WAVE MOMENTUM

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Introduction

It is commonly the case that traveling waves have associated momentum, that the momentum points in the direction of wave propagation, and that the momentum is related to the energy of the wave by a simple equation of the form

$$\text{momentum} = \frac{\text{energy}}{\text{phase velocity}}. \tag{1}$$

Throughout this paper, the term "momentum" means actual physical momentum, the kind that can push on an obstacle in its path, and not some more abstract concept like generalized momentum. Momentum is a vector, and we are interested in the component of momentum parallel to the direction of propagation of the wave.

The phenomenon of wave momentum is remarkable in several respects. First, it is not clear a priori that waves ought to have associated momentum. Waves are commonly divided into two types: transverse and longitudinal waves. In the transverse case, since nothing is moving in the direction of propagation, how can there be associated momentum in that direction? Even in the longitudinal case, since the wave motion is typically oscillatory, one would think that the *average* momentum density would be zero. How can there be net momentum in the direction of the wave?

It is also remarkable that wave momentum is related to wave energy in such a simple and seemingly universal way. As we shall see, there are many different kinds of wave motion to which some variant of Eq. 1 applies. These include electromagnetic waves, sound waves, water waves, and certain kinds of traveling waves on strings under tension. One might be tempted to conclude that there is some universal derivation of Eq. 1 that includes all of these examples as special cases. We show herein, however, that there are other examples of traveling waves in which there is (of course) associated energy, but no associated momentum at all. These examples make it seem highly unlikely that any general derivation of Eq. 1 can be devised.

Momentum is a concept normally associated with particles, and the fact that waves have momentum is perhaps the first hint, even at the classical level, of wave-particle duality. In agreement with this point of view, we shall see that Eq. 1 is also applicable to the waves that are associated with particles in quantum mechanics, not only in the non-relativistic but also in the relativistic case.

1 Electromagnetic Waves

We write Maxwell's equations in the form

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \tag{2}$$

$$\nabla \times \mathbf{B} = \frac{1}{c}\left(\frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}\right), \tag{3}$$

$$\nabla \cdot \mathbf{E} = \rho, \tag{4}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{5}$$

where $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, $\mathbf{J}$ is the current density, and ρ is the charge density. The constant c is the speed of light.

In the above system of units, the Lorentz force per unit volume takes the form

$$\mathbf{F} = \rho\mathbf{E} + \frac{1}{c}\left(\mathbf{J} \times \mathbf{B}\right). \tag{6}$$

It can be shown (see Appendix) that the momentum density and energy density of the electromagnetic field are given by

$$\boldsymbol{\mathcal{P}} = \frac{1}{c}\left(\mathbf{E} \times \mathbf{B}\right), \tag{7}$$

$$\mathcal{E} = \frac{1}{2}\left(\mathbf{E} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{B}\right). \tag{8}$$

Now consider a plane-wave solution of the free space ($\rho = 0, \mathbf{J} = 0$) Maxwell equations. Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis of $\mathbb{R}^3$. We consider in particular a wave propagating in the positive $\mathbf{e}_3$ direction with speed c. The most general such solution is of the form

$$\mathbf{E} = +f(x_3 - ct)\mathbf{e}_1 + g(x_3 - ct)\mathbf{e}_2, \tag{9}$$

$$\mathbf{B} = -g(x_3 - ct)\mathbf{e}_1 + f(x_3 - ct)\mathbf{e}_2, \tag{10}$$

where f and g are arbitrary functions of the variable $x_3 - ct$. Substituting these formulae for $\mathbf{E}$ and $\mathbf{B}$ into the general expressions for $\boldsymbol{\mathcal{P}}$ and $\mathcal{E}$ (Eqs. 7-8), we find

$$\boldsymbol{\mathcal{P}} = \frac{1}{c}\left((f(x_3 - ct))^2 + (g(x_3 - ct))^2\right)\mathbf{e}_3, \tag{11}$$

$$\mathcal{E} = (f(x_3 - ct))^2 + (g(x_3 - ct))^2. \tag{12}$$

Thus, the momentum density points in the direction of wave propagation, which is $\mathbf{e}_3$ in this case, and the scalar momentum density $\mathcal{P}$ such that $\boldsymbol{\mathcal{P}} = \mathcal{P}\mathbf{e}_3$ satisfies Eq. 1.

For comparison with other types of wave propagation that will be considered later, note that Eq. 1 holds here not just on the average but at every $(\mathbf{x}, t)$ separately. It is thus a *local* property of the traveling electromagnetic wave. Note, too, that the waveform considered above is arbitrary; there is no restriction to sinusoidal waves. In fact, Eqs. 9-10 are general enough to describe electromagnetic waves with any type of polarization. It is essential, though, that we consider waves propagating in one direction only. The foregoing relationship between momentum density and energy density is clearly invalid for superpositions of waves running in different directions. The simplest example that illustrates this is the case of a standing wave, which is a superposition of two waves of equal amplitude running in opposite directions. By symmetry, it is obvious that such a wave has zero momentum density, at least on the average, but its energy density is clearly nonzero.

2 Sound Waves

The equations of isentropic gas dynamics in one spatial dimension are as follows:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial x}p(\rho) = 0, \tag{13}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0, \tag{14}$$

Here $u(x, t)$ is the velocity of the gas, and $\rho(x, t)$ is its density (mass per unit volume). The function $p(\rho)$ gives the pressure in terms of the density.

The assumption that the gas is isentropic means that no conduction of heat is allowed. This implies that the work done on any material element of the gas must be accounted for by the change in macroscopic kinetic energy of that material element, plus the change in its internal energy. Consider for the moment compression of the gas which is done slowly under conditions of mechanical equilibrium, so that no appreciable macroscopic kinetic energy is generated. Let such a compression be applied to a unit mass of the gas, which is thermally insulated from its environment. We assume that this compression starts from a standard density ρ_0 and a standard temperature T_0. These are the density and temperature of the undisturbed gas through which a sound wave will be propagating. Because of the isentropic assumption, subsequent changes in temperature are determined by the changes in density. Thus, we do not need to regard the temperature as an independent variable. Accordingly, let $e_{\mathrm{I}}(\rho)$ be the internal energy per unit mass when the density of the gas is ρ. Then conservation of energy implies that

$$de_{\mathrm{I}} = -p(\rho)d\frac{1}{\rho} = \frac{p(\rho)}{\rho^2}d\rho, \tag{15}$$

since $1/\rho$ is the volume of a unit mass of the gas, and since $-pdv$ is, as always, the work required to effect the volume change dv. It follows from the above result that the internal energy per unit volume $\mathcal{E}_{\mathrm{I}}$ is given by

$$\mathcal{E}_{\mathrm{I}}(\rho) = \rho e_{\mathrm{I}}(\rho) = \rho \int_{\rho_0}^{\rho} \frac{p(\sigma)}{(\sigma)^2} d\sigma. \tag{16}$$

As stated above, ρ_0 is a reference density that later will be taken to be the density of the undisturbed gas through which a sound wave is propagating.

