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4/30/2018

Poisson Process

An inhomogeneous Poisson process is defined on a domain Ω (for example, $\Omega \subset \mathbb{R}^d$) and is governed by a density* $u(\underline{x})$, defined for $\underline{x} \in \Omega$, such that

(1) $$u(\underline{x}) \geq 0, \quad \text{all } \underline{x} \in \Omega$$

and

(2) $$\int_{\Omega} u(\underline{x})\, d\underline{x} = \mu < \infty$$

The process generates a finite set S of points of Ω called <u>events</u>.

This is done as follows. First, choose an integer $N \geq 0$ according to the Poisson distribution with parameter μ:

*Although $u(\underline{x})$ is a density, it is not a probability density. Its integral is equal to the expected number of events of the process.

(3) $$Pr(N=n) = \frac{\mu^n}{n!} e^{-\mu}$$

$n = 0, 1, 2, \dots$

Next, the set S is constructed in a manner that is conditioned on N:

If $N=0$, $S=\phi$. In this case, there are no events.

If $N=n>0$, the set S contains n events that are chosen randomly and independently, each with probability density function

(4) $$\frac{1}{\mu} u(\underline{x})$$

This means that if $\underline{X}$ is any one of the chosen events and if Ω' is any subset of Ω, then

(5) $$Pr(\underline{X} \in \Omega') = \frac{\mu'}{\mu}$$

Where

(6) $$\mu' = \int_{\Omega'} u(\underline{x}) d\underline{x}$$

Now let Ω_1 and Ω_2 be subsets of Ω such that

(7) $$\Omega_1 \cup \Omega_2 = \Omega$$

(8) $$\Omega_1 \cap \Omega_2 = \phi$$

For $i=1,2$, let

(9) $$\mu_i = \int_{\Omega_i} u(\underline{x})\, d\underline{x}$$

(10) $$S_i = S \cap \Omega_i$$

(11) $$N_i = \#(S_i)$$

where $\#$ denotes the number of elements of a set.

Then

(12) $$\mu_1 + \mu_2 = \mu$$

(13) $$S_1 \cup S_2 = S$$

(14) $$N_1 + N_2 = N$$

We claim that the process described above, with outcome (S_1, S_2) is equivalent to a pair of Poisson processes, each with the same definition as the process that generates S, but running <u>independently</u> on the domains Ω_1 and Ω_2, respectively, instead of Ω.

To show this, we first consider the random variables N_1 and N_2.
We have

(15) $$Pr\left(N_1 = n_1 \;\&\; N_2 = n_2\right)$$

$$= Pr\left(N_1 = n_1 \;\&\; N_2 = n_2 \,\middle|\, N = n_1 + n_2\right) Pr\left(N = n_1 + n_2\right)$$

$$= \frac{(n_1+n_2)!}{(n_1)!(n_2)!} \left(\frac{\mu_1}{\mu}\right)^{n_1} \left(\frac{\mu_2}{\mu}\right)^{n_2} \frac{\mu^{n_1+n_2}}{(n_1+n_2)!} e^{-\mu}$$

$$= \left(\frac{\mu_1^{\,n_1}}{(n_1)!} e^{-\mu_1}\right)\left(\frac{\mu_2^{\,n_2}}{(n_2)!} e^{-\mu_2}\right)$$

since $\mu = \mu_1 + \mu_2$. From this we

find the marginal distributions of N_1 and N_2 by summing over n_2 and n_1, respectively, to obtain

(16) $$Pr\left(N_1 = n_1\right) = \frac{\mu_1^{\,n_1}}{(n_1)!} e^{-\mu_1}$$

(17) $$Pr\left(N_2 = n_2\right) = \frac{\mu_2^{\,n_2}}{(n_2)!} e^{-\mu_2}$$

Thus N_1 and N_2 are each Poisson-distributed with parameters μ_1 and μ_2, respectively. Recall that μ_1 and μ_2 are defined by integration of $u(\underline{x})$ over Ω_1 and Ω_2, just as μ is defined by integration of $u(\underline{x})$ over Ω.

From (15-17) we see, moreover, that

(18) $$Pr(N_1 = n_1 \ \& \ N_2 = n_2) = Pr(N_1 = n_1)\, Pr(N_2 = n_2)$$

and this shows that N_1 and N_2 are independent.

