NOTES ON ECONOMIC GROWTH AND PRICE EQUILIBRIUM

Charles S. Peskin

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In these notes, we consider a dynamic version of an economic growth model proposed by von Neumann [1]. By “dynamic” we mean that we follow the evolution of the model over time, instead of considering only its balanced-growth state, which is a state in which the economy as a whole grows at a constant rate.

A distinctive feature of von Neumann’s model is that there is no distinction between producers and consumers. Instead, the economy is modeled as a collection of *processes*, indexed by $i = 1 \dots n$, each of which has inputs and outputs called *goods*, indexed by $j = 1 \dots m$.

The flow of time in the model economy alternates between *production periods* of unit duration, and *market days* of zero duration on which goods are exchanged to prepare for the next production period. It is during each market day that prices are determined.

The model economy starts to function at $t = 0$, with each process already having an allocation of goods so that it can begin production. The market days occur at $t = 1, 2, \dots$, and throughout these notes, t will be integer-valued.

The i^{th} process, running at constant intensity $r_i(t)$ for the production period $(t-1, t)$ consumes $r_i(t)A_{ij}$ units of good j at time $t-1$ and produces $r_i(t)B_{ij}$ units of good j at time t. Note that $r_i(t)$ is the intensity of the i^{th} process during the production period leading up to the market day at time t. How this intensity is determined will be described below.

The $n \times m$ matrices A and B characterize the economy, and the i^{th} rows of these two matrices characterize the i^{th} process. The elements of these matrices satisfy

$$A_{ij} > 0, \quad B_{ij} \geq 0, \tag{1}$$

for $i = 1 \dots n$ and $j = 1 \dots m$. Thus, each process needs some positive amount of every good (a strong but convenient assumption) in order to run, but may produce only a proper subset of all possible goods as output. It is assumed that the matrices A and B are independent of time.

Let

$$p_j(t) \geq 0, \quad j = 1 \ldots m, \tag{2}$$

be the price of one unit of good j on the market day that occurs at time t. How these prices are determined will be discussed below. For now, we regard them as given.

By selling the goods produced during the most recent production period, the i^{th} process earns cash at time t in the amount

$$\sum_{j=1}^{m} r_i(t) B_{ij} p_j(t). \tag{3}$$

We assume that all of this cash is spent on buying goods that are needed for the next production period, and that this determines the intensity $r_i(t+1)$ with which the i^{th} process will run during the production period $(t, t+1)$. This gives the equation

$$\sum_{j=1}^{m} r_i(t+1) A_{ij} p_j(t) = \sum_{j=1}^{m} r_i(t) B_{ij} p_j(t). \tag{4}$$

Note that $r_i(t+1)$ and $r_i(t)$ can be factored out of their respective sums, and then we can solve for $r_i(t+1)$ as follows:

$$r_i(t+1) = r_i(t) \frac{\sum_{j=1}^{m} B_{ij} p_j(t)}{\sum_{j=1}^{m} A_{ij} p_j(t)}. \tag{5}$$

Since the A_{ij} are all positive and the $p_j(t)$ are non-negative, the denominator in the above equation will be zero only if all of the prices are zero. We assume that this is not the case.

The only trouble with the above story is that the total amount of good j purchased at time t may exceed the total amount of good j that was produced during the prior production period. This is clearly impossible! (We are assuming that there is no mechanism for the storage of goods for later consumption; the only goods available on a given market day are those that were produced during the prior production period.) The *excess demand* for good j at time t in the above scenario is

$$
\begin{aligned}
e_j(t) &= \sum_{i=1}^{n} r_i(t+1)A_{ij} - \sum_{i=1}^{n} r_i(t)B_{ij} \\
&= \sum_{i=1}^{n} r_i(t)\left(\frac{\sum_{k=1}^{m} B_{ik}p_k(t)}{\sum_{k=1}^{m} A_{ik}p_k(t)}A_{ij} - B_{ij}\right). \qquad (6)
\end{aligned}
$$

It is useful to think of the excess demand for good j as a function of the intensity vector $r \in \mathbb{R}^n$ and the price vector $p \in \mathbb{R}^m$. Accordingly, we define

$$
E_j(r,p) = \sum_{i=1}^{n} r_i \left(\frac{\sum_{k=1}^{m} B_{ik}p_k}{\sum_{k=1}^{m} A_{ik}p_k}A_{ij} - B_{ij}\right), \qquad (7)
$$

and then we have $e_j(t) = E_j(r(t), p(t))$.