For future reference, we evaluate the first and second derivatives of the function $\mathcal{E}_{\mathrm{I}}(\rho)$. These derivatives are conveniently expressed as follows:

$$\mathcal{E}_{\mathrm{I}}'(\rho) = \frac{\mathcal{E}_{\mathrm{I}}(\rho) + p(\rho)}{\rho}, \tag{17}$$

$$\mathcal{E}_{\mathrm{I}}''(\rho) = \frac{p'(\rho)}{\rho}. \tag{18}$$

Of course, the kinetic energy density of the gas is given by

$$\mathcal{E}_{\mathrm{K}} = \frac{1}{2}\rho u^2, \tag{19}$$

and the total energy density by

$$\mathcal{E} = \mathcal{E}_{\mathrm{K}} + \mathcal{E}_{\mathrm{I}}. \tag{20}$$

Another quantity of importance is the momentum density, which is given by

$$\mathcal{P} = \rho u. \tag{21}$$

We leave it as an exercise for the reader to derive the following conservation laws from Eqs. 13-14:

$$\frac{\partial \mathcal{P}}{\partial t} + \frac{\partial}{\partial x}\left(u\mathcal{P} + p(\rho)\right) = 0, \tag{22}$$

$$\frac{\partial \mathcal{E}}{\partial t} + \frac{\partial}{\partial x}\left(u\mathcal{E} + up(\rho)\right) = 0. \tag{23}$$

It follows from these results that $\overline{\mathcal{P}}$ and $\overline{\mathcal{E}}$ are constants of the motion, where

$$\overline{\mathcal{P}} = \int_{-\infty}^{\infty} \mathcal{P}(x,t)dx, \tag{24}$$

$$\overline{\mathcal{E}} = \int_{-\infty}^{\infty} \mathcal{E}(x,t)dx. \tag{25}$$

We shall refer to these as the "total momentum" and "total energy", although strictly speaking they have units of momentum/area and energy/area, respectively. That is because we have integrated only over x, and not over the other two spatial variables.

Two other constants of the motion will also be significant in the following. They are $\overline{u}$ and $\overline{\rho - \rho_0}$ where the overbar has the same significance as above, i.e., it denotes the integral with respect to x over $(-\infty, \infty)$.

That $\overline{\rho - \rho_0}$ is a constant of the motion is nothing more than conservation of mass, and it follows directly from Eq. 14. As mentioned above, we are now assuming that ρ_0 is the density of the undisturbed gas through which a sound wave is propagating. It follows that

$$\overline{\rho - \rho_0} = \int_{-\infty}^{\infty} (\rho(x,t) - \rho_0)dx = 0. \tag{26}$$

That $\overline{u}$ is a constant of the motion is a bit more mysterious. This particular invariant, which does not seem to have a name, was first brought to this author's attention by Thomas Bringley. Its conservation law is obtained by dividing Eq. 13 by ρ and then writing the result in the form

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}u^2 + \int_{\rho_0}^{\rho} \frac{p'(\sigma)}{\sigma} d\sigma\right) = 0. \tag{27}$$

We assume in the following that

$$\overline{u} = \int_{-\infty}^{\infty} u(x,t)dx = 0. \tag{28}$$

This amounts to a choice of reference frame. It is the only frame in which the total momentum and energy, as defined above, are finite.

We now make use of the equations $\overline{\rho - \rho_0} = 0$ and $\overline{u} = 0$ to simplify the expressions for $\overline{\mathcal{P}}$ and $\overline{\mathcal{E}}$:

$$\overline{\mathcal{P}} = \overline{\rho u} = \overline{(\rho - \rho_0)u} + \rho_0 \overline{u} = \overline{(\rho - \rho_0)u}, \tag{29}$$

$$\begin{aligned}
\overline{\mathcal{E}} &= \overline{\frac{1}{2}\rho u^2 + \mathcal{E}_{\rm I}(\rho)} \\
&= \frac{1}{2}\rho_0 \overline{u^2} + \frac{1}{2}\overline{(\rho - \rho_0)u^2} + \mathcal{E}_{\rm I}'(\rho_0)\overline{(\rho - \rho_0)} + \frac{1}{2}\mathcal{E}_{\rm I}''(\rho_0)\overline{(\rho - \rho_0)^2} + \ldots \\
&= \frac{1}{2}\rho_0 \overline{u^2} + \frac{1}{2}\overline{(\rho - \rho_0)u^2} + \frac{1}{2}\frac{p'(\rho_0)}{\rho_0}\overline{(\rho - \rho_0)^2} + \ldots .
\end{aligned} \tag{30}$$

In Eq. 30, we have introduced an expansion in $\rho - \rho_0$ of the internal energy. In this expansion, The zero-order term is not present because the internal energy of the gas is measured with respect to its value at $\rho = \rho_0$. The linear term vanishes because $\overline{(\rho - \rho_0)} = 0$, and the quadratic term has been simplified with the help of Eq. 18. The kinetic energy has also been split into two terms, one involving ρ_0 and the other involving $\rho - \rho_0$.

In the acoustic limit, i.e., the limit of small amplitude disturbances about a constant state $u = 0$, $\rho = \rho_0$, the equations of motion reduce to

$$\rho_0 \frac{\partial u}{\partial t} + c_0^2 \frac{\partial \tilde{\rho}}{\partial x} = 0, \tag{31}$$

$$\frac{\partial \tilde{\rho}}{\partial t} + \rho_0 \frac{\partial u}{\partial x} = 0, \tag{32}$$

where

$$\tilde{\rho} = \rho - \rho_0, \tag{33}$$

$$c_0 = \sqrt{p'(\rho_0)} > 0. \tag{34}$$

These equations have traveling wave solutions of the form

$$u(x,t) = c_0 f(x - c_0 t), \tag{35}$$

$$\tilde{\rho}(x,t) = \rho_0 f(x - c_0 t), \tag{36}$$

where f satisfies

$$\overline{f} = 0, \tag{37}$$

so that Eqs. 26 and 28 are satisfied. The function f is otherwise arbitrary. The solution we are considering here is a traveling acoustic wave that propagates with speed c_0 in the direction of increasing x. By substituting $-c_0$ in place of c_0 in Eqs. 35-36, we get a similar wave traveling in the direction of decreasing x. Note that this substitution changes the sign of u but not that of $\tilde{\rho}$.

