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Infinite depth cochlea with active basilar membrane mechanism and fluid viscosity

$y \neq 0$:

$$\rho \frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} = \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

$$\rho \frac{\partial v}{\partial t} + \frac{\partial p}{\partial y} = \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$y = 0$:

$$u(x,0,t) = 0$$

$$v(x,0,t) = \frac{\partial h}{\partial t}(x,t)$$

$$p(x,0^-,t) - p(x,0^+,t) = s_0 e^{-\lambda x} \left(h + \beta \frac{\partial h}{\partial t} \right)(x,t)$$

active mechanism
($\beta < 0$)

Look for solutions with the following symmetry:

$$p(x,y,t) = -p(x,-y,t)$$
$$u(x,y,t) = -u(x,-y,t)$$
$$v(x,y,t) = +v(x,-y,t)$$

Then we can restrict attention to $y \le 0$, and $y=0$ will mean $y=0^-$

$y<0$:
$$\rho \frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} = \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$
$$\rho \frac{\partial v}{\partial t} + \frac{\partial p}{\partial y} = \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$y=0$:
$$u(x,0,t) = 0$$
$$v(x,0,t) = \frac{\partial h}{\partial t}(x,t)$$
$$2p(x,0,t) = S_0 e^{-\lambda x}\left(h + \beta\frac{\partial h}{\partial t}\right)(x,t)$$

$(u,v,p) \to 0$ as $y \to -\infty$

WKB method*

Introduce a small parameter ε (defined later) and a shift x_ε (also defined later). Let

$$\begin{pmatrix} u \\ v \\ p \end{pmatrix}(x,y,t,\varepsilon) = \begin{pmatrix} U \\ V \\ P \end{pmatrix}(x-x_\varepsilon, y/\varepsilon, \varepsilon)\, e^{i\left(\omega t + \frac{\Phi(x-x_\varepsilon)}{\varepsilon}\right)}$$

$$h(x,t,\varepsilon) = H(x-x_\varepsilon,\varepsilon)\, e^{i\left(\omega t + \frac{\Phi(x-x_\varepsilon)}{\varepsilon}\right)}$$

where ω is the given frequency (radians/second) of a pure tone stimulus. Note that U, V, P, H, and Φ are all complex.

Assume that U, V, P, H have expansions like

$$U(X,Y,\varepsilon) = U_0(X,Y) + \varepsilon U_1(X,Y) + \cdots$$

where $X = x - x_\varepsilon$, $Y = y/\varepsilon$

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* Applied to the cochlea by John Neu and Joe Keller

Neu JC and Keller JB: Asymptotic analysis of a viscous cochlear model. J. Acoust. Soc. Am. 77: 2107-2110, 1985 let $\xi(X) = \Phi'(X) \equiv \frac{\partial \Phi}{\partial X}(X)$

Then

$$\frac{\partial}{\partial t} = i\omega$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial X} + i\,\frac{\xi(X)}{\varepsilon}$$

$$\frac{\partial}{\partial y} = \frac{1}{\varepsilon}\frac{\partial}{\partial Y}$$

$$\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \left(\frac{\partial}{\partial X} + i\,\frac{\xi(X)}{\varepsilon}\right)\left(\frac{\partial}{\partial X} + i\,\frac{\xi(X)}{\varepsilon}\right) + \frac{1}{\varepsilon^2}\frac{\partial^2}{\partial Y^2}$$

$$= \frac{1}{\varepsilon^2}\left(-(\xi(X))^2 + \frac{\partial^2}{\partial Y^2}\right)$$

$$+ \frac{i}{\varepsilon}\left(\xi'(X) + 2\xi(X)\frac{\partial}{\partial X}\right) + \frac{\partial^2}{\partial X^2}$$

In terms of the new variables, the equations of the model cochlea may be rewritten as follows:

$Y<0$:

$$\left(i\omega\rho - \mu\Delta\right)U + \left(i\frac{\xi}{\varepsilon} + \frac{\partial}{\partial X}\right)P = 0$$

$$\left(i\omega\rho - \mu\Delta\right)V + \frac{1}{\varepsilon}\frac{\partial P}{\partial Y} = 0$$

$$\left(i\frac{\xi}{\varepsilon} + \frac{\partial}{\partial X}\right)U + \frac{1}{\varepsilon}\frac{\partial V}{\partial Y} = 0$$

$Y=0$:

$$U(X,0,\varepsilon) = 0$$

$$V(X,0,\varepsilon) = i\omega H(X,0,\varepsilon)$$

$$2P(X,0,\varepsilon) = s_0(1+i\omega\beta)e^{-\lambda X_\varepsilon}e^{-\lambda X}H(X,0,\varepsilon)$$

$(U,V,P) \to 0$ as $Y \to -\infty$

Now choose ε and x_ε as follows

$$\varepsilon^2 = \frac{\mu \lambda^2}{\rho \omega} , \qquad e^{-\lambda x_\varepsilon} = \varepsilon$$

Homework #1: Evaluate ε for a 600 Hz tone ($\omega = 2\pi \cdot 600/s$). Use the following data:

$$\frac{1}{\lambda} = 0.7 \text{ cm} \qquad \rho = 1 \frac{g}{cm^3} \qquad \mu = 0.02 \frac{g}{cm \cdot s}$$

Introduce the expansions

$$U = U_0 + \varepsilon U_1 + \cdots \qquad V = V_0 + \varepsilon V_1 + \cdots$$

$$P = \varepsilon (P_0 + \varepsilon P_1 + \cdots) \qquad H = H_0 + \varepsilon H_1 + \cdots$$

(Note that P is of order ε)

Homework #2: Substitute these expansions into the equations on the previous page, collect terms with matching powers of ε, and so derive the <u>zeroth order</u> and <u>first order</u> equations on the next two pages.

