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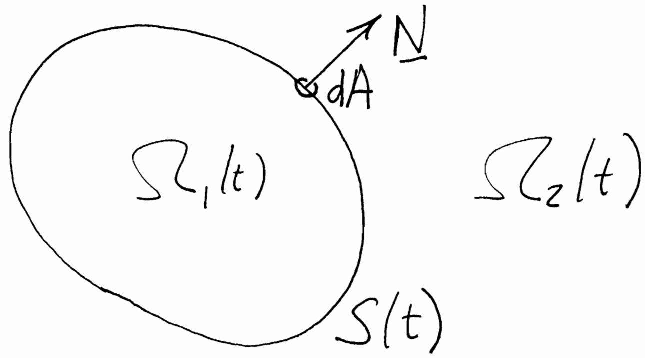
Immersed Boundary Formulation $\Rightarrow$ Classical Jump Conditions*

Consider a viscous incompressible fluid filling $\Omega = \mathbb{R}^3$ and containing an internal boundary $S(t)$ that moves with the fluid and separates Ω into two regions $\Omega_1(t)$ and $\Omega_2(t)$. Let dA be the area element on $S(t)$ and let $\underline{N}$ be the unit normal to $S(t)$ pointing out of Ω_1 and into Ω_2.



If a function $g(\underline{x},t)$ has different limiting values on the two sides of $S(t)$ these will be denoted $g^{(1)}$ and $g^{(2)}$, and we shall use the notation

$$[g] = g^{(2)} - g^{(1)} \qquad (1)$$

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* See also Lai M-C and Li ZL: A remark on jump conditions for the three-dimensional Navier-Stokes equations involving an immersed moving membrane. Applied Math Letters 14:149-154, 2001

Let $\underline{F}(\underline{x},t)$ be a force per unit area that is applied to the fluid along the surface $S(t)$. Note that $\underline{F}(\underline{x},t)$ is only defined for $\underline{x} \in S(t)$.

According to the immersed boundary (IB) formulation, the equations of motion of the fluid are then as follows:

$$\rho\left(\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j}(u_i u_j)\right) + \frac{\partial p}{\partial x_i} = \mu \frac{\partial^2 u_i}{\partial x_j^2} + f_i \qquad (2)$$

$$\frac{\partial u_j}{\partial x_j} = 0 \qquad (3)$$

where

$$f_i(\underline{x},t) = \int_{S(t)} F_i(\underline{x}',t)\,\delta(\underline{x}-\underline{x}')\,dA' \qquad (4)$$

(The notation dA' indicates that the integration is with respect to $\underline{x}'$.) In these equations, u_i, $i=1,2,3$, are the components of $\underline{u}$, and similarly for other vector quantities, and the summation convention ~~equation~~ is used. Note that the nonlinear terms are in "divergence" or "conservation" form.

The independent variables in Eqs. 2-4 are $\underline{x}$ = position and t = time. The functions of $(\underline{x}, t)$ that appear in these equations are the fluid velocity $\underline{u}(\underline{x},t)$, the fluid pressure $p(\underline{x},t)$, and the applied force density $\underline{f}(\underline{x},t)$. The notation $\delta(\underline{x})$ is shorthand for the three-dimensional Dirac delta function

$$\delta(\underline{x}) = \delta(x_1)\,\delta(x_2)\,\delta(x_3) \qquad (5)$$

Note that $\underline{f}(\underline{x},t)$ is a delta function layer with support on the moving surface $S(t)$. Integration of Eq. 4 over an arbitrary region R gives

$$\int_R f_i(\underline{x},t)\,dx = \int_{R \cap S(t)} F_i(\underline{x}',t)\,dA' \qquad (6)$$

where $dx = dx_1 dx_2 dx_3$. This says that the total force applied by the surface $S(t)$ to the region R may be found by integrating the force per unit area $\underline{F}(\underline{x},t)$ over that part of the surface that lies within R.

The constants ρ and μ in Eq. 2 are the fluid density and viscosity, respectively.

