}\big(\underline{X}(q,t),t\big) \tag{46 a}$$

$$= \int \underline{u}(\underline{x},t)\,\delta\big(\underline{x}-\underline{X}(q,t)\big)\,d\underline{x} \tag{46 b}$$

$$\delta_{var} E\big[\underline{X}(\;,t)\big] = -\int \underline{F}(q,t)\cdot\delta_{var}\underline{X}(q,t)\,dq \tag{47}$$

Eqs. 42–47 are a mixed Eulerian-Lagrangian formulation of the equations of motion of an incompressible elastic material.

Eqs. 42-43 are the Euler equations of an inviscid incompressible fluid, with a possibly non-uniform mass density $\rho(\underline{x},t)$ and an applied force density $\underline{f}(\underline{x},t)$. One would expect this system to be completed by the continuity (conservation of mass) equation for the dynamics of $\rho(\underline{x},t)$; this issue will be discussed below. Note that Eqs. 42-43 are entirely in Eulerian form, i.e., all of the variables in these equations are functions of the fixed Cartesian coordinates $\underline{x}$ and the time t.

Eq. 47 is entirely in Lagrangian form. It implicitly defines the Lagrangian force density $\underline{F}(\underline{q},t)$ with respect to the measure $d\underline{q}$ as minus the variational derivative of the elastic energy $E[\underline{X}(\ ,t)]$ with respect to the configuration $\underline{X}(\ ,t)$ of the elastic material at any given time. This is the principle of virtual work.

Eqs. 44–46 express Eulerian variables in terms of the corresponding Lagrangian variables or vice versa. In all cases, this is done by means of an integral transformation with kernel

$$\delta(\underline{x} - \underline{X}(\xi, t)) \tag{48}$$

It is important to note, however, the distinction that $\underline{x}$ appears directly in the argument of the delta function whereas ξ appears indirectly through $\underline{X}(\xi,t)$. A consequence of this is that ρ and M are not equal at corresponding points; instead, they are related as corresponding densities:

$$\rho(\underline{X}(\xi,t), t) \det\left(\frac{\partial \underline{X}}{\partial \xi}(\xi, t)\right) = M(\xi) \tag{49}$$

Similarly

$$\underline{f}(\underline{X}(\xi,t), t) \det\left(\frac{\partial \underline{X}}{\partial \xi}(\xi,t)\right) = \underline{F}(\xi,t) \tag{50}$$

In contrast to this, the Lagrangian velocity $\partial \underline{X}/\partial t$

and the Eulerian velocity $\underline{u}(\underline{x},t)$ are actually equal to each other at corresponding points, see Eq. 46a.

It is natural to ask why our system of equations does not include the familiar continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \underline{u}) = 0 \qquad (51)$$

In fact, this relationship is implied by Eq. 44. To see this, start from Eq. 44 and derive

$$\frac{\partial \rho}{\partial t} = -\int M(q)\, \nabla\delta\left(\underline{x}-\underline{X}(q,t)\right)\cdot \frac{\partial \underline{X}}{\partial t}(q,t)\, dq \qquad (52)$$

$$\nabla\cdot(\rho\underline{u}) = \int M(q)\, \nabla\cdot\left(\underline{u}(\underline{x},t)\,\delta\left(\underline{x}-\underline{X}(q,t)\right)\right) dq \qquad (53)$$

Let $\varphi(\underline{x})$ be an arbitrary test function, i.e., a smooth function which decays rapidly enough that there are no boundary terms when one integrates by parts. Multiplying by $\varphi(\underline{x})$, integrating over all $\underline{x}$, and then integrating by parts, we find

$$\int \varphi(\underline{x}) \frac{\partial \rho}{\partial t}(\underline{x},t)\, d\underline{x}$$

$$= -\int M(q) \int \varphi(\underline{x}) \nabla \delta(\underline{x} - \underline{X}(q,t))\, d\underline{x} \cdot \frac{\partial \underline{X}}{\partial t}(q,t)\, dq$$