Zeroth order equations:

$Y<0$:

$$\rho\omega\left(i-\frac{1}{\lambda^2}\left(-\xi^2+\frac{\partial^2}{\partial Y^2}\right)\right)U_0 + i\xi P_0 = 0$$

$$\rho\omega\left(i-\frac{1}{\lambda^2}\left(-\xi^2+\frac{\partial^2}{\partial Y^2}\right)\right)V_0 + \frac{\partial P_0}{\partial Y} = 0$$

$$i\xi U_0 + \frac{\partial V_0}{\partial Y} = 0$$

$Y=0$:

$$U_0(X,0) = 0$$

$$V_0(X,0) = i\omega H_0(X)$$

$$2P_0(X,0) = S_0(1+i\omega\beta)e^{-\lambda X} H_0(X)$$

$$(U_0, V_0, P_0) \to 0 \text{ as } Y \to -\infty$$

First order equations:

$Y<0$:

$$\rho\omega\left(i-\frac{1}{\lambda^2}\left(-\xi^2+\frac{\partial^2}{\partial Y^2}\right)\right)U_1 + i\xi P_1$$

$$= \frac{\rho\omega}{\lambda^2}\, i\left(\xi' + 2\xi\frac{\partial}{\partial X}\right)U_0 - \frac{\partial}{\partial X}P_0$$

$$\rho\omega\left(i-\frac{1}{\lambda^2}\left(-\xi^2+\frac{\partial^2}{\partial Y^2}\right)\right)V_1 + \frac{\partial P_1}{\partial Y}$$

$$= \frac{\rho\omega}{\lambda^2}\, i\left(\xi' + 2\xi\frac{\partial}{\partial X}\right)V_0$$

$$i\xi U_1 + \frac{\partial V_1}{\partial Y} = -\frac{\partial}{\partial X}U_0$$

$Y=0$:

$$U_1(X,0) = 0$$

$$V_1(X,0) = i\omega H_1(X)$$

$$2P_1(X,0) = S_0(1+i\omega\beta)e^{-\lambda X}H_1(X)$$

$$(U_1, V_1, P_1) \to 0 \text{ as } Y \to -\infty$$

Homework #3 : Consider the zeroth order equations ignoring (for now) the ones in which H_0 appears.

Note that these form a system of ordinary differential equations in Y at each X, together with certain boundary and decay conditions.

Use these to derive the following formulae for $P_0(X,Y)$, $U_0(X,Y)$, $V_0(X,Y)$ in terms of $P_0(X,0)$:

$$P_0(X,Y) = P_0(X,0)\, e^{\sqrt{\xi^2}\,Y}$$

$$U_0(X,Y) = -P_0(X,0)\,\frac{i\xi}{i\omega\rho}\left(e^{\sqrt{\xi^2}\,Y} - e^{\sqrt{\xi^2+i\lambda^2}\,Y}\right)$$

$$V_0(X,Y) = -P_0(X,0)\,\frac{\xi^2}{i\omega\rho}\left(\frac{e^{\sqrt{\xi^2}\,Y}}{\sqrt{\xi^2}} - \frac{e^{\sqrt{\xi^2+i\lambda^2}\,Y}}{\sqrt{\xi^2+i\lambda^2}}\right)$$

where $\sqrt{}$ denotes the root with positive real part

(Recall that $\xi = \Phi'(X)$ is complex and is a function of X.)

Now consider the two zeroth order equations involving H_0. These give two different formulae for $H_0(X)$ in terms of $P_0(X,0)$:

$$H_0(X) = \frac{V_0(X,0)}{i\omega} = P_0(X,0)\frac{\xi^2}{\omega^2\rho}\left(\frac{1}{\sqrt{\xi^2}} - \frac{1}{\sqrt{\xi^2+i\lambda^2}}\right)$$

$$H_0(X) = P_0(X,0)\frac{2e^{\alpha X}}{S_0(1+i\omega\beta)}$$

Since these two formulae must agree, we require

$$Q(X) = \xi^2\left(\frac{1}{\sqrt{\xi^2}} - \frac{1}{\sqrt{\xi^2+i\lambda^2}}\right)$$

where

$$Q(X) = \frac{2\rho\omega^2 e^{\alpha X}}{S_0(1+i\omega\beta)}$$

This is called the dispersion relation. It implicitly determines the local spatial frequency ξ in terms of X and ω. (There may be more than one solution.) If ξ is a solution, so is $-\xi$. These correspond to waves going in opposite directions.