Some important properties of the excess-demand functions $E_j(r,p)$ are noted here for future reference. All of these properties refer to the dependence on p, for any particular r. To emaphasize this, we drop the dependence on r and just write $E_j(p)$ from now on:

- The functions E_j are homogeneous functions of p of degree zero, that is

$$
E_j(\alpha p) = E_j(p) \qquad (8)
$$

for any positive scalar α. Another way to say this is that only relative prices matter. (There is a notion of "interest rate" in von Neumann's original paper that describes how prices change in overall scale as the economy grows. As far as I can see, this is completely irrelevant, since the model does not involve borrowing, and since only relative prices matter. We do not consider interest rate here.)

Because of (8), we may restrict consideration to price vectors in the set S defined as follows:

$$
S = \left\{ p \in \mathbb{R}^m : p \geq 0 \;\;\&\;\; \sum_{j=1}^{m} p_j = 1 \right\}. \qquad (9)
$$

Here and in the following, when we apply an inequality to a vector, we mean that this inequality is applicable to every component. Thus, $p \geq 0$ means that $p_j \geq 0$ for every j. Note that every $p \in S$ has $p_j > 0$ for at least one value of j, since the sum of the p_j is required to be 1.

- The functions $E_j : S \to \mathbb{R}$ are continuous with continuous first derivatives. This follows from the inequality

$$\sum_{j=1}^{m} A_{ij} p_j \geq \min_j (A_{ij}) \sum_{j=1}^{m} p_j = \min_j (A_{ij}) > 0, \tag{10}$$

  which holds for every $p \in S$.

- For any fixed r, the functions $E_j(p)$ are obviously bounded from below, since

$$E_j(p) \geq -\sum_{i=1}^{n} r_i B_{ij}. \tag{11}$$

- The excess-demand functions $E_j(p)$ collectively satisfy Walras' law [2]: that the monetary value of the total excess demand is zero *at any price vector* p. To see this, multiply both sides of (7) by p_j, sum over j, and then, on the right-hand side of the resulting equation, bring the sum over j inside the sum over i. In this way, we get

$$\begin{aligned} \sum_{j=1}^{m} p_j E_j(p) &= \sum_{i=1}^{n} r_i \left(\frac{\sum_{k=1}^{m} B_{ik} p_k}{\sum_{k=1}^{m} A_{ik} p_k} \sum_{j=1}^{m} A_{ij} p_j - \sum_{j=1}^{m} B_{ij} p_j \right) \\ &= 0. \end{aligned} \tag{12}$$

The above properties of the excess-demand functions are typical of a much wider class of models than the specific one we are considering here, and the *price equilibrium problem* can therefore be formalized, for this whole class of excess-demand functions, as follows.

Let $E : S \to \mathbb{R}^m$ be a given mapping that is continuous with continuous first derivatives, and let E satisfy Walras' law:

$$\sum_{j=1}^{m} p_j E_j(p) = 0 \tag{13}$$

for all $p \in S$. Then $p^* \in S$ is an *equilibrium price vector* if it satisfies

$$E(p^*) \leq 0, \tag{14}$$

and of course the price equilibrium problem is to find an equilibrium price vector.