We can now evaluate the total momentum and the total energy of the traveling acoustic wave constructed above. To do so, we substitute Eqs. 35-36 into Eqs. 29-30, and make use of the definition of c_0, Eq. 34. Since we are interested in the acoustic limit, we keep only the lowest order terms, i.e., those which are second order in the variables u and $\tilde{\rho}$. In particular, this means that we drop the terms summarized by "..." in the internal energy, and also the term involving $\tilde{\rho}u^2$ in the kinetic energy. The results are

$$\overline{\mathcal{P}} = c_0\rho_0\overline{f^2}, \tag{38}$$

$$\begin{aligned} \overline{\mathcal{E}} &= \frac{1}{2}\rho_0 c_0^2\overline{f^2} + \frac{1}{2}\frac{c_0^2}{\rho_0}\rho_0^2\overline{f^2} \\ &= c_0^2\rho_0\overline{f^2}. \end{aligned} \tag{39}$$

Thus,

$$\overline{\mathcal{P}} = \overline{\mathcal{E}}/c_0. \tag{40}$$

Since c_0 is the wave velocity, this is similar to the relationship mentioned in the introduction and found above for electromagnetic waves, with the interesting difference that here we are considering the total momentum and the total energy, so we have only a global relationship and not also a local one.

Note that the total momentum points in the direction of wave propagation. (It is easy to check that the sign of $\overline{\mathcal{P}}$ is reversed but the sign of $\overline{\mathcal{E}}$ remains unchanged if we consider waves propagating in the negative x direction.)

It is instructive to consider *why* there is net momentum in a traveling sound wave. As mentioned in the introduction, this is counterintuitive, since the average velocity of the air would seem to be zero, and indeed we have taken care to elevate this to a basic principle by choosing a frame of reference in which the spatial integral of $u(x,t)$ is zero for every t. It turns out, however, that there is net momentum because of the *correlation* between density and velocity in a traveling acoustic wave. In fact, those regions of space in which the air is moving forward (i.e., in the direction of wave propagation) are also the regions in which the density of the air is above average, and vice versa. It is only this effect that produces net momentum. The momentum is second order in the amplitude of the wave, but so, too, is the energy, and they turn out to be proportional, with the constant of proportionality being the wave velocity.

3 Water Waves

Although water waves and sound waves are physically quite different, we shall see that their energy-momentum relationships are strikingly similar. There is, however, this important difference: that water waves are dispersive, i.e., their speeds are wavelength dependent. Thus, in order to get a simple relationship between energy and momentum, we need to consider waves of a single wavelength, i.e., sinusoidal waves. Since these have infinite extent, it will not be possible to integrate over all x to find the total momentum and total energy, as we did in the case of (localized) sound waves. Instead, we shall integrate over one cycle of the sinusoidal water wave.

We consider an inviscid and incompressible fluid of infinite depth that moves under the influence of gravity. Let y be the vertical coordinate, so that x and z are horizontal coordinates, and let t be the time. Let u, v, w be the x, y, z components of fluid velocity, respectively, and let p be the fluid pressure. Let h be the height of the free surface, measured from its undisturbed height. We consider the special case in which $w = 0$, and u, v, p, h are independent of z. This is the case of two-dimensional motion, in the (x, y) plane. The equations of motion are as follows. For $y < h(x, t)$, we have

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) + \frac{\partial p}{\partial x} = 0, \tag{41}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) + \frac{\partial p}{\partial y} = -\rho g, \tag{42}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{43}$$

where ρ is the constant fluid density and g is the constant acceleration of a falling object under the influence of gravity.

There are two boundary conditions that need to be imposed on the free surface $y = h(x, t)$. One is simply that the pressure on the free surface is zero:

$$p(x, h(x, t), t) = 0. \tag{44}$$

The other is the kinematic condition that a particle on the free surface remains on the free surface. Let the trajectory of such a particle be denoted $X(t), Y(t)$. Then

$$Y(t) = h(X(t), t). \tag{45}$$

Differentiating the above equation with respect to time gives

$$\frac{dY}{dt} = \frac{\partial h}{\partial t}(X(t), t) + \frac{\partial h}{\partial x}(X(t), t)\frac{dX}{dt}. \tag{46}$$

By definition of the fluid velocity u, v, we have

$$\frac{dX}{dt} = u(X(t), Y(t), t), \tag{47}$$

$$\frac{dY}{dt} = v(X(t), Y(t), t). \tag{48}$$

Thus, with the help of Eq. 45, Eq. 46 may be rewritten

$$v(X(t), h(X(t), t), t) = \frac{\partial h}{\partial t}(X(t), t) + \frac{\partial h}{\partial x}(X(t), t)u(X(t), h(X(t), t), t). \tag{49}$$

Since this holds for *any* fluid particle that happens to be on the free surface, and since there is always *some* fluid particle that happens to be at position $(x, h(x, t))$ at time t, we can replace $X(t)$ by x in the above result to obtain, finally, the kinematic boundary condition

$$v(x, h(x, t), t) = \frac{\partial h}{\partial t}(x, t) + \frac{\partial h}{\partial x}(x, t)u(x, h(x, t), t). \tag{50}$$

The boundary conditions at infinite depth are that u, v, and $p+\rho g y$ all approach zero as $y \to -\infty$. Note in particular that this singles out a frame of reference, since the addition of a nonzero constant to u would spoil the condition that $u \to 0$ as $y \to -\infty$.

We now define the momentum density (in the x direction) and also the energy density of the fluid. The x-momentum density is given by

$$\mathcal{P}(x, t) = \rho \int_{-\infty}^{h(x,t)} u(x, y, t)dy. \tag{51}$$

The energy density is the sum of the kinetic energy density $\mathcal{E}_{\mathrm{K}}$ and the gravitational potential energy density, which we denote $\mathcal{E}_{\mathrm{G}}$. The kinetic energy density is given by

$$\mathcal{E}_{\mathrm{K}}(x, t) = \frac{1}{2}\rho \int_{-\infty}^{h(x,t)} \left((u(x, y, t))^2 + (v(x, y, t))^2\right) dy. \tag{52}$$

Because of the infinite depth of the fluid, we need to be careful in making a definition of the gravitational potential energy that gives a finite result. For this purpose, we introduce a depth $d_0 > 0$ which is sufficiently large that

$$h(x,t) > -d_0 \tag{53}$$

for all x, t. Now we ignore the potential energy of that part of the fluid for which $y < -d_0$, since this potential energy, although infinite, is constant. Thus, we make the definition

$$\mathcal{E}_{\mathrm{G}}(x,t) = \rho g \int_{-d_0}^{h(x,t)} y dy - \rho g \int_{-d_0}^{0} y dy. \tag{54}$$

In this equation, the second term on the right-hand side is present because we want to measure the potential energy relative to that of the undisturbed fluid, for which $h(x,t) = 0$. Of course, the integrals on the right-hand side are easily evaluated, and the result is

$$\mathcal{E}_{\mathrm{G}} = \frac{1}{2}\rho g (h(x,t))^2. \tag{55}$$

Note that the artificial depth d_0 no longer appears.