$$= \int M(q) \int \nabla\varphi\, \delta(\underline{x} - \underline{X}(q,t))\, d\underline{x} \cdot \frac{\partial \underline{X}}{\partial t}(q,t)\, dq$$

$$= \int M(q)\, \nabla\varphi(\underline{X}(q,t)) \cdot \frac{\partial \underline{X}}{\partial t}(q,t)\, dq \qquad (54)$$

$$\int \varphi(\underline{x})\, \nabla\cdot(\rho \underline{u})\, d\underline{x}$$

$$= \int M(q) \int \varphi(\underline{x})\, \nabla\cdot\left(\underline{u}(\underline{x},t)\, \delta(\underline{x} - \underline{X}(q,t))\right) d\underline{x}\, dq$$

$$= -\int M(q) \int \nabla\varphi(\underline{x}) \cdot \underline{u}(\underline{x},t)\, \delta(\underline{x} - \underline{X}(q,t))\, d\underline{x}\, dq$$

$$= -\int M(q)\, \nabla\varphi(\underline{X}(q,t)) \cdot \underline{u}(\underline{X}(q,t), t)\, dq \qquad (55)$$

Adding Eqs. 54 and 55, and making use of Eq. 46a, we see that

$$\int \varphi(\underline{x}) \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) \right)(\underline{x},t)\, d\underline{x} = 0 \qquad (56)$$

from which Eq. 51 follows, since φ is arbitrary.

Even though Eqs. 42-47 are derived for an incompressible elastic material, they suggest an interpretation in which such a material is regarded as an idealized hydroelastic composite with two components:

1) An incompressible fluid with non-uniform mass density $\rho(\underline{x},t)$, and

2) An elastic component which displaces <u>no volume</u> and has <u>no mass</u>.

The elastic component is a pure force generator. It moves at the local fluid velocity (Eq. 46) and generates elastic forces (Eq. 47) which are applied locally to the fluid (Eqs. 45 and 42).

Because the elastic component moves with the fluid and displaces no additional volume, the incompressibility of the fluid (Eq. 43) is enough to ensure the incompressibility of the material as a whole. In particular, our formulation has the feature that the bulk modulus of the elastic component is irrelevant, since only incompressible deformations are allowed in any case.

Under the above interpretation, our variables have the following meanings:

$\underline{u}(\underline{x},t)$ = fluid velocity

$p(\underline{x},t)$ = fluid pressure

$\underline{X}(q,t)$ = configuration (at fixed t) or motion (at fixed q) of the elastic component

$\underline{F}(q,t)\,dq = \underline{f}(\underline{x},t)\,d\underline{x}$ = force applied locally by elastic component to fluid

$M(q)\,dq = \rho(\underline{x},t)\,d\underline{x}$ = mass ~~density~~ of fluid element = mass ~~density~~ of material element

Let us now discuss some generalizations of the above formulation that are useful in applications:

1) Viscosity

In order to model a visco-elastic material, all we need to do is add a viscous term on the right-hand side of Eq. 42, so that Eqs. 42-43 become the incompressible Navier-Stokes equations:

$$\rho(\underline{x},t)\,\frac{D\underline{u}}{Dt} + \nabla p = \mu \Delta \underline{u} + \underline{f}(\underline{x},t) \qquad (57)$$

$$\nabla \cdot \underline{u} = 0 \qquad (58)$$

where μ is the constant viscosity. The rest of the formulation remains unchanged.

It is also of interest to consider variable viscosity $\mu(\underline{x},t)$. This raises two issues. The first is the correct form of the viscous force when μ is not constant. In an incompressible fluid with non-uniform viscosity, the viscous stress tensor is

$$\mu(\underset{\sim}{x},t)\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right) \qquad (59)$$

and the viscous force per unit volume is the divergence of the above expression, i.e.

$$\frac{\partial}{\partial x_j}\left[\mu(\underset{\sim}{x},t)\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\right] \qquad (60)$$

where we are using the summation convention that repeated indices are summed over 1, 2, 3. It is interesting that the term $\partial u_j/\partial x_i$ makes a nonzero contribution in the case of variable viscosity even though $\partial u_j/\partial x_j = 0$.