Homework #4: Let ζ be any solution of the dispersion relation, and let $\eta = \sqrt{\zeta^2}$. Show that η satisfies the following cubic equation:

$$\eta^3 - \frac{1}{2}\left(Q + \frac{i\lambda^2}{Q}\right)\eta^2 + i\lambda^2\eta - \frac{1}{2}\, i\lambda^2 Q = 0$$

But not every solution of this equation yields solutions of the dispersion relation. From the cubic, one can derive

$$Q = \eta \pm \frac{\eta^2}{\sqrt{\eta^2 + i\lambda^2}}$$

but only the minus sign is acceptable. Also η cannot equal $\sqrt{\zeta^2}$ unless $Re(\eta) > 0$.

Thus, we can find all solutions of the dispersion relation by finding the roots of the cubic and then keeping only those that satisfy both of the following tests

$$Q = \eta - \frac{\eta^2}{\sqrt{\eta^2 + i\lambda^2}}$$

$$Re(\eta) > 0$$

Given such a root, one may set $\zeta = \pm \eta$.

Suppose we have found a $\xi(X)$ that solves the dispersion relation. Then we need to find corresponding $U_0(X,Y)$, $V_0(X,Y)$, $P_0(X,Y)$, $H_0(X)$. By the results on pages 9–10, however, this is reduced to finding $P_0(X,0)$. It turns out that we can find an equation for $P_0(X,0)$ by considering the solvability of the first-order equations (see page 8). The function $\xi(X)$ has been chosen to make these equations singular, so the right hand sides are not arbitrary but must satisfy a certain condition.

To find out what this condition is, we multiply the interior first-order equations by U_0^*, V_0^*, and P_0^*, respectively, where these "*" functions of (X,Y) are to be determined below. After multiplying, we integrate over $(-\infty, 0)$ with respect to Y.

(Note: * does <u>not</u> denote complex conjugate; it is just a label to distinguish U_0^*, V_0^*, P_0^* from the related functions U_0, V_0, P_0.)

The results are

$$\rho\omega\int_{-\infty}^{0} U_0^*\left(i-\frac{1}{\lambda^2}\left(-\xi^2+\frac{\partial^2}{\partial Y^2}\right)\right)U_1\,dY + i\xi\int_{-\infty}^{0} U_0^* P_1\,dY$$

$$= \frac{i\omega\rho}{\lambda^2}\int_{-\infty}^{0} U_0^*\left(\xi^2+2\xi\frac{\partial}{\partial X}\right)U_0\,dY - \int_{-\infty}^{0} U_0^*\frac{\partial P_0}{\partial X}\,dY$$

$$\rho\omega\int_{-\infty}^{0} V_0^*\left(i-\frac{1}{\lambda^2}\left(-\xi^2+\frac{\partial^2}{\partial Y^2}\right)\right)V_1\,dY + \int_{-\infty}^{0} V_0^*\frac{\partial P_1}{\partial Y}\,dY$$

$$= \frac{i\omega\rho}{\lambda^2}\int_{-\infty}^{0} V_0^*\left(\xi^2+2\xi\frac{\partial}{\partial X}\right)V_0\,dY$$

$$i\xi\int_{-\infty}^{0} P_0^* U_1\,dY + \int_{-\infty}^{0} P_0^*\frac{\partial V_1}{\partial Y}\,dY = -\int_{-\infty}^{0} P_0^*\frac{\partial V_0}{\partial X}\,dY$$

Now suppose that $U_0^*, V_0^*, P_0^*, H_0^*$ satisfy the same equations as U_0, V_0, P_0, H_0 but with ξ replaced by $-\xi$. (Recall that the dispersion relation depends only on ξ^2, so if $\xi(X)$ is a solution, so is $-\xi(X)$.) That is

$Y<0$:

$$\rho\omega\left(i-\frac{1}{\lambda^2}\left(-\xi^2+\frac{\partial^2}{\partial Y^2}\right)\right)U_0^* - i\xi P_0^* = 0$$

$$\rho\omega\left(i-\frac{1}{\lambda^2}\left(-\xi^2+\frac{\partial^2}{\partial Y^2}\right)\right)V_0^* + \frac{\partial P_0^*}{\partial Y} = 0$$

$$-i\xi U_0^* + \frac{\partial V_0^*}{\partial Y} = 0$$

$Y=0$:

$$U_0^*(X,0) = 0$$

$$V_0^*(X,0) = i\omega H_0^*(X)$$

$$2P_0^*(X,0) = s_0(1+i\omega\beta)e^{-\lambda X} H_0^*(X)$$

$$(U_0^*, V_0^*, P_0^*) \to 0 \text{ as } Y \to -\infty$$

Homework #5

In the equations on page 13, use integration by parts at every opportunity to apply the operator $\partial/\partial Y$ to U_0^*, V_0^*, P_0^* instead of U_1, V_1, P_1. Use the equations on page 14 and all relevant boundary conditions (see also pages 7 and 8) to simplify the results. Finally, combine the simplified equations to obtain

$$0 = \frac{i\omega\rho}{\lambda^2}\int_{-\infty}^{0}\left[U_0^*\left(\xi' + 2\xi\frac{\partial}{\partial X}\right)U_0 + V_0^*\left(\xi' + 2\xi\frac{\partial}{\partial X}\right)V_0\right]dY$$

$$+ \int_{-\infty}^{0}\left(P_0^*\frac{\partial U_0}{\partial X} - U_0^*\frac{\partial P_0}{\partial X}\right)dY$$

Hint: When considering boundary conditions, note that $\partial V/\partial Y$ can be related to $i\xi U$ by making use of the equation that comes from divergence $(u,v)=0$.