Combining the gravitational energy density and the kinetic energy density, we get the overall energy density

$$\mathcal{E} = \frac{1}{2}\rho g (h(x,t))^2 + \frac{1}{2}\rho \int_{-\infty}^{h(x,t)} \left((u(x,y,t))^2 + (v(x,y,t))^2 \right) dy. \tag{56}$$

Note that the momentum density and the energy density defined above have units of momentum/area and energy/area, respectively. This is because we have integrated over the depth. These units reflect the fundamental nature of water waves as surface phenomena.

We leave it as an exercise for the reader to show, starting from Eqs. 41-44 together with Eq. 50, that

$$\frac{\partial \mathcal{P}}{\partial t} + \frac{\partial}{\partial x} \int_{-\infty}^{h(x,t)} \left(p + \rho u^2 \right) dy = 0, \tag{57}$$

$$\frac{\partial \mathcal{E}}{\partial t} + \frac{\partial}{\partial x} \int_{-\infty}^{h(x,t)} u \left(p + \rho g y + \frac{1}{2}\rho \left(u^2 + v^2 \right) \right) dy = 0. \tag{58}$$

These are the equations of conservation of momentum and energy for water waves.

In the following we consider waves that are periodic in x with period L, and we let an overbar denote the integral over any one period, e.g.,

$$\overline{\mathcal{P}} = \int_0^L \mathcal{P} dx, \tag{59}$$

$$\overline{\mathcal{E}} = \int_0^L \mathcal{E} dx. \tag{60}$$

It follows from Eqs. 57-58 that $\overline{\mathcal{P}}$ and $\overline{\mathcal{E}}$ are independent of time. We call these constants of the motion the "total momentum" and the "total energy". More precisely, they are the momentum per unit length carried by one period of the wave, and the energy per unit length carried by one period of the wave, respectively. (The "length" in "per unit length" here runs in the horizontal direction perpendicular to the plane of the motion, i.e., parallel to the crests of the waves.)

Let us now consider small amplitude water waves. The linearized equations of motion are obtained by treating u, v, h, and $\tilde{p} = p + \rho g y$ as quantities of first order, and retaining only first-order terms in the equations of motion, which then become

$$\rho \frac{\partial u}{\partial t} + \frac{\partial \tilde{p}}{\partial x} = 0, \tag{61}$$

$$\rho \frac{\partial v}{\partial t} + \frac{\partial \tilde{p}}{\partial y} = 0, \tag{62}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{63}$$

$$\tilde{p}(x, 0, t) = \rho g h(x, t), \tag{64}$$

$$\frac{\partial h}{\partial t}(x, t) = v(x, 0, t). \tag{65}$$

Equations 61-63 are valid on the domain $y < 0$. An important consequence of linearization is that our problem is now defined on this fixed domain (instead of $y < h(x, t)$). The linearized boundary conditions on the upper surface $y = 0$ are Eqs. 64-65, and the boundary conditions at extreme depth are that u, v, and $\tilde{p}$ all approach zero as $y \to -\infty$.

We seek a solution to the linearized equations in the form of a sinusoidal wave that decays exponentially with depth:

$$\begin{aligned} u(x,y,t) &= U\sin\left(\omega t - \frac{2\pi x}{L}\right)\exp\left(\frac{2\pi y}{L}\right), & (66)\\ v(x,y,t) &= V\cos\left(\omega t - \frac{2\pi x}{L}\right)\exp\left(\frac{2\pi y}{L}\right), & (67)\\ \tilde{p}(x,y,t) &= P\sin\left(\omega t - \frac{2\pi x}{L}\right)\exp\left(\frac{2\pi y}{L}\right), & (68)\\ h(x,t) &= H\sin\left(\omega t - \frac{2\pi x}{L}\right). & (69) \end{aligned}$$

It is easy to check that the linearized equations of motion are satisfied provided that we set

$$\begin{aligned} P &= \rho g H, & (70)\\ U &= \omega H, & (71)\\ V &= \omega H, & (72)\\ \omega &= \pm\sqrt{\frac{2\pi}{L}g}. & (73) \end{aligned}$$

The construction given by Eqs. 66-73 is a sinusoidal wave with amplitude H and wavelength L which travels in the direction of increasing x if $\omega > 0$, and in the direction of decreasing x if $\omega < 0$. The phase velocity of this traveling wave is easily seen to be

$$v_{\mathrm{p}} = \pm\sqrt{\frac{Lg}{2\pi}}. \qquad (74)$$

For future reference, we note that

$$\omega v_{\mathrm{p}} = g. \qquad (75)$$

We now seek to evaluate the momentum and energy associated with the traveling wave constructed above. Consistent with our small-amplitude approximation, we evaluate these quantities only to lowest order in the wave amplitude. First, we consider the momentum density, which we expand as a Taylor series in $h(x,t)$:

$$\begin{aligned} \mathcal{P} &= \rho\int_{-\infty}^{h(x,t)} u(x,y,t)dy \\ &= \rho\int_{-\infty}^{0} u(x,y,t)dy + \rho h(x,t)u(x,0,t) + \ldots, & (76) \end{aligned}$$

where $\ldots$ denotes terms of third order and higher, since these terms are at least second order in $h(x,t)$ and first order in u. Next we integrate the above result over one period in x. Clearly, the integral of the first-order term is zero, and we are left with the following result, to lowest (i.e., second) order:

$$\begin{aligned}
\overline{\mathcal{P}} &= \rho\overline{h(x,t)u(x,0,t)} \\
&= \rho U H \overline{\sin^2\left(\omega t - \frac{2\pi x}{L}\right)} \\
&= \frac{1}{2}\rho U H L \\
&= \frac{1}{2}\rho\omega H^2 L.
\end{aligned} \tag{77}$$

To lowest (second) order, the energy density is given by

$$\begin{aligned}
\mathcal{E} &= \frac{1}{2}\rho g(h(x,t))^2 + \frac{1}{2}\rho\int_{-\infty}^{0}\left((u(x,y,t))^2 + (v(x,y,t)^2\right) dy \\
&= \frac{1}{2}\rho g H^2 \sin^2\left(\omega t - \frac{2\pi x}{L}\right) + \frac{1}{2}\omega^2 H^2 \frac{L}{4\pi}.
\end{aligned} \tag{78}$$