Thus, in the case of non-uniform viscosity, the Navier-Stokes equations of an incompressible fluid become

$$\rho(\underset{\sim}{x},t)\frac{Du_i}{Dt}+\frac{\partial p}{\partial x_i}=\frac{\partial}{\partial x_j}\left(\mu(\underset{\sim}{x},t)\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\right)+f_i(\underset{\sim}{x},t) \qquad (61)$$

$$\frac{\partial u_j}{\partial x_j}=0 \qquad (62)$$

The second issue in the case of variable viscosity is how to determine the dynamics of $\mu(\underline{x},t)$. We assume here that $\mu(\underline{x},t)$ remains constant for a given material point, i.e., that

$$\frac{\partial}{\partial t}\mu\left(\underline{X}(q,t),t\right) = 0 \qquad (63)$$

This is equivalent to

$$\frac{\partial \mu}{\partial t} + \underline{u}\cdot\nabla\mu = 0 \qquad (64)$$

which, in an incompressible fluid can also be written

$$\frac{\partial \mu}{\partial t} + \nabla\cdot(\underline{u}\mu) = 0 \qquad (65)$$

As shown above for $\rho(\underline{x},t)$, this equation is automatically satisfied if we set

$$\mu(\underline{x},t) = \int \mu^L(q)\,\delta\left(\underline{x}-\underline{X}(q,t)\right)dq \qquad (66)$$

where $\mu^L(q)$ is a Lagrangian viscosity that is independent of t. Eq. 66 is a relationship between densities, and it may seem strange to think of viscosity as a density. Recall, however, the relationships between the "dynamic viscosity" μ and the "kinematic viscosity" ν, namely $\mu = \rho\nu$, where ρ is the mass density. Let us postulate the same relationships on the Lagrangian side and set

$$\mu^L(q) = M(q)\,\nu^L(q) \qquad (67)$$

If we now make the <u>kinematic</u> viscosities numerically equal at corresponding points

$$\nu(\underline{X}(q,t),t) = \nu^L(q) \qquad (68)$$

then the dynamic viscosities will indeed be related as densities. In fact, we can derive Eq. 66 from Eq. 68 in the following way. Start from the equation for $\rho(\underline{x},t)$ in terms of $M(q)$, and multiply both sides by $\nu(\underline{x},t)$:

$$\rho(\underline{x},t)\,\nu(\underline{x},t) = \int M(q)\,\nu(\underline{x},t)\,\delta(\underline{x}-\underline{X}(q,t))\,dq \qquad (69)$$

On the right-hand side, we may replace $\nu(\underline{x},t)$ by $\nu(\underline{X}(q,t),t) = \nu^L(q)$, since $\nu(\underline{x},t)$ is multiplied by $\delta(\underline{x}-\underline{X}(q,t))$. Once this has been done, we see that Eq. 69 is the same as Eq. 66.

In summary, in the case of non-uniform viscosity, our formulation becomes

$$\rho(\underline{x},t)\frac{Du_i}{Dt} + \frac{\partial p}{\partial x_i} = \frac{\partial}{\partial x_j}\left(\mu(\underline{x},t)\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\right) + f_i(\underline{x},t) \tag{70}$$

$$\frac{\partial u_j}{\partial x_j} = 0 \tag{71}$$

$$\rho(\underline{x},t) = \int M(q)\,\delta(\underline{x}-\underline{X}(q,t))\,dq \tag{72}$$

$$\mu(\underline{x},t) = \int \mu^L(q)\,\delta(\underline{x}-\underline{X}(q,t))\,dq \tag{73}$$

$$\underline{f}(\underline{x},t) = \int \underline{F}(q,t)\,\delta(\underline{x}-\underline{X}(q,t))\,dq \tag{74}$$

$$\frac{\partial \underline{X}}{\partial t}(q,t) = \underline{u}(\underline{X}(q,t),t) = \int \underline{u}(\underline{x},t)\,\delta(\underline{x}-\underline{X}(q,t))\,d\underline{x} \tag{75}$$

$$\delta_{var} E[\underline{X}(\,,t)] = -\int \underline{F}(q,t)\,\delta_{var}\underline{X}(q,t)\,dq \tag{76}$$

2) Incompressible elastic material immersed in fluid.

