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Differential Geometry II — Final Exam — May 8, 2025

Solve the following seven questions. You may use one 3" × 5" index card of notes, but no other aids.

1. Let $f\colon M \to \mathbb{R}$ be a smooth function. If $W \in \mathcal{V}(M)$ satisfies

$$Xf = \langle X \mid W \rangle$$

for any $X \in \mathcal{V}(M)$, we call W the *gradient* of f and write $\nabla f = W$.

- Show that the gradient of f is well-defined and smooth and that if $\gamma\colon [a,b] \to M$ is smooth, then

$$f(b) - f(a) = \int_a^b \langle \gamma' \mid \nabla f \rangle.$$

- Show that if $\|\nabla f\| = 1$ on all of M and if γ is a smooth curve satisfying $\gamma'(t) = X_{\gamma(t)}$ for all t (an *integral curve* of X), then γ is a geodesic.

2. Let M be a Riemannian manifold with $K_M \leq 0$ and let $\gamma \colon \mathbb{R} \to M$ be a geodesic. Let $J \in \mathcal{V}(\gamma)$ be a Jacobi field. Show that if $\|J\|$ is bounded, then J is a parallel vector field.

3. Let M be a Riemannian manifold of dimension 2 and let $p \in M$. For $r < \text{injrad}(p)$, let $C(r)$ be the circumference of the disc of radius r around p. Show (give a full calculation) that the Gaussian curvature of M at p is given by

$$K(p) = \lim_{r \to 0} \frac{3}{\pi} \frac{2\pi r - C(r)}{r^3}.$$

4. Let M be a complete Riemannian n–manifold and let $k > 0$. Suppose that there is a compact subset $D \subset M$ such that for all $p \notin D$, $\mathrm{Ric}(U, U) \geq (n-1)k$ for all $U \in T_pM$ such that $\|U\| = 1$. Show that M is compact.

5. Let M be a complete Riemannian manifold with $K_M \leq 0$ and let $\gamma\colon [0, \infty) \to M$ be a geodesic ray. Let $p \in M$. Show that there is a geodesic ray $\lambda\colon [0, \infty) \to M$ such that $d(\gamma(t), \lambda(t)) \leq d(\gamma(0), \lambda(0))$ for all $t > 0$.

   (Hint: For $i \geq 0$, let λ_i be the geodesic from p to $\gamma(i)$, parameterized with unit speed. Consider a limit of the λ_i's.)

6. Suppose that M and N are surfaces embedded in $\mathbb{R}^n$ and that M and N intersect along a smooth curve γ. Suppose that $T_{\gamma(t)}M = T_{\gamma(t)}N$ for all t. Show that γ is a geodesic in M if and only if it is a geodesic in N.

7. Equip $M = \mathbb{R}^2$ with the standard coordinate system (x, y) and let g be the metric

$$dg^2 = dx^2 + f(x)^2 dy^2,$$

where $f\colon \mathbb{R} \to \mathbb{R}$ is a smooth function with $f(x) > 0$ for all x.

- Recall that for any y_0, the map $\gamma_{y_0}(t) = (t, y_0)$ is a geodesic. Use these geodesics to construct a normal Jacobi field J on γ_0 with $J(0) = \partial_y$.
- Use the Jacobi field you constructed to compute the sectional curvature of M.

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