An important remark is that if p^* is an equilibrium price vector and if $E_j(p^*) < 0$ for some j, then $p_j^* = 0$. The proof of this is very simple. For an equilibrium price vector p^*, the sum in Walras' law involves only non-positive terms, since $E_j(p^*) \leq 0$ (by definition of price equilibrium) and since $p_j^* \geq 0$ (since $p^* \in S$). A sum of non-positive terms can only be zero if every term is zero. This gives the result that

$$p_j^* E_j(p^*) = 0, \quad j = 1 \ldots m, \tag{15}$$

from which it follows that if $E_j(p^*) < 0$ then $p_j^* = 0$, as claimed.

The significance of the above remark is that any good for which there is negative excess demand (i.e., positive excess supply) at equilibrium is a *free good*, i.e., its equilibrium price is zero. Note, however, that it is not known until the price equilibrium problem is solved whether any particular good will be a free good or not.

It should also be mentioned that price equilibrium is sometimes defined in a more restrictive way than here, to rule out the possibility of excess supply as well as excess demand. This is called *market clearing*. By this more restrictive definition, an equilibrium price vector is any $p \in S$ that satisfies $E(p) = 0$. Since this is a stronger condition than $E(p) \leq 0$, the existence of an equilibrium price vector is correspondingly harder to prove, and in fact further conditions on the mapping E are then required to guarantee existence beyond those assumed in our statement of the price equilibrium problem, above.

The existence of an equilibrium price vector $p^* \in S$ that satisfies $E(p^*) \leq 0$ is proved in the following way [3]. Let $P : S \to S$ be defined by

$$P_j(p) = \frac{p_j + \max(0, E_j(p))}{\sum_{k=1}^{m} (p_k + \max(0, E_k(p)))}, \quad j = 1 \ldots m. \tag{16}$$

To see that $P(p) \in S$ for all $p \in S$, note that $P_j(p) \geq 0$, and sum both sides of (16) over j to obtain $\sum_{j=1}^{m} P_j(p) = 1$. The mapping P is continuous, since the functions E_j are continuous, since $\max(0,x)$ is a continuous function of x, and since the denominator in (16) is bounded from below by 1 for all $p \in S$.

The set S is a closed, bounded, convex subset of $\mathbb{R}^m$. It follows, then, from Brouwer's fixed-point theorem [4], that there exists at least one price vector $p^* \in S$ such that $p^* = P(p^*)$. Writing out what this means, we have

$$p_j^* = \frac{p_j^* + \max\left(0, E_j(p^*)\right)}{\sum_{k=1}^{m} \left(p_k^* + \max\left(0, E_k(p^*)\right)\right)}, \quad j = 1 \ldots m. \tag{17}$$

We claim that p^* is an equilibrium price vector. To prove this, multiply both sides of (17) by $E_j(p^*)$, sum over j, and apply Walras' law, equation (13). The denominator on the right-hand side drops out because the left-hand side becomes zero, and we are left with the equation

$$0 = \sum_{j=1}^{m} E_j(p^*) \max\left(0, E_j(p^*)\right). \tag{18}$$

Because of the $\max$ function, we have here a sum that is equal to zero in which every term is non-negative, so every term must be equal to zero, and this implies that $E_j(p^*) \le 0$ for $j = 1 \ldots m$, as required. This completes the proof of existence of a price equilibrium.

For the computation of a price equilibrium, it is helpful to reformulate the problem as one of minimization. Let $\phi : S \to \mathbb{R}$ be defined by

$$\phi(p) = \frac{1}{2} \sum_{j : E_j(p) > 0} E_j^2(p). \tag{19}$$

It may seem redundant to use E_j^2 instead of E_j in the definition of ϕ, since the sum is only over terms for which E_j is positive anyway, but the choice E_j^2 is needed to make the gradient of ϕ be continuous, see below.