Homework #6

The equation derived in #5 must hold for any choice of U_0^*, V_0^*, P_0^* that satisfies the equations on page 14. In particular, we may choose

$$U_0^* = -U_0 \qquad V_0^* = V_0 \qquad P_0^* = P_0$$

(Check that the equations on page 14 are then satisfied.)

Once this choice has been made, show that

$$0 = \frac{\partial}{\partial X} \int_{-\infty}^{0} \left[\frac{i\omega\rho\zeta}{\lambda^2} \left(V_0^2 - U_0^2 \right) + U_0 P_0 \right] dY$$

so that

$$\int_{-\infty}^{0} \left[\frac{i\omega\rho\zeta}{\lambda^2} \left(V_0^2 - U_0^2 \right) + U_0 P_0 \right] dY = C_0$$

where C_0 is a constant, independent of X.

Homework #7

We already have formulae for $U_0(X,Y)$, $V_0(X,Y)$, and $P_0(X,Y)$ in terms of $P_0(X,0)$, see page 9. Making use of these, we can reduce the last result to a formula for $P_0(X,0)$. Show that

$$\left(P_0(X,0)\right)^2 =$$

$$\frac{2\omega\rho\, C_0 \left(\sqrt{\xi^2+i\lambda^2}\right)^3\sqrt{\xi^2}}{\xi\left(\sqrt{\xi^2+i\lambda^2}-\sqrt{\xi^2}\right)\left(\sqrt{\xi^2+i\lambda}^2\sqrt{\xi^2}-i\lambda^2\right)}$$

Hint: When doing #7, it helps to note that

$$\left(\sqrt{\xi^2+i\lambda^2}\right)^2-\left(\sqrt{\xi^2}\right)^2 = i\lambda^2.$$

This completes the construction of the zeroth order solution.

Suggested computing project

1) For each of several frequencies ω within the range of hearing, plot the solution constructed above. Refer everything back to the original physical variables. Plot the cochlea map: the position of peak basilar membrane displacement as a function of $\log \omega$.

2) Do the foregoing first for $\beta = 0$ and then for selected negative values of β. Note the effect of negative β in sharpening the response of the cochlea.

3) Optional: Calculate the frequency response (amplitude and phase) of a fixed point x on the basilar membrane as the frequency ω is varied. Do this for several selected values of x. Observe the effect of negative β.

4) Optional: Take the (inverse) Fourier transform of the results of (3) and so find the impulse response of selected points on the basilar membrane.

(see remarks, next page)

Remarks on suggested computing project

i) Use data from HW#1. The value of s_0 is not given there, but a change in s_0 is equivalent to a shift of origin, so the value of s_0 is not too important. The constant β is not known. Start with $\beta = 0$ and experiment with various negative β.

ii) When you solve the dispersion relation, there may be more than one acceptable value of η at some X. Choose the one with smallest real part (longest wavelength.) For each η, there are two possible choices of $\xi = \pm\eta$. To get a wave traveling into the cochlea choose $\xi = -\eta$, so that $Re(\xi) < 0$. (You may also want to look at the other case to see how rapidly waves going the wrong way are damped.)

iii) To do parts (3) and (4) of the project, it is necessary to choose a normalization at each frequency. That is, the constant C_0 must be chosen for each ω. I believe this can be done by setting

$$\lim_{X \to -\infty} P_0(X,0) = P_{source}$$

where P_{source} is a given constant independent of ω.

CSPeskin 2/11/2018

Analysis of the role of the active mechanism

In the foregoing, we have determined the solution to the cochlea problem to lowest order in the parameter ε, but the solution has to be evaluated numerically because of the need to solve the equation of the dispersion relation for $\xi(X)$.

Here, however, we show how it is possible to analyze the factor

(1) $$\left| e^{i\left(\omega t + \frac{1}{\varepsilon}\Phi(X)\right)} \right| = e^{-\frac{1}{\varepsilon}\operatorname{Im}\left(\Phi(X)\right)}$$

which affects all of the variables and which turns out to have interesting behavior as a consequence of the active mechanism.

Our starting point is to expand Φ in a Taylor series about some point X_p through terms of second order. Recall that $\xi(X) = \Phi'(X)$. Therefore,

(2) $$\left| e^{i\left(\omega t + \frac{1}{\varepsilon}\Phi(X)\right)} \right|$$

$$= e^{-\frac{1}{\varepsilon} Im\left(\Phi(X_p) + \xi(X_p)(X - X_p) + \frac{1}{2}\xi'(X_p)(X-X_p)^2\right)}$$

If $\xi(X_p)$ is real and if $Im(\xi'(X_p)) > 0$, this has the form of a Gaussian

(3) $$C_p\, e^{-\frac{1}{2\varepsilon} Im(\xi'(X_p))(X - X_p)^2}$$

where C_p is the constant $e^{-\frac{1}{\varepsilon} Im(\Phi(X_p))}$.