Now we condition on $N_1 = n_1$ & $N_2 = n_2$, and we set $n = n_1 + n_2$. We want to show that the sets S_1 and S_2 each have essentially the same properties as the parent set S but with reference to the domains Ω_1 and Ω_2, respectively, and also that the sets S_1 and S_2 are independent of each other.

Let $\underline{X}^{(1)} \dots \underline{X}^{(n)}$ be the elements of S, in arbitrary order and let $i(k)$ be a function from

(19) $$\{1 \dots n\} \longrightarrow \{1, 2\}$$

This function is arbitrary except that

(20) $$\#\{k : i(k) = 1\} = n_1$$

(21) $$\#\{k : i(k) = 2\} = n_2$$

For each $k \in \{1 \dots n\}$, choose a set $\Omega^{(k)}$ which is arbitrary except for the restriction that

(22) $$\Omega^{(k)} \subseteq \Omega_{i(k)}$$

Let

(23) $$\mu^{(k)} = \int_{\Omega^{(k)}} u(\underline{x})\, d\underline{x}$$

We would like to evaluate

(24) $$\Pr\left(\underline{X}^{(k)} \in \Omega^{(k)}, k=1\cdots n \;\middle|\; \underline{X}^{(k)} \in \Omega_{i(k)}, k=1\cdots n\right)$$

To do so, we note that

(25) $$\Pr\left(\underline{X}^{(k)} \in \Omega^{(k)}, k=1\cdots n \;\middle|\; \underline{X}^{(k)} \in \Omega_{i(k)}, k=1\cdots n\right)$$

$$\cdot \Pr\left(\underline{X}^{(k)} \in \Omega_{i(k)}, k=1\cdots n\right)$$

$$= \Pr\left(\left(\underline{X}^{(k)} \in \Omega^{(k)} \;\&\; \underline{X}^{(k)} \in \Omega_{i(k)}\right), k=1\cdots n\right)$$

$$= \Pr\left(\underline{X}^{(k)} \in \Omega^{(k)}, k=1\cdots n\right)$$

$$= \prod_{k=1}^{n} \frac{\mu^{(k)}}{\mu}$$

In the next-to-last step of (25), we used the fact that $\Omega^{(k)} \subset \Omega_{i(k)}$, by construction, and in the last step we used the independence and given identical distribution of $\underline{X}_1 \cdots \underline{X}_N$, conditioned on $N=n$, see (4-6).

But again using (4-6), we also have

(26) $$Pr\left(\underline{X}_k \in \Omega_{i(k)},\ k=1\cdots n\right)$$

$$= \prod_{k=1}^{n} \frac{\mu_{i(k)}}{\mu} = \frac{\mu_1^{\,n_1}\mu_2^{\,n_2}}{\mu^n}$$

Substituting this into (25) and solving for the conditional probability (24), we find

(27) $$Pr\left(\underline{X}^{(k)} \in \Omega^{(k)},\ k=1\cdots n \,\middle|\, \underline{X}^{(k)} \in \Omega_{i(k)},\ k=1\cdots n\right)$$

$$= \left(\prod_{k:\,i(k)=1} \frac{\mu^{(k)}}{\mu_1}\right)\left(\prod_{k:\,i(k)=2} \frac{\mu^{(k)}}{\mu_2}\right)$$

Thus, if we condition on $N_1 = n_1$ and $N_2 = n_2$, then the sets S_1 and S_2 are independent and, moreover, the elements of S_i are independent and identically distributed on Ω_i with probability density function

(28) $$\frac{1}{\mu_i} u(\underline{x})$$

for each element of S_i.

In summary, we have shown that a Poisson process P, defined on a domain $\Omega = \Omega_1 \cup \Omega_2$ with $\Omega_1 \cap \Omega_2 = \phi$ and governed by a density function $u(\underline{x})$, is equivalent to a pair of <u>independent</u> Poisson processes P_1 on Ω_1 and P_2 on Ω_2, with P_1 governed by the restriction of $u(\underline{x})$ to Ω_1, and P_2 governed by the restriction of $u(\underline{x})$ to Ω_2.

The above result can be applied recursively to partition the domain Ω into a large number of subdomains Ω_k, each of which is so small that we don't care about spatial localization within Ω_k. These subdomains may be called pixels (or voxels in 3D). Each pixel is characterized by a non-negative real number μ_k, and the output of the process is a vector of non-negative integers

(29) $$N_1 \quad \cdots \quad N_{k_{max}}$$

The N_k are independent but not identically distributed. Their distributions are

(30) $$Pr\left(N_k = n\right) = \frac{\mu_k^n}{n!} e^{-\mu_k}$$

This is the appropriate Poisson process for modeling a digital camera, some of which actually operate as photon counters under low-light conditions.