Note that $\phi(p) \ge 0$ for all $p \in S$. Also, ϕ is continuous on S, with continuous first derivatives. To see this, recall, first, that the functions E_j are continuous on S with continuous first derivatives, so the only possible discontinuities are related to the discrete nature of the condition $j : E_j(p) > 0$ that decides whether or not a term is to be included in ϕ. The dangerous points are those for which $E_j(p) = 0$ for one or more values of j. At precisely such points, however, any term that might or might not be included in the sum has the value zero, and likewise (since E_j^2 was used instead of E_j in the definition of ϕ), all of the first derivatives of such a term are zero at such points as well.

From the definition of ϕ, equation (19), it is obvious that $\phi(p) = 0$ if and only if $E_j(p) \leq 0$ for all j, which means that p by definition is an equilibrium price vector. (Recall that the domain of ϕ is S, so $p \in S$ is implied by $\phi(p) = 0$.) Since $\phi(p) \geq 0$ for all $p \in S$, it follows that every equilibrium price vector is a global minimizer of ϕ, with the value $\phi = 0$. We can therefore compute price equilibria by minimizing ϕ, and we can distinguish actual price equilibria from any non-global minima that may be found simply by evaluating ϕ.

We anticipate that global minima of ϕ will sometimes occur on the boundary of S, see the discussion of free goods, above. Although minimization software can deal with this, it is a complication that we can avoid by making a change of variables. Let $p(q)$ be defined by

$$p_j = q_j^2, \;\; j = 1 \ldots m. \tag{20}$$

The domain of q is the unit sphere in $\mathbb{R}^m$:

$$\overline{S} = \left\{ q \in \mathbb{R}^m : \sum_{j=1}^{m} q_j^2 = 1 \right\}, \tag{21}$$

which of course has no boundaries. If $q \in \overline{S}$, then $p(q) \in S$. Note that this method of putting the problem on the unit sphere is different from another approach that is sometimes used, which is to normalize the prices by setting the sums of the squares of the prices equal to 1, see for example [5]. Here we retain the more natural normalization that the sum of the prices themselves, not their squares, is equal to 1.

The function that we need to minimize over the unit sphere in $\mathbb{R}^m$ is

$$\overline{\phi}(q) = \frac{1}{2} \sum_{j:\overline{E}_j(q)>0} \overline{E}_j^2(q), \tag{22}$$

where

$$\begin{aligned} \overline{E}_j(q) &= E_j(p(q)) \\ &= \sum_{i=1}^{n} r_i \left(\frac{\sum_{k=1}^{m} B_{ik} q_k^2}{\sum_{k=1}^{m} A_{ik} q_k^2} A_{ij} - B_{ij} \right). \end{aligned} \tag{23}$$

The components of the gradient of $\overline{\phi}$ are therefore given by

$$\frac{\partial\overline{\phi}}{\partial q_k}(q) = \sum_{j:\overline{E}_j(q)>0} \overline{E}_j(q)\frac{\partial\overline{E}_j}{\partial q_k}(q) \tag{24}$$

where

$$\frac{\partial\overline{E}_j}{\partial q_k}(q) = 2q_k\sum_{i=1}^{n} r_i A_{ij}\frac{B_{ik}\sum_{\ell=1}^{m} A_{i\ell}q_\ell^2 - A_{ik}\sum_{\ell=1}^{m} B_{i\ell}q_\ell^2}{\left(\sum_{\ell=1}^{m} A_{i\ell}q_\ell^2\right)^2} \tag{25}$$

Note the important point that

$$\sum_{k=1}^{m} q_k\frac{\partial\overline{E}_j}{\partial q_k}(q) = 0, \tag{26}$$

which implies that

$$\sum_{k=1}^{m} q_k\frac{\partial\overline{\phi}}{\partial q_k}(q) = 0. \tag{27}$$

The meaning of this result (which is a consequence of the homogeneity of degree zero of the excess-demand functions) is that the gradient of $\overline{\phi}$ is tangent to the unit sphere in $\mathbb{R}^m$.