The equation that determines $\xi(X)$ is the dispersion relation, which we write here as follows

(4) $$\frac{2\rho\omega^2}{S_0\lambda(1+i\omega\beta)} e^{\lambda X} = f\left(\frac{\xi^2}{\lambda^2}\right)$$

where

(5) $$f(z) = z\left(\frac{1}{\sqrt{z}} - \frac{1}{\sqrt{z+i}}\right)$$

Thus

(6) $$z = \frac{\xi^2(X)}{\lambda^2}$$

By definition, $\mathrm{Re}(\sqrt{}) \geq 0$. This makes $\sqrt{}$ well defined unless it is applied to a negative real number. We have used this property of $\sqrt{}$ to ensure that our solution decays with distance away from the basilar membrane.

Now if we differentiate on both sides of (4) with respect to X and then make use of (4) itself to simplify the result, we get

(7) $$\lambda f\left(\frac{\xi^2(X)}{\lambda^2}\right) = f'\left(\frac{\xi^2(X)}{\lambda^2}\right)\frac{2}{\lambda^2}\xi(X)\xi'(X)$$

or

(8) $$\frac{\lambda^3}{3} = (\log f)'(z)\ \xi(X)\xi'(X)$$

Here we are using z as shorthand for $\xi^2(X)/\lambda^2$.

Multiplying both sides of (8) by the complex conjugate of $(\log f)'(z)$, we find

(9) $$\frac{\lambda^3}{3}\overline{(\log f)'(z)} = \left|(\log f)'(z)\right|^2 \xi(X)\xi'(X)$$

Now consider a point X_p at which $\xi(X_p)$ is real, so that (3) is applicable.

At such a point, equation (9) determines the sign of the imaginary part of ξ' as follows:

(10) $$-\operatorname{sign}\left(\operatorname{Im}\left((\log f)'(z_p)\right)\right)$$

$$= \operatorname{sign}\left(\xi(X_p)\right)\ \operatorname{sign}\left(\operatorname{Im}\left(\xi'(X_p)\right)\right)$$

Therefore, at a point X_p where $\xi(X_p)$ is real, the sign of the imaginary part of $\xi'(X_p)$ will be different for each of the two choices of $\xi(X_p)$. There are two such choices since the dispersion relation only involves ξ^2, and for that same reason both choices have the same value of z_p, see equation (6).

The sign of $\xi(X_p)$ determines the direction of wave propagation, with $\xi(X_p) < 0$ corresponding to a wave running into the cochlea, since $\omega > 0$ throughout our analysis. (Recall that our definition of ε involved $\sqrt{\omega}$.)

Thus, the expression (3) is a genuine Gaussian, and not an inverted Gaussian, for one and only one of the two possible directions of wave propagation. In order to find out which one, we need to evaluate the left-hand side of (10).

First, however, we look for solutions of (4) in which ξ is real, in which case $z = \xi^2/\lambda^2$ is both real and positive. Thus, we write

(11) $$z = r > 0$$

and note that

(12) $$f(r) = r\left(\frac{1}{\sqrt{r}} - \frac{1}{\sqrt{r+i}}\right)$$

$$= \sqrt{r}\left(1 - \frac{1}{\sqrt{1+ir^{-1}}}\right)$$

It is straightforward to show (or check) that

$$(13)\qquad \frac{1}{\sqrt{1+ir^{-1}}} =$$

$$\sqrt{\frac{1+\sqrt{1+r^{-2}}}{2(1+r^{-2})}} - i\sqrt{\frac{-1+\sqrt{1+r^{-2}}}{2(1+r^{-2})}}$$

Therefore,

$$(14)\qquad Re\big(f(r)\big) = \sqrt{r}\left(1 - \sqrt{\frac{1+\sqrt{1+r^{-2}}}{2(1+r^{-2})}}\right)$$

$$(15)\qquad Im\big(f(r)\big) = \sqrt{r}\left(\sqrt{\frac{-1+\sqrt{1+r^{-2}}}{2(1+r^{-2})}}\right)$$

In equation (14)

(16) $$\frac{1+\sqrt{1+r^{-2}}}{2(1+r^{-2})}$$

$$= \frac{1}{2(1+r^{-2})} + \frac{1}{2\sqrt{1+r^{-2}}}$$

$$< \frac{1}{2} + \frac{1}{2} = 1$$

Therefore

(17) $$Re(f(r)) > 0$$

and it is obvious from (15) that

(18) $$Im(f(r)) > 0$$

Since $e^{\alpha X} > 0$ for all X, the left-hand side of equation (4) lies on a ray in the complex plane emanating from the origin and passing through the point

(19) $$1 - i\omega\beta$$

In fact, by adjusting X, we can make the left-hand side of (4) be any point of this ray. Thus, we require that the right-hand side also be a point of this ray. From (18), we see that this is impossible if $\beta \geq 0$. (Recall that $\omega > 0$.)