Equations 2-4 are relationships among distributions. To "decode" them, we apply the standard recipe of multiplying by smooth test functions and then integrating by parts as many times as may be needed to apply all differential operators to the test functions only. The integration by parts is purely formal, and is performed on the whole domain, without regard to the internal boundary $S(t)$. Also, the test functions decay sufficiently rapidly at ∞ that there are no boundary terms.

Following this procedure, we arrive at the following decoded version (weak formulation) of Eqs. 2-4:

$$-\int_0^T\int_\Omega\left(\rho\left(\frac{\partial w_i}{\partial t}+u_j\frac{\partial w_i}{\partial x_j}\right)u_i+\frac{\partial w_i}{\partial x_i}\,p\right)dx\,dt$$

$$=\int_0^T\int_\Omega \mu\,\frac{\partial^2 w_i}{\partial x_j^2}\,u_i\,dx\,dt+\int_0^T\int_{S(t)} w_i F_i\,dA\,dt \qquad (7)$$

$$-\int_0^T\int_\Omega \frac{\partial \phi}{\partial x_j}\,u_j\,dx\,dt=0 \qquad (8)$$

Equations 7-8 are supposed to hold for all sufficiently smooth $w_i(\underline{x},t)$ and $\phi(\underline{x},t)$ that approach zero sufficiently rapidly as $|\underline{x}| \to \infty$, and moreover such that $w_i(\underline{x},0) = w_i(\underline{x},T) = 0$ for $i=1,2,3$.

Since all derivatives in Eqs. 7-8 are applied to the test functions, the only a priori assumptions we need on $\underline{u}, p$ is that they are bounded. Then the surface $S(t)$ makes no finite contribution to any integrals over Ω, and we may write

$$\int_{\Omega} (\quad) dx = \int_{\Omega_1(t)} (\quad) dx + \int_{\Omega_2(t)} (\quad) dx \tag{9}$$

We shall sometimes denote the right-hand side of Eq. 9 as

$$\int_{\Omega_1(t)+\Omega_2(t)} (\quad) dx \tag{10}$$

Our general strategy will be to split up integrals over Ω in this way, and then to use integration by parts in each domain separately to move derivatives back to $\underline{u}, p$.

Begin with Eq. 8. We have

$$-\int_{\Omega_1(t)} \frac{\partial \phi}{\partial x_j} u_j \, dx = -\int_{\Omega_1(t)} \frac{\partial}{\partial x_j}(\phi u_j) dx + \int_{\Omega_1(t)} \phi \frac{\partial u_j}{\partial x_j} dx$$

$$= -\int_{S(t)} \phi u_j^{(1)} N_j \, dA + \int_{\Omega_1(t)} \phi \frac{\partial u_j}{\partial x_j} dx \qquad (11)$$

Similarly

$$-\int_{\Omega_2(t)} \frac{\partial \phi}{\partial x_j} u_j \, dx = +\int_{S(t)} \phi u_j^{(2)} N_j \, dA + \int_{\Omega_2(t)} \phi \frac{\partial u_j}{\partial x_j} dx \qquad (12)$$

Adding these equations, integrating over time, and making use of Eq. 8, we find

$$0 = \int_0^T \int_{S(t)} \phi [u_j] N_j \, dA \, dt + \int_0^T \int_{\Omega_1(t)+\Omega_2(t)} \phi \left(\frac{\partial u_j}{\partial x_j} \right) dx \, dt \qquad (13)$$

From the arbitrariness of ϕ, it then follows that

$$[u_j]N_j = [u_j N_j] = 0 \quad \text{on } S(t) \tag{14}$$

$$\frac{\partial u_j}{\partial x_j} = 0 \quad \text{in } \Omega_1(t) \text{ and } \Omega_2(t) \tag{15}$$

Eq. 14 is useful because it means that we do not have to distinguish between $u_j^{(1)}N_j$ and $u_j^{(2)}N_j$.