Integrating this result over one period in x, we get

$$\begin{aligned}
\overline{\mathcal{E}} &= \frac{1}{2}\rho g H^2 \frac{L}{2} + \frac{1}{2}\omega^2 H^2 \frac{L}{4\pi} L \\
&= \frac{1}{4}\rho g H^2 L + \frac{1}{4}\rho g H^2 L \\
&= \frac{1}{2}\rho g H^2 L.
\end{aligned} \tag{79}$$

Now recalling Eq. 75, and comparing Eqs. 78 and 79, we see that

$$\overline{\mathcal{E}} = v_{\mathrm{p}}\overline{\mathcal{P}} \tag{80}$$

Thus, the same basic relationship between energy and momentum that we have seen so far in every case holds for water waves as well. Here, just as in the case of sound waves, the relationship is only a global one; it does not hold at each separate position and time (as it did in the case of traveling electromagnetic waves). The similarity to the case of sound waves becomes even more striking when we consider why it is that traveling water waves have momentum at all. The motion of fluid particles in water waves is circular, and one might think that the net momentum in the direction of propagation would be zero. This reasoning is incorrect, however, because of the correlation between the height of the water and the direction of horizontal motion. As any swimmer knows, the water is moving forward (i.e., in the direction of the wave) at the crest of the wave, and backward in the trough. This asymmetry is the fundamental source of net momentum in the direction of wave propagation, as the foregoing mathematical argument shows.

Despite the above similarity between water waves and sound waves, there is this important difference: sound waves (in the acoustic, or small-amplitude limit) all travel at the same velocity, regardless of their wavelength. Water waves (in deep water), by contrast, are dispersive, i.e., their velocity is wave-length dependent. This gives us the opportunity to address the following basic question: Which is the relevant velocity in the relationship between energy and momentum, the phase velocity or the group velocity? (Recall that the phase velocity is ω/k, and the group velocity is $d\omega/dk$, where ω is the temporal frequency in radians per unit time, and k is the spatial frequency in radians per unit length. In our notation $k = 2\pi/L$, where L is the wavelength.) Considering the standard interpretation of the group velocity as the velocity at which a localized disturbance carrying energy and momentum propagates, one would certainly be tempted to guess that in any relationship like Eq. 80, it would be the group velocity that would appear. Note that this is *not* the case. It seems from the water wave example that the *phase* velocity is the relevant one. We shall return to this issue when considering the momenta of particles and the velocity of their associated waves in quantum mechanics.

4 Nonlinear Vibrating String

In this section we consider traveling waves on an elastic string under tension. We do not linearize the problem, but instead consider the full, nonlinear, equations of motion. Let the motion of the string be described by a function

$$\mathbf{x} = \mathbf{X}(s, t), \tag{81}$$

where s is a material coordinate (not necessarily arclength), and t is the time, so that $\mathbf{X}(s, t)$ is the position in $\mathbb{R}^3$ of the material point whose label is s at time t. We assume that the material properties of the string are time-independent and homogeneous, and that the material coordinates have been chosen in a way that reflects this homogeneity, so that equal intervals of s correspond to equal amounts of material with identical properties. It follows that the mass density ρ with respect to the material coordinate s is constant, and also that the elastic energy density $\mathcal{E}_{\mathrm{E}}$ with respect to the material coordinate s is a function only of $|\partial \mathbf{X}/\partial s|$, and does not depend explicitly on s or t.

Let $'$ denote the derivative of a function of one variable with respect to its argument. Then $\mathcal{E}_{\mathrm{E}}'(|\partial \mathbf{X}/\partial s|)$ is the tension in the string. The direction associated with this tension is the unit tangent to the string, which is given by $(\partial \mathbf{X}/\partial s)/|\partial \mathbf{X}/\partial s|$. It points, of course in the direction of increasing s.

With these considerations in mind, it is straightforward to write down the equation of motion of the string, which is:

$$\rho \frac{\partial^2 \mathbf{X}}{\partial t^2} = \frac{\partial}{\partial s} \left(\mathcal{E}_{\mathrm{E}}' \left(\left| \frac{\partial \mathbf{X}}{\partial s} \right| \right) \frac{\partial \mathbf{X}/\partial s}{|\partial \mathbf{X}/\partial s|} \right). \tag{82}$$

Equation 82 is, in fact, the equation of momentum conservation, with momentum density (with respect to the material coordinate s) given by

$$\boldsymbol{\mathcal{P}} = \rho \frac{\partial \mathbf{X}}{\partial t}. \tag{83}$$

The energy density (again, with respect to the material coordinate s) is

$$\mathcal{E} = \frac{1}{2} \rho \left| \frac{\partial \mathbf{X}}{\partial t} \right|^2 + \mathcal{E}_{\mathrm{E}} \left(\left| \frac{\partial \mathbf{X}}{\partial s} \right| \right), \tag{84}$$

and we leave it as an exercise for the reader to show that Eq. 82 implies the equation of conservation of energy, which takes the form

$$\frac{\partial \mathcal{E}}{\partial t} = \frac{\partial}{\partial s} \left(\frac{\partial \mathbf{X}}{\partial t} \cdot \mathcal{E}_{\mathrm{E}}' \left(\left| \frac{\partial \mathbf{X}}{\partial s} \right| \right) \frac{\partial \mathbf{X}/\partial s}{|\partial \mathbf{X}/\partial s|} \right). \tag{85}$$

Equations 82 and 85 imply that $\overline{\boldsymbol{\mathcal{P}}}$ and $\overline{\mathcal{E}}$ are constants of the motion, where in this section the overbar denotes the integral from $s = -\infty$ to $s = \infty$. In saying this, we are assuming that the behavior of $\mathbf{X}(s,t)$ as $s \to \pm\infty$ is such that these constants of the motion have finite values. This will be the case for the localized solutions that we construct below.