When $\beta = 0$, our model has fluid viscosity as an energy-consuming mechanism, with no energy source to balance it, other than the incident sound wave. When $\beta > 0$, the situation is even worse, since there are two dissipative mechanisms, fluid viscosity and basilar membrane friction. Thus, it is not surprising that with $\beta \geq 0$ we cannot achieve a real spatial frequency ξ at a real temporal frequency ω, even locally, since that would correspond to a wave that is locally undamped.

With $\beta < 0$, however, the basilar membrane plays the role of an energy source in our model, and it is then possible to achieve a local balance between this input of energy and the dissipation of energy by fluid viscosity. The cells that perform this function are the outer hair cells, which simultaneously sense basilar membrane motion and apply forces that amplify the sensed motion.

Let

(20) $$\theta = -\beta\omega > 0$$

We seek $r > 0$ such that $f(r)$ lies on the ray emanating from the origin an passing through the point $1 + i\theta$. From equations (14–15), such an r satisfies

(21) $$\theta = \frac{\mathrm{Im}(f(r))}{\mathrm{Re}(f(r))} = \frac{\sqrt{(-1+A)/2}}{A - \sqrt{(1+A)/2}}$$

where

(22) $$A = \sqrt{1 + r^{-2}} \in (1, \infty)$$

Also let

(23) $$B = \frac{A-1}{2} \in (0, \infty)$$

Then (21) becomes

(24) $$\theta = \frac{\sqrt{B}}{2B+1-\sqrt{B+1}}$$

$$= \frac{\sqrt{B}\,(2B+1+\sqrt{B+1})}{4B^2+4B+1-(B+1)}$$

$$= \frac{2B+1+\sqrt{B+1}}{\sqrt{B}\,(4B+3)}$$

$$= \frac{g(B)}{\sqrt{B}}$$

where

(25) $$g(B) = \frac{2B+1+\sqrt{B+1}}{4B+3}$$

Note that

$$g(0) = \frac{2}{3}, \quad g(\infty) = \frac{1}{2}$$

This suggests that $g(B)$ is decreasing and indeed

(26) $$(4B+3)^2 g'(B) =$$

$$(4B+3)\left(2 + \tfrac{1}{2}(B+1)^{-1/2}\right) - \left(2B+1+(B+1)^{1/2}\right)4$$

$$= 8B+6 + \left(2B+\tfrac{3}{2}\right)(B+1)^{-1/2} - 8B - 4 - 4(B+1)^{1/2}$$

$$= 2 + \left(2B+\tfrac{3}{2}\right)(B+1)^{-1/2} - 4(B+1)^{1/2}$$

$$= (B+1)^{-1/2}\left(2(B+1)^{1/2} + \left(2B+\tfrac{3}{2}\right) - 4(B+1)\right)$$

$$< (B+1)^{-1/2}\left(2(B+1) + \left(2B+\tfrac{3}{2}\right) - 4(B+1)\right)$$

$$= (B+1)^{-1/2}\left(-\tfrac{1}{2}\right) < 0$$

It follows from the above properties of g that (24) has a unique solution B for each positive θ, since the right-hand side is a continuous and strictly decreasing function with limits of $+\infty$ and 0 as $B \to 0$ and ∞, respectively.

Also, it is easy to see from (22) & (23) that B uniquely determines r, since $r > 0$. To be specific, (22) & (23) $\Rightarrow$

(27) $$(2B+1)^2 = 1 + r^{-2}$$

(28) $$4(B^2+B) = r^{-2}$$

(29) $$r = \frac{1}{2\sqrt{B}\,\sqrt{B+1}}$$

From (14-15), (22-23), and (29), we can evaluate $|f(r)|^2$ as follows:

(30) $$|f(r)|^2 = r\left(\left(1 - \frac{1}{A}\sqrt{B+1}\right)^2 + \left(\frac{1}{A}\sqrt{B}\right)^2\right)$$

$$= r\left(1 - \frac{2}{A}\sqrt{B+1} + \frac{1}{A^2}(2B+1)\right)$$

$$= r\left(1 - \frac{2}{A}\sqrt{B+1} + \frac{1}{A}\right)$$

$$= \frac{r}{A}\left(A+1-2\sqrt{B+1}\right) = \frac{r}{A}\left(2B+2-2\sqrt{B+1}\right)$$

$$= \frac{2r\sqrt{B+1}}{A}\left(\sqrt{B+1}-1\right)$$

$$= \frac{1}{A}\,\frac{\left(\sqrt{B+1}-1\right)}{\sqrt{B}}$$

$$= \frac{\sqrt{B+1}-1}{(2B+1)\sqrt{B}} = \frac{\sqrt{B}}{(2B+1)\left(\sqrt{B+1}+1\right)}$$

Therefore, if we take the square of the magnitude of both sides of equation (4) at the point $X = X_p$ where $\xi^2/\lambda^2 = r$, we get

(31) $$\left(\frac{2\rho}{S_0\lambda\beta^2}\right)^2 \frac{\theta^4}{1+\theta^2} e^{2\lambda X_p} = \frac{\sqrt{B}}{(2B+1)(\sqrt{B+1}+1)}$$