Now we turn to Eq. 7 and begin by considering the time-derivative terms. For these, we need the transport theorem*

$$\frac{d}{dt}\int_{\Omega_1(t)} g\,dx = \int_{\Omega_1(t)} \frac{\partial g}{\partial t}\,dx + \int_{S(t)} g^{(1)} u_j N_j\,dA \tag{16}$$

$$\frac{d}{dt}\int_{\Omega_2(t)} g\,dx = \int_{\Omega_2(t)} \frac{\partial g}{\partial t}\,dx - \int_{S(t)} g^{(2)} u_j N_j\,dA \tag{17}$$

These are applied as follows:

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* This is where we use the assumption that the surface $S(t)$ moves with the fluid.

$$-\int_{\Omega_1(t)} \rho \frac{\partial w_i}{\partial t} u_i \, dx = -\int_{\Omega_1(t)} \rho \frac{\partial}{\partial t}(w_i u_i)\, dx + \int_{\Omega_1(t)} \rho w_i \frac{\partial u_i}{\partial t}\, dx$$

$$= -\frac{d}{dt}\int_{\Omega_1(t)} \rho w_i u_i \, dx + \int_{S(t)} \rho w_i u_i^{(1)} u_j N_j \, dA$$

$$+ \int_{\Omega_1(t)} \rho w_i \frac{\partial u_i}{\partial t}\, dx \qquad (18)$$

Integrating over $(0,T)$ and making use of the initial and final conditions on w_i, we get

$$-\int_0^T \int_{\Omega_1(t)} \rho \frac{\partial w_i}{\partial t} u_i \, dx\, dt = \int_0^T \int_{S(t)} \rho w_i u_i^{(1)} u_j N_j \, dA\, dt$$

$$+ \int_0^T \int_{\Omega_1(t)} \rho w_i \frac{\partial u_i}{\partial t}\, dx\, dt \qquad (19)$$

Similarly

$$-\int_0^T \int_{\Omega_2(t)} \rho \frac{\partial w_i}{\partial t} u_i \, dx\, dt = -\int_0^T \int_{S(t)} \rho w_i u_i^{(2)} u_j N_j \, dA\, dt$$

$$+ \int_0^T \int_{\Omega_2(t)} \rho w_i \frac{\partial u_i}{\partial t}\, dx\, dt \qquad (20)$$

Adding Eqs. 19 and 20, we get

$$-\int_0^T\int_{\Omega} \rho \frac{\partial w_i}{dt} u_i \, dx\, dt = -\int_0^T\int_{S(t)} \rho w_i [u_i] u_j N_j \, dA\, dt$$

$$+\int_0^T\int_{\Omega_1(t)+\Omega_2(t)} \rho w_i \frac{\partial u_i}{\partial t} \, dx\, dt \quad (21)$$

The rest of the terms are simpler because they do not involve time derivatives. We have

$$-\int_{\Omega_1(t)} \rho u_i u_j \frac{\partial w_i}{\partial x_j} dx = -\int_{S(t)} \rho u_i^{(1)} w_i u_j N_j \, dA$$

$$+\int_{\Omega_1(t)} \rho w_i \frac{\partial}{\partial x_j}(u_i u_j)\, dx \quad (22)$$

$$-\int_{\Omega_2(t)} \rho u_i u_j \frac{\partial w_i}{\partial x_j} dx = +\int_{S(t)} \rho u_i^{(2)} w_i u_j N_j \, dA$$

$$+\int_{\Omega_2(t)} \rho w_i \frac{\partial}{\partial x_j}(u_i u_j)\, dx \quad (23)$$

Adding Eqs. 22 and 23, and integrating over time, we get

$$-\int_0^T \int_\Omega \rho u_i u_j \frac{\partial w_i}{\partial x_j}\, dx\, dt = \int_0^T \int_{S(t)} \rho [u_i] w_i u_j N_j \, dA\, dt$$

$$+ \int_0^T \int_{\Omega_1(t)+\Omega_2(t)} \rho w_i \frac{\partial}{\partial x_j}(u_i u_j)\, dx\, dt \qquad (24)$$