Equation 82 is a nonlinear system in the three components of $\mathbf{X}(s,t)$. It is a remarkable fact that this nonlinear system has traveling wave solutions. To construct such traveling waves, we look for solutions of Eq. 82 with the property that

$$\left| \frac{\partial \mathbf{X}}{\partial s} \right| = r, \tag{86}$$

where r is a positive constant. It is not obvious a priori that there are such solutions, but if there are, they satisfy

$$\rho \frac{\partial^2 \mathbf{X}}{\partial t^2} = \frac{\mathcal{E}'_{\mathrm{E}}(r)}{r} \frac{\partial^2 \mathbf{X}}{\partial s^2}. \tag{87}$$

This is nothing but the wave equation

$$\frac{1}{c^2} \frac{\partial^2 \mathbf{X}}{\partial t^2} = \frac{1}{r^2} \frac{\partial^2 \mathbf{X}}{\partial s^2} \tag{88}$$

with wave velocity

$$c = \sqrt{\frac{\mathcal{E}'_{\mathrm{E}}(r)}{\rho/r}}. \tag{89}$$

Note that rs measures arclength along the string (see Eq. 86) and that c is the wave speed expressed as arclength per unit time. Since $\mathcal{E}'_{\mathrm{E}}(r)$ is the tension in the string, and ρ/r is the mass per unit arclength, Eq. 89 is the standard formula for the speed of a small-amplitude transverse wave on a vibrating string. It is interesting that this formula is also applicable here, even though we have not made any small-amplitude approximation.

Now consider solutions of Eq. 89 of the following form:

$$\mathbf{X}(s,t) = rs\mathbf{e}_1 + \sum_{i=1}^{3} f_i(rs - ct)\mathbf{e}_i, \tag{90}$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the standard basis of $\mathbb{R}^3$, and where the second derivatives of the three functions f_i are continuous and the first derivatives have bounded support, so that the wave described by Eq. 90 may be said to be localized. At any given time, then, for sufficiently large $|s|$, we have simply

$$\mathbf{X}(s,t) = rs\mathbf{e}_1 + \mathbf{c}, \tag{91}$$

where $\mathbf{c}$ is a constant vector which may have one value for large positive s and another value for large negative s. Eq. 91 describes a string under tension $\mathcal{E}'_{\mathrm{E}}(r)$ that is stretched out parallel to the x_1 axis. Thus, we are considering the limit of an infinitely long string with "ends" to which a force parallel to the x_1 axis has been applied. Of couse the forces at the two ends are opposite to each other, and their directions are such as to stretch the string, i.e., the force at $s = +\infty$ points in the positive x_1 direction, and the force at $-\infty$ points in the negative x_1 direction.

Although Eq. 88 is automatically satisfied by any $\mathbf{X}(s,t)$ of the form given by Eq. 90, we can only be assured that Eq. 82 is satisfied if the functions f_i are chosen in such a way that Eq. 86 is *also* satisfied. This leads to the condition

$$1 = (1 + f_1')^2 + (f_2')^2 + (f_3')^2, \tag{92}$$

which can also be written

$$-2f_1' = (f_1')^2 + (f_2')^2 + (f_3')^2 \tag{93}$$

Equation 92, or alternatively Eq. 93, can be thought of as a quadratic equation for f_1' in terms of f_2' and f_3'. The solutions are real if and only if $(f_2')^2 + (f_3')^2 \leq 1$. For simplicity, we assume from now on that the corresponding strict inequality is always satisfied, i.e., that

$$(f_2')^2 + (f_3')^2 < 1. \tag{94}$$

When this inequality is satisfied, there are exactly two solutions f_1' for each pair f_2', f_3'. Only one of these has the property that $f_1' = 0$ when $f_2' = f_3' = 0$, and that is the solution we want. It is characterized by

$$1 + f_1' > 0, \tag{95}$$

which means that the string never "turns back" on itself, but instead has the property that $X_1(s,t)$ is a strictly increasing function of s for each t.

The solution for f_1' that we have chosen can be written explicitly as

$$f_1' = -\left(1 - \sqrt{1 - (f_2')^2 - (f_3')^2}\right) \tag{96}$$

We see, then, that our traveling wave solution is completely determined, up to a constant of integration, if we specify the functions f_2 and f_3. These functions are arbitrary, except that their second derivatives should be continuous, and their first derivatives should have bounded support and satisfy $(f_2')^2 + (f_3')^2 < 1$.

It is quite remarkable that the nonlinear system given by Eq. 82 has exact traveling wave solutions, and moreover that these have so much in common with the solutions of the linear wave equation. In particular, the solutions we have constructed have an essentially arbitrary waveform (except for an upper bound on amplitude), at least insofar as the transverse components are concerned, although the longitudinal component is then determined by the transverse components. Note, however, that there is no principle of superposition here. In particular, although we can construct traveling waves running in either direction, the sum of two such waves running in opposite directions will *not* be a solution, in general. Indeed, if two such waves collide, they will interact in a complicated way.

Let us now consider the relationship between energy and momentum for traveling waves of the type considered above. Since these waves, by construction, do not stretch the string, they do not involve any changes in the elastic energy, $\mathcal{E}_{\mathrm{E}}$. To avoid the introduction of an infinite constant, we therefore redefine the elastic energy as

$$\mathcal{E}_{\mathrm{E}}\left(\left|\frac{\partial \mathbf{X}}{\partial s}\right|\right) - \mathcal{E}_{\mathrm{E}}\left(r\right), \tag{97}$$

and note that this redefined elastic energy is zero for the traveling waves that we have constructed. Thus, in the following, we ignore elastic energy and consider kinetic energy only.

The x_1 component of the momentum density (with respect to s) is given by

$$\mathcal{P}_1 = \rho \frac{\partial X_1}{\partial t} = -\rho c f_1'(rs - ct). \tag{98}$$

The energy density (also with respect to s) is

$$\mathcal{E} = \frac{1}{2}\rho \left|\frac{\partial \mathbf{X}}{\partial t}\right|^2 = \frac{1}{2}\rho c^2 \sum_{i=1}^{3} \left(f_i'\right)^2 (rs - ct). \tag{99}$$

From Eq. 93, we see immediately that

$$\mathcal{P}_1 = \frac{\mathcal{E}}{c} \tag{100}$$

Although the densities that appear in this equation are defined with respect to s, the same result is obviously valid for densities defined with respect to arclength. To obtain the latter result, we need only divide both sides of Eq. 100 by the arclength per unit s, which is r. What is significant, however, is that the velocity c which appears in Eq. 100 is the arclength per unit time traversed by the wave. This has a meaning that is independent of the arbitrary scale of the s variable.

Of course, if we had considered a wave moving in the direction of decreasing x_1, we would have obtained the same result except that the sign of $\mathcal{P}_1$ would have been negative. Thus the momentum density points in the direction of wave propagation, and is related to the energy density by an equation of the same kind as we have seen in each type of wave propagation that we have considered up to now. Note in particular that here we have not only a global relationship, but also a local one, i.e., a relationship that holds at every position and time separately. This is similar to what we found for Maxwell's equations, and unlike what we found in the cases of sound waves and water waves.