From the first line of (24)

(32) $$\theta^2 = \frac{B}{4B^2+4B+1-\sqrt{B+1}(2)(2B+1)+(B+1)}$$

$$= \frac{B}{4B^2+5B+2-2(2B+1)\sqrt{B+1}}$$

(33) $$1+\theta^2 = \frac{4B^2+6B+2-2(2B+1)\sqrt{B+1}}{4B^2+5B+2-2(2B+1)\sqrt{B+1}}$$

(34) $$\frac{1+\theta^2}{\theta^2} = \frac{2(2B+1)((B+1)-\sqrt{B+1})}{B}$$

$$= \frac{2(2B+1)\sqrt{B+1}(\sqrt{B+1}-1)}{B}$$

Thus, equation (31) becomes

(35) $$\left(\frac{2\rho}{s_0 \lambda \beta^2}\right)^2 \theta^2 e^{2\lambda X_p} = \frac{2\sqrt{B+1}\left(\sqrt{B+1}-1\right)}{\sqrt{B}\left(\sqrt{B+1}+1\right)}$$

Taking the square root of both sides, and writing out $\theta = (-\beta)\omega$, we get

(36) $$\frac{2\rho}{s_0 \lambda (-\beta)} \omega e^{\lambda X_p} = \sqrt{\frac{2\sqrt{B+1}\left(\sqrt{B+1}-1\right)}{\sqrt{B}\left(\sqrt{B+1}+1\right)}}$$

Now recall the change of variables made earlier, when setting up the WKB analysis, that

(37) $$X = x - x_\varepsilon$$

where

(38) $$e^{-\lambda x_\varepsilon} = \varepsilon = \left(\frac{\mu}{\rho\omega}\right)^{1/2} \lambda$$

Accordingly, we set $X_p = x_p - x_\varepsilon$, and then (36) becomes

(39) $$\left(\frac{\omega}{\omega_0}\right)^{1/2} e^{\lambda x_p} = \sqrt{\frac{\sqrt{B+1}\left(\sqrt{B+1}-1\right)}{\sqrt{B}\left(\sqrt{B+1}+1\right)}}$$

where

(40) $$\omega_0 = \frac{S_0^2 \beta^2}{2\rho\mu}$$

When B is large, the right-hand side of (39) is approximately equal to 1, and in this limit we have an explicit formula for x_p in terms of ω, namely

(41) $$x_p = -\frac{1}{2\lambda} \log\left(\frac{\omega}{\omega_0}\right)$$

Even without making this approximation, we have the explicit formulae needed to get a plot of X_p vs ω by using B as a parameter. For each $B \in (0,\infty)$ we can use (24) to evaluate ω and then (39) to evaluate X_p:

(42) $$\omega = \frac{1}{(-\beta)} \frac{\sqrt{B}}{2B+1-\sqrt{B+1}}$$

(43) $$X_p = -\frac{1}{2\lambda} \log\left(\left(\frac{\omega}{\omega_0}\right)\left(\frac{\sqrt{B}\left(\sqrt{B+1}+1\right)}{\sqrt{B+1}\left(\sqrt{B+1}-1\right)}\right)\right)$$

Each such pair (ω, X_p) gives one point of the plot.

Now we return to the question raised earlier about the sign of the imaginary part of $\xi'(X_p)$. We need to evaluate $(\log f)'(z_p)$, where

(44) $$\log f(z) = \log(z) + \log\left(\frac{1}{\sqrt{z}} - \frac{1}{\sqrt{z+i}}\right)$$

We have

(45) $$(\log f)'(z)$$

$$= \frac{1}{z} - \frac{1}{2}\left(\frac{\left(\frac{1}{\sqrt{z}}\right)^3 - \left(\frac{1}{\sqrt{z+i}}\right)^3}{\left(\frac{1}{\sqrt{z}}\right) - \left(\frac{1}{\sqrt{z+i}}\right)}\right)$$

$$= \frac{1}{z} - \frac{1}{2}\left(\left(\frac{1}{\sqrt{z}}\right)^2 + \frac{1}{\sqrt{z}}\frac{1}{\sqrt{z+i}} + \left(\frac{1}{\sqrt{z+i}}\right)^2\right)$$

$$= \frac{1}{2}\left(\frac{1}{z} - \frac{1}{\sqrt{z}}\frac{1}{\sqrt{z+i}} - \frac{1}{z+i}\right)$$

When $z = r > 0$, this becomes

(46) $$(\log f)'(r) =$$

$$\frac{1}{2r}\left(1 - \frac{1}{\sqrt{1+ir^{-1}}} - \frac{1}{1+ir^{-1}}\right)$$

From Equation (13),

(47) $$\operatorname{Im}\left(\frac{1}{\sqrt{1+ir^{-1}}}\right) = -\sqrt{\frac{-1+\sqrt{1+r^{-2}}}{2(1+r^{-2})}}$$

and we also have

(48) $$\operatorname{Im}\frac{1}{1+ir^{-1}} = -\frac{r^{-1}}{1+r^{-2}}$$

Therefore,

(49) $$\operatorname{Im}\left((\log f)'(r)\right) = \frac{1}{2r}\left(\sqrt{\frac{-1+\sqrt{1+r^{-2}}}{2(1+r^{-2})}} + \frac{r^{-1}}{1+r^{-2}}\right)$$

Since the right-hand side of (49) is positive for all $r > 0$, we may conclude from (10) that

$$\xi(X_p)\, \mathrm{Im}(\xi'(X_p)) < 0 \tag{50}$$

and this implies that (3) is a genuine Gaussian if and only if $\xi(X_p) < 0$, i.e., if and only if the direction of wave propagation is into the cochlea. (This is the direction of increasing x, which is the direction of decreasing stiffness of the basilar membrane.)