Adding Eqs. 21 and 24, we find that <u>the boundary terms cancel</u>, with the result that

$$-\int_0^T \int_\Omega \rho \left(\frac{\partial w_i}{\partial t} + u_j \frac{\partial w_i}{\partial x_j} \right) u_i \, dx\, dt$$

$$= \int_0^T \int_{\Omega_1(t)+\Omega_2(t)} \rho \left(\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j}(u_i u_j) \right) w_i \, dx\, dt \qquad (25)$$

Next, consider the pressure terms:

$$-\int_{\Omega_1(t)} \frac{\partial w_i}{\partial x_i} p \, dx = -\int_{S(t)} p^{(1)} w_i N_i \, dA + \int_{\Omega_1(t)} w_i \frac{\partial p}{\partial x_i} dx \quad (26)$$

$$-\int_{\Omega_2(t)} \frac{\partial w_i}{\partial x_i} p \, dx = +\int_{S(t)} p^{(2)} w_i N_i \, dA + \int_{\Omega_2(t)} w_i \frac{\partial p}{\partial x_i} dx \quad (27)$$

Adding these, and integrating over t we get

$$-\int_0^T\int_{\Omega} \frac{\partial w_i}{\partial x_i} p \, dx\, dt = \int_0^T\int_{S(t)} [p] w_i N_i \, dA + \int_{\Omega_1(t)+\Omega_2(t)} w_i \frac{\partial p}{\partial x_i} dx \quad (28)$$

Finally, consider the viscous terms:

$$\int_{\Omega_1(t)} \mu \frac{\partial^2 w_i}{\partial x_j^2} u_i \, dx = \int_{S(t)} \mu N_j \frac{\partial w_i}{\partial x_j} u_i^{(1)} dA - \int_{\Omega_1(t)} \mu \frac{\partial w_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} dx \quad (29)$$

$$\int_{\Omega_1(t)} \mu w_i \frac{\partial^2 u_i}{\partial x_j^2} dx = \int_{S(t)} \mu N_j \left(\frac{\partial u_i}{\partial x_j}\right)^{(1)} w_i \, dA - \int_{\Omega_1(t)} \mu \frac{\partial w_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} dx \quad (30)$$

Similarly, for $\Omega_2(t)$, we have

$$\int_{\Omega_2(t)} \mu \frac{\partial^2 w_i}{\partial x_j^2} u_i \, dx = -\int_{S(t)} \mu N_j \frac{\partial w_i}{\partial x_j} u_i^{(2)} dA - \int_{\Omega_2(t)} \mu \frac{\partial w_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} dx \quad (31)$$

$$\int_{\Omega_2(t)} \mu w_i \frac{\partial^2 u_i}{\partial x_j^2} dx = -\int_{S(t)} \mu N_j \left(\frac{\partial u_i}{\partial x_j}\right)^{(2)} w_i dA - \int_{\Omega_2(t)} \mu \frac{\partial w_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} dx \quad (32)$$

Combining the last four equations as (29)−(30)+(31)−(32), and then integrating over $(0,T)$, we get

$$\int_0^T \int_{\Omega} \mu \frac{\partial^2 w_i}{\partial x_j^2} u_i \, dx\, dt = \int_0^T \int_{\Omega_1(t)+\Omega_2(t)} \mu w_i \frac{\partial^2 u_i}{\partial x_j^2} dx\, dt$$

$$+ \int_0^T \int_{S(t)} \mu w_i \left[N_j \frac{\partial u_i}{\partial x_j} \right] dA\, dt$$

$$- \int_0^T \int_{S(t)} \mu N_j \frac{\partial w_i}{\partial x_j} [u_i] \, dA\, dt \quad (33)$$

