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Differential Geometry II — Final Exam — May 14, 2024

Solve any six of the following seven problems.

1. Let M be a smooth manifold and let $X, Y \in \mathcal{V}(M)$ be vector fields.

   (a) Prove that $[X,Y] = XY - YX$ is a vector field, i.e., that it satisfies the Leibniz rule

   $$[X,Y](fg) = [X,Y](f) \cdot g + f \cdot [X,Y](g).$$

   (b) Let $f\colon M \to \mathbb{R}$ be a smooth function. Prove the product rule for the Lie bracket, i.e.,

   $$[X, fY] = Xf \cdot Y + f[X,Y].$$

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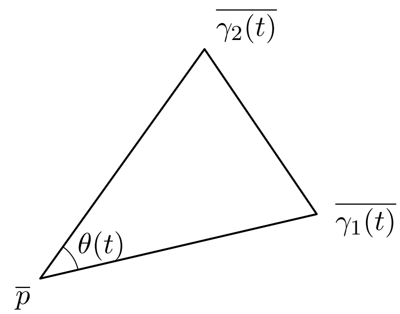
2. Let M be a CAT(0) space. Let $p \in M$ and let $\gamma_1, \gamma_2 \colon [0, \infty) \to M$ be two unit-speed geodesics such that $\gamma_1(0) = \gamma_2(0) = p$. For $t \geq 0$, let $\theta(t) \in [0, \pi]$ be the angle at $\overline{p}$ of the comparison triangle $\Delta(\overline{p}, \overline{\gamma_1(t)}, \overline{\gamma_2(t)})$. Show that θ is a nondecreasing function.



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3. Let $p \in M$ and suppose that $\exp_p(V)$ is defined for all $V \in T_pM$. Show that M is geodesically complete.

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4. Let M be a Riemannian manifold of dimension 2 and let $p \in M$. For $r < \text{injrad}(p)$, let $C(r)$ be the circumference of the disc of radius r around p. Show (give a full calculation) that the Gaussian curvature of M at p is given by

$$K(p) = \lim_{r \to 0} \frac{3}{\pi} \frac{2\pi r - C(r)}{r^3}.$$

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5. Let M be a complete Riemannian n–manifold and let $k > 0$. Suppose that there is a compact subset $D \subset M$ such that for all $p \notin D$, $\mathrm{Ric}(U, U) \geq (n-1)k$ for all $U \in T_pM$ such that $\|U\| = 1$. Show that M is compact.

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6. Suppose that $\kappa > 0$ and that M is a Riemannian manifold with sectional curvature $K(X, Y) \leq \kappa$ for all $X, Y \in TM$. Show that any geodesic in M of length less than $\frac{\pi}{\sqrt{\kappa}}$ has no conjugate points.

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7. Let M be a complete manifold with nonpositive sectional curvature, i.e., $K(X,Y) \leq 0$ for all $p \in M$ and all $X, Y \in T_pM$. Let $p \in M$ and let r be the injectivity radius of M at p.

   (a) Show that there must be unit vectors $v, w \in T_pM$ such that $v \neq w$, but $\exp_p(rv) = \exp_p(rw)$.

   (b) Show that

   $$\gamma(t) = \begin{cases} \exp_p(tv) & t \in [0, r] \\ \exp_p((2r - t)w) & t \in (r, 2r] \end{cases}$$

   is a geodesic. (Hint: Start by showing that if γ is not a geodesic, then there is a sequence of curves λ_i such that $\ell(\lambda_i) < 2r$ and $\lambda_i \to \gamma$. Show that there are curves $\alpha_i \colon [0, 2r] \to T_pM$ such that $\lambda_i(t) = \exp_p(\alpha_i(t))$ and that $\ell(\alpha_i) < 2r$ and try to deduce a contradiction.)

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