Previously, we considered an incompressible elastic material filling all of space. We now consider the case in which an incompressible elastic material fills only part of the space; the rest being occupied by an incompressible fluid. We assume that the fluid is homogeneous, i.e., that it has a constant density ρ_0 and a constant viscosity μ_0.

Clearly, in the above circumstances, the Lagrangian coordinates are only needed within the elastic material. We have to be careful, however, about the sharp interface between the elastic material and the fluid. In particular, when we integrate by parts over the Lagrangian variables we get boundary terms there. This can influence the form of the force that is applied to the fluid by the elastic material, as pointed out by Luca Heltai.

To study this problem, we have to be more specific about the form of the elastic energy functional. Let

$$E\left[\underline{X}(\,,t)\right] = \int_{\Omega} \mathcal{E}\left(\frac{\partial \underline{X}}{\partial \xi}\right) d\xi \qquad (77)$$

where Ω is the subset of ξ-space that is occupied by the elastic material. Of course, Ω is independent of time, since $\xi = \xi_1, \xi_2, \xi_3$ are material coordinates.

As before $\partial \underline{X} / \partial \xi$ denotes the 3×3 matrix with elements $\partial X_i / \partial \xi_j$. Thus, the local elastic energy density $\mathcal{E}$ with respect to the measure $d\xi$ is a function of the 9 variables

$$\partial X_1 / \partial \xi_1 \quad \cdots \quad \partial X_3 / \partial \xi_3$$

We shall use the notation

$$\mathcal{E}_{ij} = \frac{\partial \mathcal{E}}{\partial (\partial X_i / \partial \xi_j)} \qquad (78)$$

We remark that the assumed form of the elastic energy density in Eq. 77 is very general. It allows for nonlinearity and anisotropy in the elastic response of the material. The only restriction is homogeneity in space and time, since there is no explicit dependence of $\mathcal{E}$ on ξ or t. In fact, it would not be difficult to generalize further and allow for $\mathcal{E}(\partial \underline{X} / \partial \xi, \xi, t)$, but we avoid that for now.

Applying the variation operator δ_{var} to the elastic energy defined by Eq. 77, we get

$$\delta_{var} E = \int_{\Omega} \mathcal{E}_{ij}\left(\frac{\partial X}{\partial \xi}\right) \frac{\partial}{\partial \xi_j} \delta_{var} X_i \; d\xi$$

$$= \int_{\Omega} \frac{\partial}{\partial \xi_j}\left[\mathcal{E}_{ij}\left(\frac{\partial X}{\partial \xi}\right)\delta_{var} X_i\right] d\xi - \int_{\Omega}\left[\frac{\partial}{\partial \xi_j}\mathcal{E}_{ij}\left(\frac{\partial X}{\partial \xi}\right)\right]\delta_{var} X_i \, d\xi \tag{79}$$

The first term can be reduced to a surface integral by applying the divergence theorem. The result is

$$\delta_{var} E = \int_{\partial\Omega} N_j \,\mathcal{E}_{ij}\left(\frac{\partial X}{\partial \xi}\right)\delta_{var} X_i \; dA(\xi)$$

$$- \int_{\Omega}\left[\frac{\partial}{\partial \xi_j}\mathcal{E}_{ij}\left(\frac{\partial X}{\partial \xi}\right)\right]\delta_{var} X_i \; d\xi \tag{80}$$

In the first term of this result, $\partial\Omega$ is the surface that bounds the elastic material (in ξ-space), (N_1, N_2, N_3) is the unit outward normal to this surface, and $dA(\xi)$ is the area element of the surface.