The key results above are Eqs. 25, 28, and 33. Making use of them, we find that Eq. 7 may be rewritten

$$0 = \int_0^T \int_{\Omega_1(t)+\Omega_2(t)} \left(\rho\left(\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j}(u_i u_j)\right) + \frac{\partial p}{\partial x_i} - \mu \frac{\partial^2 u_i}{\partial x_j^2} \right) w_i \, dx\, dt$$

$$+ \int_0^T \int_{S(t)} \left([p] N_i - \mu \left[N_j \frac{\partial u_i}{\partial x_j} \right] - F_i \right) w_i \, dA\, dt$$

$$+ \int_0^T \int_{S(t)} \mu \, N_j \frac{\partial w_i}{\partial x_j} [u_i] \, dA\, dt \qquad (34)$$

The arbitrariness of w_i now implies that

$$\rho\left(\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j}(u_i u_j)\right) + \frac{\partial p}{\partial x_i} - \mu \frac{\partial^2 u_i}{\partial x_j^2} = 0 \quad \text{in } \Omega_1(t) \text{ and in } \Omega_2(t) \qquad (35)$$

$$[p] N_i - \mu \left[N_j \frac{\partial u_i}{\partial x_j} \right] = F_i \quad \text{on } S \qquad (36)$$

$$[u_i] = 0 \quad \text{on } S \qquad (37)$$

Eq. 35 is just the restriction to the interiors of $\Omega_1(t)$ and $\Omega_2(t)$ of the Navier-Stokes equation that was our starting point. Recall that we also found $\partial u_j / \partial x_j = 0$ in $\Omega_1(t)$ and $\Omega_2(t)$, Eq. 15. The continuity of $\underline{u} \cdot \underline{N}$ that we found previously (Eq. 14) is now superceded by Eq. 37, which states that all components of $\underline{u}$ are continuous across the internal boundary.

Our results, then, are Eqs. 36 and 37, which may be written in vector form as follows:

$$[p]\,\underline{N} - \mu \left[\frac{\partial \underline{u}}{\partial N} \right] = \underline{F} \qquad (38)$$

$$[\underline{u}] = 0 \qquad (39)$$

where we are using the shorthand

$$\frac{\partial}{\partial N} = N \cdot \nabla = N_j \frac{\partial}{\partial x_j} \qquad (40)$$

for the normal derivative.

It is important to note that $[\partial \underline{u}/\partial N]$ is tangential to the boundary $S(t)$. To see this, consider

$$\underline{N} \cdot [\partial \underline{u}/\partial \underline{N}] = N_i N_j \left[\frac{\partial u_i}{\partial x_j}\right] \qquad (41)$$

Let $\underline{A}$ and $\underline{B}$ be mutually orthogonal unit vectors tangent to $S(t)$. Then $(\underline{N}, \underline{A}, \underline{B})$ is an orthonormal triad at each point of $S(t)$. As such, it satisfies the completeness relation

$$N_i N_j + A_i A_j + B_i B_j = \delta_{ij} \qquad (42)$$

Therefore

$$\underline{N} \cdot [\partial \underline{u}/\partial N] = (\delta_{ij} - A_i A_j - B_i B_j)\left[\frac{\partial u_i}{\partial x_j}\right]$$

$$= \left[\frac{\partial u_i}{\partial x_i}\right] - A_i \left[A_j \frac{\partial u_i}{\partial x_j}\right] - B_i \left[B_j \frac{\partial u_i}{\partial x_j}\right] \qquad (43)$$

But $\partial u_i/\partial x_i = 0$ on both sides of the boundary, and $\left(A_j \frac{\partial}{\partial x_j}, B_j \frac{\partial}{\partial x_j}\right)$ are <u>tangential</u> derivatives

Therefore

$$\underline{N} \cdot [\partial \underline{u}/\partial N] = -\underline{A} \cdot \left(\underline{A} \cdot \nabla [\underline{u}]\right) - \underline{B} \cdot \left(\underline{B} \cdot \nabla [\underline{u}]\right)$$
$$= 0 \qquad (44)$$

since $[\underline{u}] = 0$. Since $[\partial \underline{u}/\partial N]$ is tangential, Eq. 38 may be written

$$[p] = \underline{F} \cdot \underline{N} \qquad (45)$$

$$-\mu \left[\frac{\partial u}{\partial N}\right] = \underline{F} - \left(\underline{F} \cdot \underline{N}\right)\underline{N} \qquad (46)$$