We can also go in the other direction and define a Poisson process on large domain in terms of independent Poisson processes on its subdomains. This could be useful for parallel processing, or if the large domain has an inconvenient shape. This idea can also be used to extend our definition of a Poisson process to a case in which

(31) $$\int_{\Omega} u(\underline{x})\, d\underline{x} = \infty$$

For example, suppose $u(\underline{x}) = u_0$, independent of $\underline{x}$, and $\Omega = \mathbb{R}^2$. This is a homogeneous Poisson process on the plane. Our original definition is inapplicable to such a case, but we can tile the plane by a countable number of tiles, e.g., unit squares, each of finite area, and we can then generate a Poisson process on each tile according to our definition, and finally we can pool the resulting sets of events.

One unsatisfactory feature of our definition of a Poisson process is that the Poisson distribution appears in the definition without any derivation or motivation. Could some other distribution have been used instead?

It turns out that the Poisson distribution is the only one that yields independence under binomial splitting, and hence the independence properties of the Poisson process as derived above. To show this, consider an integer-valued random variable N such that

(32) $$\Pr(N=n) = f(n)$$

where $n = 0, 1, 2, \ldots$, $f(n) \geq 0$, and

(33) $$\sum_{n=0}^{\infty} f(n) = 1$$

Let N_1 and N_2 be random variables related to N by the following. First choose N_1 according to

(34) $$Pr(N_1 = n_1 \mid N = n) = \binom{n}{n_1} p^{n_1} (1-p)^{n-n_1}$$

for $n_1 = 0 \cdots n$, and $Pr(N_1 = n_1 \mid N = n) = 0$, otherwise. Here p is some given parameter in $(0, 1)$. Then, with N_1 chosen, set

(35) $$N_2 = N - N_1$$

The above binomial splitting first chooses a number N according to the probability distribution f. It then takes N objects and assigns each of them independently to one of two bins with probability p of choosing the first bin and probability $1-p$ of choosing the second one. When all N objects have been assigned, N_1 is the number in the first bin and N_2 is the number in the second one.

We seek f such that N_1 and N_2 are independent, and we claim that the only such f is a Poisson distribution. The proof is due to R. Varadhan:

First, we note that

(36)
$$\Pr(N_1 = n_1 \ \& \ N_2 = n_2)$$

$$= \Pr(N = n_1 + n_2)\Pr(N_1 = n \mid N = n_1 + n_2)$$

$$= f(n_1 + n_2)\frac{(n_1+n_2)!}{(n_1)!(n_2)!}\, p^{n_1}(1-p)^{n_2}$$

On the other hand, if N_1 and N_2 are independent, then $\Pr(N_1 = n_1 \ \& \ N_2 = n_2)$ must be a product of a function of n_1 and a function of n_2. The factors

(37)
$$\frac{p^{n_1}}{(n_1)!} \quad \frac{(1-p)^{n_2}}{(n_2)!}$$

are already of this form, so we require

(38) $$f(n_1+n_2)(n_1+n_2)! = g(n_1)h(n_2)$$

Now set $n_2 = 1$ and $n_2 = 0$:

(39) $$f(n_1+1)(n_1+1)! = g(n_1)h(1)$$

(40) $$f(n_1)\ (n_1)! = g(n_1)h(0)$$

Dividing (39) by (40) gives

(41) $$\frac{f(n_1+1)(n_1+1)}{f(n_1)} = \frac{h(1)}{h(0)}$$

This recursion relation uniquely determines $f(n)$ in terms of $f(0)$ and the parameter $h(1)/h(0)$, which we denote by μ. The result is

(42) $$f(n) = f(0)\ \frac{\mu^n}{n!}$$

Finally, $f(0)$ is determined by making use of the normalization condition (33) with the result that $f(0) = e^{-\mu}$, and

(43) $$f(n) = \frac{\mu^n}{n!} e^{-\mu}$$

which is a Poisson distribution, as claimed.

Campbell's Theorems

Let $\underline{X}_1 \cdots \underline{X}_N$ be the events of a Poisson process on Ω with density $u(\underline{x})$, and consider the function

(44) $$f(\underline{x}) = \sum_{k=1}^{N} h(\underline{x} - \underline{X}_k)$$

in which h is some given function.