Let

$$F_i(\xi,t) = \frac{\partial}{\partial \xi_j} \mathcal{E}_{ij}\left(\frac{\partial \underline{X}}{\partial \xi}\right), \quad \xi \in \Omega \tag{81}$$

$$G_i(\xi,t) = -N_j \mathcal{E}_{ij}\left(\frac{\partial \underline{X}}{\partial \xi}\right), \quad \xi \in \partial\Omega \tag{82}$$

Then

$$-\delta_{var} E = \int_{\Omega} \underline{F} \cdot \delta_{var} \underline{X} \, d\xi + \int_{\partial\Omega} \underline{G} \cdot \delta_{var} \underline{X} \, dA(\xi) \tag{83}$$

Thus, we see that Eq. 6 for $\delta_{var} E$ is generalized by the addition of a surface term. It follows that the equation for $\underline{f}(\underline{x},t)$ must be similarly generalized

$$\underline{f}(\underline{x},t) = \int_{\Omega} \underline{F}(\xi,t)\, \delta(\underline{x} - \underline{X}(\xi,t))\, d\xi + \int_{\partial\Omega} \underline{G}(\xi,t)\, \delta(\underline{x} - \underline{X}(\xi,t))\, dA(\xi) \tag{84}$$

It is important to note that the two terms in Eq. 84 are of drastically different character. Both involve the three-dimensional Dirac delta function, but the volume integral $d\xi$ in the first term gives a result that is finite. The surface integral $dA(\xi)$, on the other hand, gives a result that is still singular like a one-dimensional Dirac delta function. That is, it described a delta-function layer with support on the surface of the elastic material. Despite the above distinction, both terms give finite results when integrated over an arbitrary region, or when multiplied by a test function and then integrated over all space.

Another issue that we must face, now that the elastic material does not occupy all of space but is surrounded by fluid, is how to define $\rho(\underline{x},t)$ and $\mu(\underline{x},t)$. Clearly, we cannot use Eqs. 72 and 73, since these give zero when $\underline{x}$ is outside the region occupied by the elastic material at any particular time. Instead, we use

$$\rho(\underline{x},t) = \rho_0 + \int_\Omega \tilde{M}(\underline{\xi})\,\delta\big(\underline{x}-\underline{X}(\underline{\xi},t)\big)\,d\underline{\xi} \qquad (85)$$

$$\mu(\underline{x},t) = \mu_0 + \int_\Omega \tilde{\mu}^L(\underline{\xi})\,\delta\big(\underline{x}-\underline{X}(\underline{\xi},t)\big)\,d\underline{\xi} \qquad (86)$$

where ρ_0 and μ_0 are the density and viscosity of the surrounding fluid and where

$$\tilde{M}(\underline{\xi}) = M(\underline{\xi}) - \rho_0 \det\left(\frac{\partial \underline{X}}{\partial \underline{\xi}}\right) \qquad (87)$$

$$\tilde{\mu}^L(\underline{\xi}) = \mu^L(\underline{\xi}) - \mu_0 \det\left(\frac{\partial \underline{X}}{\partial \underline{\xi}}\right) \qquad (88)$$

Here $\left(\tilde{M}(\underline{\xi})d\underline{\xi}\right)$ is the "buoyant mass" of the element $d\underline{\xi}$ of the elastic material, i.e., its mass minus the mass of the ambient fluid that it displaces. In the special case of a neutrally buoyant material, $\tilde{M}=0$. Note that $\tilde{M}$ may be negative, subject only to the restriction that $\rho(\underline{x},t) > 0$ everywhere. The quantity $\tilde{\mu}^L$ is similarly the excess viscosity contributed by the elastic material, which may be visco-elastic. If $\tilde{\mu}^L=0$, the viscosity of the elastic material matches that of the ambient fluid. Also, $\tilde{\mu}^L$ may be negative, subject to the restriction that $\mu(\underline{x},t) \geq 0$. In summary, we have the following system of equations:

$$\rho(\underline{x},t)\frac{Du_i}{Dt} + \frac{\partial p}{\partial x_i} = \frac{\partial}{\partial x_j}\left(\mu(\underline{x},t)\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right) + f_i(\underline{x},t) \quad (89)$$

$$\frac{\partial u_j}{\partial x_j} = 0 \quad (90)$$

$$\rho(\underline{x},t) = \rho_0 + \int_\Omega \tilde{M}(\xi)\,\delta\big(\underline{x}-\underline{X}(\xi,t)\big)\,d\xi \quad (91)$$

$$\mu(\underline{x},t) = \mu_0 + \int_\Omega \tilde{\mu}^L(\xi)\,\delta\big(\underline{x}-\underline{X}(\xi,t)\big)\,d\xi \quad (92)$$

$$\underline{f}(\underline{x},t) = \int_\Omega \underline{F}(\xi,t)\,\delta\big(\underline{x}-\underline{X}(\xi,t)\big)\,d\xi + \int_{\partial\Omega} \underline{G}(\xi,t)\,\delta\big(\underline{x}-\underline{X}(\xi,t)\big)\,dA(\xi) \quad (93)$$

$$\frac{\partial \underline{X}}{\partial t}(\xi,t) = \underline{u}\big(\underline{X}(\xi,t),t\big) = \int \underline{u}(\underline{x},t)\,\delta\big(\underline{x}-\underline{X}(\xi,t)\big)\,d\underline{x} \quad (94)$$

$$F_i(\xi,t) = \frac{\partial}{\partial \xi_j}\,\mathcal{E}_{ij}\left(\frac{\partial \underline{X}}{\partial \xi}\right), \quad \xi \in \Omega \quad (95)$$

$$G_i(\xi,t) = -N_j\,\mathcal{E}_{ij}\left(\frac{\partial \underline{X}}{\partial \xi}\right), \quad \xi \in \partial\Omega \quad (96)$$

where (N_1, N_2, N_3) is the unit normal to $\partial\Omega$ in q-space,

$$\mathcal{E}_{ij} = \frac{\partial \mathcal{E}}{\partial(\partial X_i / \partial q_j)} \qquad (97)$$

and $\mathcal{E}$ is the local density of elastic energy with respect to the measure dq.

An important conceptual point concerning the above formulation is that the fluid equations hold throughout space, including the region occupied by the elastic material as well as the region that is not so occupied and contains only fluid. As discussed previously, we regard the elastic material as an idealized composite with a massive and incompressible fluid component as well as a massless elastic component, the volume fraction of the elastic component being equal to zero.

There is an important special case in which the surface term in Eq. 93 vanishes because $\sigma_i(q,t)$ as given by Eq. 96 is equal to zero. This is the case in which the elastic material is comprised of a system of fibers permeated by incompressible fluid, and in which the fibers are never cut by the surface of the elastic material but instead run along that surface.

Although such a material may be comprised of multiple fiber families, we consider a single such family for simplicity. This means that there is exactly one fiber through any given point of the material. Let the parameters ξ_1, ξ_2, ξ_3 be chosen in such a manner that ξ_1 and ξ_2 are constant along any fiber. Thus, in ξ-space, the fibers run in the ξ_3 coordinate direction. It follows that a surface which the fibers do not cross is of the form

$$S(\xi_1, \xi_2) = 0 \tag{98}$$

Such a surface has a unit normal with the property that

$$N_3 = 0 \tag{99}$$

Now the elastic energy of the fibers is of the form

$$E = \int_\Omega \mathcal{E}\left(\left| \frac{\partial \underline{X}}{\partial \xi_3} \right| \right) d\xi_1 \, d\xi_2 \, d\xi_3 \tag{100}$$

That is, $\mathcal{E}$ is independent of $\partial \underline{X} / \partial \xi_1$ and $\partial \underline{X} / \partial \xi_2$. It follows that

$$\varepsilon_{i1} = \varepsilon_{i2} = 0 \qquad (101)$$

But Eq. 96 gives

$$G_i = -N_j \, \varepsilon_{ij} = -\left(N_1 \varepsilon_{i1} + N_2 \varepsilon_{i2} + N_3 \varepsilon_{i3}\right) \qquad (102)$$

The first two terms are zero because of Eq. 101, and the third term is zero because of Eq. 99. Thus $G_i = 0$ (for all i) and the surface term in Eq. 93 vanishes.