These, together with $[\underline{u}] = 0$ (Eq. 38) are the classical jump conditions across an interface in a viscous, incompressible fluid. Note that $\underline{F}$ is the force <u>per unit area</u> applied to the fluid along the interface. If a different measure besides area is used, these results must be adjusted accordingly. For example, if the surface is given in parametric form

$$\underline{x} = \underline{X}(r,s,t) \tag{47}$$

and if the applied force is expressed in terms of (r,s), so that

$$\underline{\mathcal{F}}(r,s,t)\,dr\,ds \tag{48}$$

is the force applied to the fluid by the patch $dr\,ds$ of the surface, then

$$dA = \left| \frac{\partial \underline{X}}{\partial r} \times \frac{\partial \underline{X}}{\partial s} \right| dr\,ds \tag{49}$$

and since $\underline{F}\,dA = \underline{\mathcal{F}}\,dr\,ds$ at corresponding points, we have

$$\underline{F}(\underline{X}(r,s,t),t) = \frac{\underline{\mathcal{F}}(r,s,t)}{\left| \frac{\partial \underline{X}}{\partial r} \times \frac{\partial \underline{X}}{\partial s} \right|} \tag{50}$$

Appendix 1

Justification of the formal integration by parts that is used to derive Eqs. 7-8.

Start out by using a smoothed version of the Dirac δ function, $\delta_h(\underline{x})$, where δ_h is smooth but $\delta_h \to \delta$ as $h \to 0$.

For any $h > 0$, there is no singularity on the immersed boundary and the integrations by parts ignoring the immersed boundary are fully justified. However, the integral on the right-hand side of Eq. 7 involving F_i becomes

$$\int_0^T \int_\Omega \int_{S(t)} W_i(\underline{x}) F_i(\underline{x}', t)\, \delta_h(\underline{x} - \underline{x}')\, dA'\, d\underline{x}\, dt$$

where $dA' = dA(\underline{x}')$, i.e., the area element surrounding the point $\underline{x}'$. At this point we can let $h \to 0$ and obtain Eqs. 7-8. (After the integration by parts, there are no derivatives of $\underline{u}$ or p, so we do not have to worry about the limits of those derivatives as $h \to 0$.)

Appendix 2: Classical derivation of the jump conditions across an interface in a viscous, incompressible fluid

In this Appendix we recall the classical (i.e., not based on the immersed boundary formulation in terms of delta-function forces) derivation of the jump conditions on an elastic, massless immersed boundary in a viscous incompressible fluid.

According to the classical no-slip condition, the viscous fluid sticks to both sides of the immersed boundary, and it follows from this that there cannot be any jump in velocity across the immersed boundary. Thus

(A2.1) $$[\underline{u}] = 0$$

Since the immersed boundary is massless, the elastic force per unit area that it applies to the fluid has to be balanced by the jump in normal stress. This gives

(A2.2) $$[\sigma_{ij}]N_j + F_i = 0$$

where

(A2.3) $$\sigma_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Substituting (A2.3) into (A2.2) gives

(A2.4) $$-[p]N_i + \mu\left[\frac{\partial u_i}{\partial x_j}\right]N_j + \mu\left[\frac{\partial u_j}{\partial x_i}\right]N_j + F_i = 0$$

Note that the second term in (A2.4) can alternatively be written as $[\underline{N}\cdot\nabla u_i]$ or $[\partial u_i/\partial N]$. We now show that the normal component of this term is zero, and also that the third term in (A2.4) is identically zero. For this purpose, we introduce an orthonormal triad of vectors

(A2.5) $$(\underline{\tau}^{(1)}, \underline{\tau}^{(2)}, \underline{N})$$

and recall that any such triad satisfies the identity

(A2.6) $$\tau_i^{(1)}\tau_j^{(1)} + \tau_i^{(2)}\tau_j^{(2)} + N_i N_j = \delta_{ij}$$