We seek to evaluate $E[f(\underline{x})]$ and $E[f^2(\underline{x})]$, where E denotes the expected value. Our two-step definition of the Poisson process is well suited to these tasks. We take the expectation first with N given, and then we take the expectation over N.

Applying this strategy to $f(\underline{x})$, we get

(45) $$E[f(\underline{x})|N] = \frac{N}{\mu}\int_{\Omega} h(\underline{x}-\underline{X})\,u(\underline{X})\,d\underline{X}$$

since the random variables $\underline{X}_1 \cdots \underline{X}_N$ given N all have the same probability density function, which is $\frac{1}{\mu}u(\underline{x})$. Note that $\underline{X}$ on the right-hand side of (45) is not a random variable, but is simply the variable of integration. Next, we take the expectation of both sides of (45) over the only remaining random variable, N. Since

(46) $$E[N] = \mu$$

we get

(47) $$E[f(\underline{x})] = \int_{\Omega} h(\underline{x}-\underline{X})\,u(\underline{X})\,d\underline{X}$$

as might have been expected.

Evaluation of $E[f^2(\underline{x})]$ is a little more complicated. First

(48) $$f^2(\underline{x}) = \sum_{j,k=1}^{N} h(\underline{x}-\underline{X}_j)\,h(\underline{x}-\underline{X}_k)$$

$$= \sum_{j=1}^{N} h^2(\underline{x}-\underline{X}_j) + \sum_{\substack{j,k=1\\ j\neq k}}^{N} h(\underline{x}-\underline{X}_j)\,h(\underline{x}-\underline{X}_k)$$

Therefore, since $\underline{X}_j$ and $\underline{X}_k$ are independent for $j\neq k$, given N.

(49) $$E\left[f^2(\underline{x})\,|\,N\right] = \frac{N}{\mu}\int_{\Omega} h^2(\underline{x}-\underline{X})\,u(\underline{X})\,d\underline{X}$$

$$+\frac{N^2-N}{\mu^2}\iint_{\Omega} h(\underline{x}-\underline{X}')\,h(\underline{x}-\underline{X}'')\,u(\underline{X}')\,u(\underline{X}'')\,d\underline{X}'\,d\underline{X}''$$

Finally, using the property of the Poisson distribution that

(50) $$E\left[N^2-N\right] = \left(E[N]\right)^2 = \mu^2$$

we get the unconditioned result

(51) $$E[f^2(\underline{x})] = \int_{\Omega} h^2(\underline{x}-\underline{X})\,u(\underline{X})\,d\underline{X}$$

$$+ \iint_{\Omega} h(\underline{x}-\underline{X}')\,h(\underline{x}-\underline{X}'')\,u(\underline{X}')\,u(\underline{X}'')\,d\underline{X}'\,d\underline{X}''$$

The double integral in this expression is the square of the right-hand side of (47), so (51) can also be written as

(52) $$E[f^2(\underline{x})] - \left(E[f(\underline{x})]\right)^2 = \int_{\Omega} h^2(\underline{x}-\underline{X})\,u(\underline{X})\,d\underline{X}$$

Although the derivation given above assumes that

(53) $$\int_{\Omega} u(\underline{x})\,d\underline{x} < \infty$$

but the results make sense under more

general conditions. Suppose, for example, that $\Omega = \mathbb{R}^2$, that u is bounded, and that h and h^2 are integrable. Then, as discussed above, we can tile the plane with finite-area tiles and apply our derivation to the Poisson process on each of them. Then, finally, we can sum over the tiles to obtain the desired result.

```
function [N,X,Y] = ipp (mu, ubound)
% inhomogeneous Poisson process
% on Omega, with density u(x,y)
% mu = integral_Omega u(x,y) dx dy
% u(x,y) <= ubound for all x,y in Omega
% N = number of events
% X(i), Y(i) = coordinates of i-th event
%   for i = 1 ... N

% choose N from Poisson distribution
% with mean mu

N=0;
sum = -log(rand);
while (sum < mu)
    N = N+1;
    sum = sum - log(rand);
end
X = zeros(1,N); Y = zeros(1,N);
% choose N events with pdf u/mu by rejection
for i = 1:N
    [XX,YY] = rptOmega; ur = rand * ubound;
    while (ur > u(XX,YY))
        [XX,YY] = rptOmega; ur = rand * ubound;
    end
    X(i) = XX; Y(i) = YY;
end
```