5 Linear Vibrating String

There is a special case of the foregoing theory in which the equations of motion, Eq. 82, are exactly linear, without any auxiliary assumptions like Eq. 86, and also without any restriction on the amplitude of the vibration. This special case is characterized by a quadratic elastic energy function

$$\mathcal{E}_{\mathrm{E}}(r) = \frac{1}{2}Kr^2. \tag{101}$$

Then

$$\mathcal{E}'_{\mathrm{E}}(r) = Kr, \tag{102}$$

and

$$\frac{\mathcal{E}'_{\mathrm{E}}(r)}{r} = K, \tag{103}$$

independent of r. It follows that Eq. 82 reduces to the linear wave equation

$$\rho\frac{\partial^2 \mathbf{X}}{\partial t^2} = K\frac{\partial^2 \mathbf{X}}{\partial s^2}. \tag{104}$$

Note in particular that the three components of $\mathbf{X}(s,t)$ are uncoupled in Eq. 104, and that each of them satisfies the wave equation. Therefore, it is perfectly possible to have solutions of Eq. 104 which are purely transverse waves, i.e., solutions of the following form

$$X_1(s,t) = rs, \tag{105}$$
$$X_2(s,t) = f_2(rs - ct), \tag{106}$$
$$X_3(s,t) = f_3(rs - ct), \tag{107}$$

where r is a positive constant, and

$$c = r\sqrt{\frac{K}{\rho}} \tag{108}$$

These are transverse waves on a stretched string. Clearly, they have energy (both kinetic and elastic), and they propagate in the x_1 direction, but their x_1 component of momentum is obviously zero, since $\partial X_1/\partial t = 0$.

More generally, if we allow constraints on the motion, it seems clear that we can easily generate other examples, both linear and nonlinear, in which there are traveling waves (which of course have energy), and in which there cannot be any momentum at all in the direction of the wave. All we have to do is impose constraints that prevent motion in the direction of propagation. Any wave motion that occurs will then be purely transverse. Such waves carry energy but cannot carry momentum in the direction of wave propagation.

These examples show that the relationship postulated in Eq. 1 cannot be universal after all, and this makes it all the more mysterious that Eq. 1 seems to be satisfied for so many different types of wave propagation.

6 Quantum Mechanics

In quantum mechanics, particles are associated with waves, and since the particles also have energy and momentum, it is fair to ask whether a relationship like Eq. 1 is valid or not.

The fundamental relationships that translate between particle properties and wave properties in quantum mechanics are as follows:

$$E = \hbar\omega, \tag{109}$$
$$P = \hbar k, \tag{110}$$

where E is the energy of the particle, P is its momentum (for simplicity, we consider motion in a specified direction, so that momentum is a scalar), ω is the temporal frequency (radians per unit time) of the associated wave, k is the spatial frequency (radians per unit length) of the associated wave, and $\hbar = h/(2\pi)$, where h is Planck's constant.

Clearly, it is a consequence of these fundamental relationships that

$$\frac{E}{P} = \frac{\omega}{k}. \tag{111}$$

Since ω/k is the phase velocity of a wave, this establishes Eq. 1 for the waves that are associated with particles in quantum mechanics. Note the interesting detail that the relevant velocity is the phase velocity of the wave, not the group velocity. (Recall that in the case of water waves, we also found that the phase velocity was the correct velocity to use in Eq. 1.)

The above fact is particularly remarkable when we consider that the means by which we assign an "energy" or "momentum" to a wave in quantum mechanics is completely different from how this is done for a classical wave. In the classical case, the energy, and the momentum if there is any, definitely increase with the amplitude of the wave. In quantum mechanics, however, any multiple of a wave function represents the same state as the original wave function, and therefore has the same associated energy and momentum.

Although we have already reached the main goal of this section, let us now see how this works in greater detail, paying particular attention to the relationship between the phase velocity of the wave and particle velocity, where we identify the particle velocity with the group velocity of the particle-associated wave.

Consider, first, the case of a free, non-relativistic particle. In classical mechanics, such a particle has

$$P = mv, \tag{112}$$

$$E = \frac{1}{2}mv^2, \tag{113}$$

where m is the mass of the particle and v is its velocity. It follows that

$$E = \frac{P^2}{2m}. \tag{114}$$

This relationship between energy and momentum holds in non-relativistic quantum mechanics as well as in classical mechanics. Combining Eqs. 111 and 114, we see that the phase velocity of the associated wave should be $P/(2m) = v/2$. Let us identify the classical velocity v of the particle with the group velocity of the wave, $d\omega/dk$, and ask whether it is indeed the case that the phase velocity is half of the group velocity.

To check, recall that the Schroedinger equation for a free particle moving in one dimension is

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}. \tag{115}$$

This has solutions

$$\psi(x,t) = \exp\left(i(kx-\omega t)\right). \tag{116}$$

Substituting this into the Schroedinger equation, we get

$$\hbar\omega = \frac{(\hbar k)^2}{2m}. \tag{117}$$

It follows that

$$\frac{\omega}{k} = \frac{\hbar k}{2m}, \tag{118}$$

$$\frac{d\omega}{dk} = \frac{\hbar k}{m}. \tag{119}$$

Thus, the phase velocity is indeed half the group velocity.

In the relativistic case, we have the momentum and energy

$$P = \frac{mv}{\sqrt{1-(v^2/c^2)}}, \tag{120}$$

$$E = \frac{mc^2}{\sqrt{1-(v^2/c^2)}}, \tag{121}$$

where m is the rest mass of the particle, v is its velocity, and c is the speed of light. This gives

$$\frac{E}{P} = \frac{c^2}{v}, \tag{122}$$

and comparison with Eq. 111 shows that the phase velocity of the associated wave must be

$$\frac{\omega}{k} = \frac{c^2}{v}, \tag{123}$$

which is greater than the velocity of light, and is inversely related to the particle velocity! As in the non-relativistic case, we want to associate v with the group velocity, $d\omega/dk$, so we need to check whether the associated wave equation indeed has the property that the product of the phase velocity and the group velocity is equal to c^2.

To check whether this is the case, we consider the Klein-Gordon equation for a relativistic free particle:

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} + \left(\frac{mc}{\hbar}\right)^2\phi = 0 \tag{124}$$

Again, we look for a solution of the form

$$\phi(x,t) = \exp\left(i(kx-\omega t)\right), \tag{125}$$

and we find the dispersion relation

$$-\frac{\omega^2}{c^2} + k^2 + \left(\frac{mc}{\hbar}\right)^2 = 0. \tag{126}$$

Differentiating with respect to k and dividing by 2, we find

$$-\frac{\omega}{c^2}\frac{d\omega}{dk} + k = 0, \tag{127}$$

from which it follows that

$$\frac{\omega}{k} = \frac{c^2}{d\omega/dk}, \tag{128}$$

just as predicted by Eq. 123.