Therefore,

(A2.7) $$N_i N_j \left[\frac{\partial u_i}{\partial x_j}\right] = \delta_{ij}\left[\frac{\partial u_i}{\partial x_j}\right] - \tau_i^{(1)}\tau_j^{(1)}\left[\frac{\partial u_i}{\partial x_j}\right] - \tau_i^{(2)}\tau_j^{(2)}\left[\frac{\partial u_i}{\partial x_j}\right]$$

$$= [\nabla\cdot\underline{u}] - \underline{\tau}^{(1)}\cdot[\underline{\tau}^{(1)}\cdot\nabla\underline{u}] - \underline{\tau}^{(2)}\cdot[\underline{\tau}^{(2)}\cdot\nabla\underline{u}]$$

The first term is zero because $\nabla\cdot\underline{u} = 0$ on both sides of the immersed boundary. The other two terms are zero because $\underline{\tau}^{(1)}\cdot\nabla$ and $\underline{\tau}^{(2)}\cdot\nabla$ are tangential derivatives and $[\underline{u}] = 0$. Thus

(A2.8) $$N_i N_j \left[\frac{\partial u_i}{\partial x_j}\right] = 0$$

Note that we can also interchange i,j in (A2.8) to get

(A2.9) $$N_i N_j \left[\frac{\partial u_j}{\partial x_i}\right] = 0$$

Moreover

(A2.10) $$\tau_i^{(1)} N_j \left[\frac{\partial u_j}{\partial x_i}\right] = \underline{N}\cdot[\underline{\tau}^{(1)}\cdot\nabla\underline{u}] = 0$$

(A2.11) $$\tau_i^{(2)} N_j \left[\frac{\partial u_j}{\partial x_i}\right] = \underline{N}\cdot[\underline{\tau}^{(2)}\cdot\nabla\underline{u}] = 0$$

Equation (A2.8) shows that $N_j \left[\frac{\partial u_i}{\partial x_j}\right]$ is purely tangential, since its normal component is zero.

Equations (A2.9)–(A2.11) shows that $N_j \left[\frac{\partial u_j}{\partial x_i}\right]$ is zero, since all of its components are zero.

Thus, equation (A2.4) separates into normal and tangential components as follows:

(A2.12) $$[p] = \underline{F} \cdot \underline{N}$$

(A2.13) $$-\mu \left[\partial \underline{u} / \partial \underline{N}\right] = \underline{F} - (\underline{F} \cdot \underline{N})\underline{N}$$

These, together with (A2.1) are the jump conditions across a massless immersed boundary.

Appendix 3: Jump condition on $\partial p/\partial N$.

As pointed out by Lai & Li (cited above), there is also a jump condition on $\partial p/\partial N$. This jump condition should not be considered as an additional boundary condition, however, since it is a simple consequence of the Navier-Stokes equations together with the other jump conditions.

On each side of the immersed boundary, we have

(A3-1) $$\rho \frac{Du}{Dt} + \nabla p = \mu \Delta \underline{u} = -\mu \nabla \times (\nabla \times \underline{u})$$

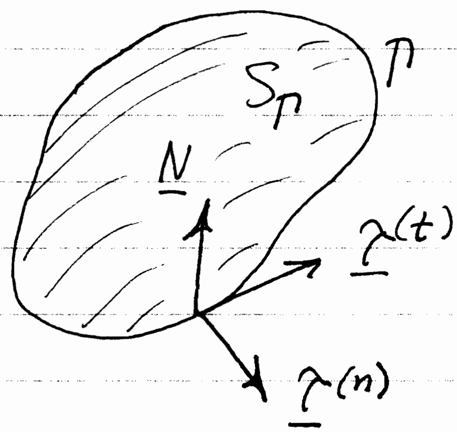
in which we have made use of $\nabla \cdot \underline{u} = 0$ to rewrite the viscous term. Here D/Dt is the material derivative. Applying the jump operator $[\ \ ]$ to every term, and noting that $[\underline{u}] = 0$, we get

(A3.2) $$[\nabla p] = -\mu \left[\nabla \times (\nabla \times \underline{u})\right]$$

the normal component of which is

(A3.3) $$[\partial p/\partial N] = -\mu\, \underline{N} \cdot \left[\nabla \times (\nabla \times \underline{u})\right]$$

Now consider a closed curve Γ lying on the immersed boundary, and let S_Γ be the part of the immersed boundary that is enclosed by Γ.