We can therefore use gradient descent on the unit sphere in $\mathbb{R}^m$ to find minima of the function $\overline{\phi}$. One way to think about gradient descent is in terms of the initial-value problem

$$\frac{dq}{d\tau} = -\left(\nabla\overline{\phi}\right)(q(\tau)), \tag{28}$$

$$q(0) = q^{(0)}, \tag{29}$$

in which τ is a time-like parameter (not to be confused with actual time), and $q^{(0)}$ is a given initial point on the unit sphere. A natural choice for $q^{(0)}$ (except on the first market day) is the value of q corresponding to the prices that prevailed on the previous market day. On the first market day, the choice of $q^{(0)}$ can be made by choosing a random point on the unit sphere in $\mathbb{R}^m$.

Because of equation (27), solutions to (28) remain on the unit sphere if they start on the unit sphere. Also,

$$
\begin{aligned}
\frac{d}{d\tau}\overline{\phi}\left(q(\tau)\right) &= \sum_{k=1}^{m} \frac{\partial\overline{\phi}}{\partial q_k}\frac{dq_k}{d\tau} \\
&= -\sum_{k=1}^{m}\left(\frac{\partial\overline{\phi}}{\partial q_k}\right)^2 \\
&\leq 0, \qquad (30)
\end{aligned}
$$

with equality if and only if $\nabla\overline{\phi} = 0$. Since the unit sphere in R^m is closed and bounded, it follows that any solution to that starts from any point on the unit sphere will converge as $\tau \to \infty$ to a stationary point of $\overline{\phi}$, i.e., a point at which $\nabla\overline{\phi} = 0$. Such a stationary point may or may not be one of the global minima that we seek, but we can easily tell whether it is or not by evaluating $\overline{\phi}$ at the point in question. If $\overline{\phi} > 0$ (in practice, we have to allow some tolerance, say ϵ, so if $\overline{\phi} > \epsilon$), then we need to try again to find a global minimum by starting the search from a different initial point. We could, for example, choose a random initial point on the unit sphere, and continue to do so until a global minimum is found. (Recall that the existence of a global minimum, with $\overline{\phi} = 0$, is guaranteed). Another strategy that might be better is to choose multiple random points on the unit sphere until a starting point is found at which the value of $\overline{\phi}$ is below the value at the latest stationary point that has been found. This would guarantee that the same stationary point will not be found again.

The numerical solution of (28) can be done by Euler's method with projection onto the sphere, that is

$$
q(\tau + \Delta\tau) = \frac{q(\tau) - (\Delta\tau)(\nabla\overline{\phi})(q(\tau))}{\left\| q(\tau) - (\Delta\tau)(\nabla\overline{\phi})(q(\tau)) \right\|}. \qquad (31)
$$

Another possibility is to use a Matlab ODE solver, and yet another possibility is to avoid using the ODE formulation, and to conduct the search by means of the Matlab function fmincon, which does constrained minimization (the constraint here being that we are on the unit sphere), and which can take advantage of having an analytic gradient. All of these methods have in common that a starting point needs to be specified, so the strategies discussed above for choosing a starting point are applicable to all of them.

Once the prices have been found for the market day at time t, we can evaluate the intensities $r_i(t + 1)$ of all of the processes for the next production period from equation (5), and proceed to the next market day. The whole process is deterministic except for the random initial guess concerning the prices on the first market day, and also the random re-start procedure if it turns out that this procedure actually has to be used. The effects of either of these sources of randomness can be assessed by running multiple simulations and comparing the results.

It would be an interesting project to implement the above model with random choices of the matrices A and B. For example, one could choose all of the elements of these matrices as independent random variables, uniformly distibuted on $(0, 1)$. Many other ways of choosing these matrices will no doubt occur to the reader. I suggest choosing $n >> m$, so that there are many more processes than goods. What I expect to happen over time is selection, in which unproductive processes are gradually eliminated in favor of more productive ones. Some goods may also, in effect, be eliminated by becoming free goods. It will be interesting to see whether the economy as a whole settles into a state of balanced growth, as predicted by von Neumann, or whether more complicated behaviors such as cycling or chaos are observed.

References

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