A further remark about this surface term is that it will be automatically included at the discrete level without any special consideration, provided that one follows the strategy of discretizing the elastic energy functional $E[\underline{X}(\,,t)]$ to obtain an energy function of the form $E(\underline{X}_1 \cdots \underline{X}_N)$, where $\underline{X}_1 \cdots \underline{X}_N$ are the nodes of the discretization, and then differentiating $E(\underline{X}_1 \cdots \underline{X}_N)$ to find the forces that should be applied to the fluid at each node. In the typical case, there will be unbalanced forces at surface nodes that are substantially larger than the nearly balanced forces at interior nodes.

3) Immersed elastic boundaries

We now come to the example that gives the immersed boundary method its name. We consider here an elastic material that takes the form of a two-dimensional manifold immersed in a viscous incompressible fluid. Here, only two Lagrangian parameters are needed: $q = (q_1, q_2)$. The Lagrangian description of the motion of the elastic material is

(103) $$\underline{x} = \underline{X}(q_1, q_2, t)$$

As usual, we assume that the elastic energy is of the form

(104) $$E[\underline{X}(\ ,\ ,t)]$$

and we define a Lagrangian force density $\underline{F}(q_1, q_2, t)$ with respect to the measure $dq_1\, dq_2$ by taking the variational derivative of E:

(105) $$\delta_{var} E[\underline{X}(\ ,\ ,t)] = -\iint \underline{F}(q_1, q_2, t) \cdot \delta_{var} \underline{X}(q_1, q_2, t)\, dq_1\, dq_2$$

The equations of motion of the whole system may now be written as follows

(106) $$\rho(\underline{x},t)\,\frac{D\underline{u}}{Dt} + \nabla p = \mu_0 \Delta \underline{u} + \underline{f}(\underline{x},t)$$

(107) $$\nabla\cdot\underline{u} = 0$$

(108) $$\rho(\underline{x},t) = \rho_0 + \iint M(\xi_1,\xi_2)\,\delta\big(\underline{x}-\underline{X}(\xi_1,\xi_2,t)\big)\,d\xi_1\,d\xi_2$$

(109) $$\underline{f}(\underline{x},t) = \iint \underline{F}(\xi_1,\xi_2,t)\,\delta\big(\underline{x}-\underline{X}(\xi_1,\xi_2,t)\big)\,d\xi_1\,d\xi_2$$

(110) $$\frac{\partial \underline{X}}{\partial t}(\xi_1,\xi_2,t) = \underline{u}\big(\underline{X}(\xi_1,\xi_2,t),t\big)$$

$$= \int \underline{u}(\underline{x},t)\,\delta\big(\underline{x}-\underline{X}(\xi_1,\xi_2,t)\big)\,d\underline{x}$$

(111) $$\delta_{\mathrm{var}} E\big[\underline{X}(\,,\,,t)\big] = -\iint \underline{F}(\xi_1,\xi_2,t)\cdot\delta_{\mathrm{var}}\underline{X}(\xi_1,\xi_2,t)\,d\xi_1\,d\xi_2$$

In these equations, $M(s_1, s_2)$ is the mass density, if any, of the immersed elastic boundary, with respect to the measure $ds_1 ds_2$. We do not need to distinguish the mass density and the bouyant mass density, since the two-dimensional boundary displaces no volume. In the special case that $M(s_1,s_2)=0$, we have the major simplification that $\rho(\underline{x},t)=\rho_0$. Otherwise $\rho(\underline{x},t)$ includes a delta-function layer with support on the immersed boundary.