To summarize this section, it appears that the relationship between energy and momentum that we have found to be valid for many (but not for all!) types of classical traveling waves is also valid for the waves that are associated with free particles in quantum mechanics. This relationship is simply $E = Pv_{\mathrm{p}}$, where $v_{\mathrm{p}} = \omega/k$ is the phase velocity of the associated wave. Note that this is not the same as the particle velocity v, which instead is equal to the group velocity, $v = d\omega/dk$.

Summary and Conclusions

It seems to be a recurring theme in wave propagation that traveling waves have associated momentum as well as energy, that the momentum points in the direction of wave propagation, and that it is related to the energy by a simple equation in which the momentum is equal to the energy divided by the phase velocity of the traveling wave.

As we have seen, these basic facts hold for traveling electromagnetic waves, traveling sound waves, traveling water waves, a particular kind of large-amplitude traveling wave on a string under tension, and even for the traveling waves associated with free particles in non-relativistic and also in relativistic quantum mechanics.

Yet the thought that there should be a general theory that embraces all of these cases is frustrated by the simple example of a purely transverse mechanical wave, in which there is clearly associated energy, but no associated momentum in the direction of wave propagation. We have seen in detail how such momentum-free traveling waves exist as exact solutions to the equations of motion of a particular kind of elastic string, one in which the elastic energy of any material interval of the string is a homogeneous quadratic function of its length. Besides this example, it is easy to envision other mechanical arrangements in which longitudinal motion is prohibited by constraints, but in which wave propagation can nevertheless occur. All such examples of purely transverse mechanical waves will necessarily have zero momentum in the direction of wave propagation. These counterexamples make the seemingly universal character of wave momentum all the more mysterious.

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Appendix: Momentum Density and Energy Density of the Electromagnetic Field

In most of the cases that we have considered, the expressions for momentum density and energy density are immediate consequences of the mechanical definitions of momentum and energy. For the electromagnetic field, however, this is not the case. The purpose of this appendix, then, is to derive the conservation laws that justify the expressions for momentum density and energy density that we have used. For convenience, we repeat here the Maxwell equations and the expression for the Lorentz force in the units that we are using:

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \tag{129}$$

$$\nabla \times \mathbf{B} = \frac{1}{c}\left(\frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}\right), \tag{130}$$

$$\nabla \cdot \mathbf{E} = \rho, \tag{131}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{132}$$

$$\mathbf{F} = \rho \mathbf{E} + \frac{1}{c}\left(\mathbf{J} \times \mathbf{B}\right). \tag{133}$$

Here $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, $\mathbf{J}$ is the current density, ρ is the charge density, and $\mathbf{F}$ is the force per unit volume applied by the electromagnetic field to the charges and currents. The constant c is the speed of light.

We claim that the momentum density and energy density of the electromagnetic field are given by

$$\begin{aligned} \boldsymbol{\mathcal{P}} &= \frac{1}{c}\left(\mathbf{E}\times\mathbf{B}\right), && (134)\\ \mathcal{E} &= \frac{1}{2}\left(\mathbf{E}\cdot\mathbf{E}+\mathbf{B}\cdot\mathbf{B}\right). && (135)\end{aligned}$$

To justify this claim, we derive and interpret the corresponding conservation laws.

With the help of the first two Maxwell equations, it is easy to see that

$$\frac{\partial\boldsymbol{\mathcal{P}}}{\partial t} = (\nabla\times\mathbf{B})\times\mathbf{B} + (\nabla\times\mathbf{E})\times\mathbf{E} - \frac{1}{c}(\mathbf{J}\times\mathbf{B}). \tag{136}$$

To proceed further, we need the vector identity

$$(\nabla\times\mathbf{V})\times\mathbf{V} = \mathbf{V}\cdot\nabla\mathbf{V} - \frac{1}{2}\nabla\left(|\mathbf{V}|^2\right). \tag{137}$$

In components, this can be rewritten as follows:

$$((\nabla\times\mathbf{V})\times\mathbf{V})_i = \frac{\partial}{\partial x_k}\left(V_iV_k - \frac{1}{2}\delta_{ik}|\mathbf{V}|^2\right) - V_i\nabla\cdot\mathbf{V} \tag{138}$$

We use this identity twice, once with $\mathbf{V}=\mathbf{E}$ and once with $\mathbf{V}=\mathbf{B}$. Taking into account the last two Maxwell equations, and also the formula for the Lorentz force, we get

$$\frac{\partial\mathcal{P}_i}{\partial t} = \frac{\partial\sigma_{ik}}{\partial x_k} - F_i, \tag{139}$$

where

$$\begin{aligned}\sigma_{ik} &= E_iE_k + B_iB_k - \frac{1}{2}\delta_{ik}\left(|\mathbf{E}|^2+|\mathbf{B}|^2\right),\\ &= E_iE_k + B_iB_k - \delta_{ik}\mathcal{E}. && (140)\end{aligned}$$

Equation 139 is the standard statement of momentum conservation in continuum mechanics, with σ_{ik} as the stress tensor and $-\mathbf{F}$ as a body force applied to the system. (The reason for the minus sign is that $\mathbf{F}$ was defined as the force per unit volume applied by the electromagnetic field to the charges and currents, but we want the force per unit volume applied by the charges and currents to the electromagnetic field.)

Next, we turn our attention to the conservation of energy. We have

$$\begin{aligned}\frac{\partial \mathcal{E}}{\partial t} &= \mathbf{E}\cdot\frac{\partial \mathbf{E}}{\partial t} + \mathbf{B}\cdot\frac{\partial \mathbf{B}}{\partial t} \\ &= c\left(\mathbf{E}\cdot(\nabla\times\mathbf{B}) - \mathbf{B}\cdot(\nabla\times\mathbf{E})\right) - \mathbf{E}\cdot\mathbf{J} \\ &= -c\nabla\cdot(\mathbf{E}\times\mathbf{B}) - \mathbf{E}\cdot\mathbf{J}, \end{aligned} \tag{141}$$

where we have used the first two Maxwell equations and then the vector identity

$$\nabla\cdot(\mathbf{E}\times\mathbf{B}) = -\mathbf{E}\cdot(\nabla\times\mathbf{B}) + \mathbf{B}\cdot(\nabla\times\mathbf{E}) \tag{142}$$

Equation 141 is the statement of conservation of energy. On the last line we see that $c(\mathbf{E}\times\mathbf{B})$ is the energy flux, and that $\mathbf{E}\cdot\mathbf{J}$ is the rate at which the electromagnetic field does work on the charges and currents. (The magnetic field does no work because the force generated by the magnetic field is always perpendicular to the currents.)