Let $\{\underline{\tau}^{(n)}, \underline{\tau}^{(t)}, \underline{N}\}$ be a right-handed orthonormal triad at each point of Γ such that $\underline{\tau}^{(t)}$ is tangent to Γ, where $\underline{N}$ is the unit normal to the immersed boundary.

By Stokes' theorem

(A3-4) $$\int_{S_\Gamma} [\partial p/\partial N]\, dA = -\int_{\Gamma} \underline{\tau}^{(t)} \cdot [\nabla \times \underline{u}]\, ds$$

where ds = arclength on Γ.

But

$$(A3.5) \quad \underline{\tau}^{(t)} \cdot [\nabla \times \underline{u}] = (\underline{N} \times \underline{\tau}^{(n)}) \cdot [\nabla \times \underline{u}]$$

$$= -(\underline{\tau}^{(n)} \times \underline{N}) \cdot [\nabla \times \underline{u}]$$

$$= -(\underline{\tau}^{(n)} \cdot [\underline{N} \times (\nabla \times \underline{u})])$$

and

$$(A3.6) \quad (\underline{N} \times (\nabla \times \underline{u}))_i = \varepsilon_{ijk} N_j \varepsilon_{klm} \frac{\partial}{\partial x_l} u_m$$

$$= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) N_j \frac{\partial}{\partial x_l} u_m$$

$$= N_m \frac{\partial u_m}{\partial x_i} - N_j \frac{\partial u_i}{\partial x_j}$$

We have previously shown (see Appendix 2) that $N_m [\partial u_m / \partial x_i] = 0$. Therefore

$$(A3.7) \quad [\underline{N} \times (\nabla \times \underline{u})] = -[\partial \underline{u} / \partial N]$$

Combining (A3.4), (A3.5), (A3.7), and (A2.13), we see that

(A3.8) $$\int_{S_\Gamma} [\partial p / \partial N]\, dA = \int_{\Gamma} \underline{\tau}^{(n)} \cdot \underline{F}^{(tan)}\, ds$$

where

(A3.9) $$\underline{F}^{(tan)} = \underline{F} - (\underline{F} \cdot \underline{N})\underline{N}$$

is the tangential part of $\underline{F}$. (Of course, we could replace $\underline{F}^{(tan)}$ by $\underline{F}$ in (A3.8), since $\underline{\tau}^{(n)}$ is tangential in any case, but we write $\underline{F}^{(tan)}$ to emphasize that the normal component plays no role here.)

Let the surface divergence operator $(\nabla_s \cdot)$ be defined by

(A3.10) $$\int_{S_\Gamma} (\nabla_s \cdot \underline{v})\, dA = \int_{\Gamma} (\underline{\tau}^{(n)} \cdot \underline{v})\, ds$$

where S is a smooth surface, Γ is an arbitrary smooth closed curve on S, S_Γ is the part of S enclosed by Γ, $\underline{\tau}^{(n)}$ is a unit vector field defined on Γ such that $\underline{\tau}^{(n)}$ is normal to Γ and tangent to S, and $\underline{v}$ is a vector field defined on S and tangent to S at every point of S.

Comparison of (A3.8) and (A3.10) shows, since Γ is arbitrary, that

(A3.11) $$\left[\frac{\partial p}{\partial N}\right] = \nabla_s \cdot \underline{F}^{(tan)}$$

Thus, the surface divergence of the tangential force generates a jump in the normal derivative of the pressure across the immersed boundary.