It is also of interest to evaluate $\mathrm{Im}(\xi'(X_p))$, and not merely its sign, since this quantity determines how narrow the Gaussian (3) is. This can be done from equation (8) or (9). Either way, we see that it is not enough to know the imaginary part of $(\log f)'$. The real part is also involved.

Thus, we return to equation (46) and rewrite it by making use of equation (12), which we use in the form

(51) $$1 - \frac{1}{\sqrt{1+ir^{-1}}} = \frac{f(r)}{\sqrt{r}}$$

(52) $$\frac{1}{\sqrt{1+ir^{-1}}} = 1 - \frac{f(r)}{\sqrt{r}}$$

(53) $$\frac{1}{1+ir^{-1}} = \left(1 - \frac{f(r)}{\sqrt{r}}\right)$$

Substituting (51) & (53) into (46) gives

(54) $$(\log f)'(r) = \frac{1}{2r}\left(\frac{f(r)}{\sqrt{r}} - \left(1 - \frac{f(r)}{\sqrt{r}}\right)^2\right)$$

From (14-15) & (22-23), we have

(55) $$\frac{f(r)}{\sqrt{r}} = 1 - \frac{\sqrt{B+1}}{2B+1} + i\,\frac{\sqrt{B}}{2B+1}$$

(56) $$1 - \frac{f(r)}{\sqrt{r}} = \frac{\sqrt{B+1}}{2B+1} - i\,\frac{\sqrt{B}}{2B+1}$$

(57) $$\left(1 - \frac{f(r)}{\sqrt{r}}\right)^2 = \frac{1 - 2i\sqrt{B}\sqrt{B+1}}{(2B+1)^2}$$

Also from (29)

(58) $$\frac{1}{2r} = \sqrt{B}\sqrt{B+1}$$

Therefore

(59) $$(\log f)'(r) =$$

$$\sqrt{B}\sqrt{B+1}\left(1 - \frac{\sqrt{B+1}}{2B+1} - \frac{1}{(2B+1)^2}\right.$$

$$\left. + i\,\frac{\sqrt{B}}{2B+1}\left(1 + \frac{2\sqrt{B+1}}{2B+1}\right)\right)$$

We consider this further only for large B.
In that case

$$(\log f)'(r) \sim B\left(1 + i\frac{1}{2\sqrt{B}}\right) \tag{60}$$

$$\overline{(\log f)'(r)} \sim B\left(1 - i\frac{1}{2\sqrt{B}}\right) \tag{61}$$

$$\left|(\log f)'(r)\right|^2 \sim B^2 \tag{62}$$

Also

$$\zeta(X_p) = -\lambda\sqrt{r} \sim \frac{-\lambda}{\sqrt{2B}} \tag{63}$$

see (29). Substituting these results into (9) and taking minus the imaginary parts of both sides gives

$$\frac{\lambda^3}{3}\,\frac{B}{2\sqrt{B}} = B^2\,\frac{\lambda}{\sqrt{2B}}\,\operatorname{Im}\left(\zeta'(X_p)\right) \tag{64}$$

or

(65) $$Im(\xi'(X_p)) \sim \frac{\lambda^2}{3\sqrt{2}\,B}$$

From (24) with B large we also have

(66) $$\theta = (-\beta)\omega \sim \frac{1}{2\sqrt{B}}$$

so

(67) $$(\omega\beta)^2 \sim \frac{1}{4B}$$

Thus, for small $\omega\beta$, we have

(68) $$Im(\xi'(X_p)) \sim \frac{4\lambda^2}{3\sqrt{2}}(\omega\beta)^2$$

and the Gaussian amplitude (3) becomes

(69) $$C_p\, e^{-\frac{1}{2}\frac{(\omega\beta)^2}{\varepsilon}\left(\frac{4}{3\sqrt{2}}\right)\lambda^2(X-X_p)^2}$$

Although it is derived only for $(\omega\beta)^2$ small, equation (69) illuminates the role of the active mechanism in narrowing the spatial distribution of the response of the cochlea to a pure tone, since the width of the Gaussian in (69) is inversely proportional to $(-\beta)$.

We remark that all of the results derived in this <u>analysis of the role of the active mechanism</u> are implicit in the more general WKB analysis on which this section is based. In that more general analysis, however, we have to proceed numerically beyond a certain point because we do not have a nice formula for the solution $\xi(X)$ of the dispersion relation. Our aim here has been to bring out the behavior of an important part of that numerical solution for the interesting case in which $\beta < 0$.