Differential Geometry II — Midterm Exam — March 14, 2024

Solve any five of the following six problems.

1. Let γ be a geodesic and let J be a Jacobi field on γ. Show (without using theorems or results from the problem sets) that if $\langle J(0) \mid \gamma'(0) \rangle = 0$, and $\langle D_t J(0) \mid \gamma'(0) \rangle = 0$, then $J(t)$ is orthogonal to $\gamma'(t)$ for all t.

2. Let $\lambda = (\lambda^x, \lambda^z)\colon \mathbb{R} \to \mathbb{R}^2$ be a smooth curve with $\|\lambda'(t)\| = 1$ and $\lambda^x(t) > 0$ for all t. Let $M \subset \mathbb{R}^3$ be the surface of revolution obtained by rotating λ around the z–axis.

   We parameterize M by the map $F\colon \mathbb{R}^2 \to \mathbb{R}^3$,

$$F(\theta, \tau) = (\lambda^x(\tau)\cos\theta, \lambda^x(\tau)\sin\theta, \lambda^z(\tau)).$$

   Let $\tau_0 \in \mathbb{R}$ and let $\gamma(t) = F(t, \tau_0)$. Calculate $D_t\gamma'$ in terms of ∂_τ and ∂_θ and find necessary and sufficient conditions for γ to be a geodesic.

3. Let $z \in (0,1)$ and for $r > 0$, $\theta \in \mathbb{R}$, let

$$u(r,\theta) = (rz\cos\theta, rz\sin\theta, r\sqrt{1-z^2}).$$

The image of u is a cone which we denote M. Let $p = u(1,0) = (z, 0, \sqrt{1-z^2})$ and let $q = u(1,\pi) = (-z, 0, \sqrt{1-z^2})$.

- Let R be the ray from the origin through q; Find an isometry between $M \setminus R$ and a subset of $\mathbb{R}^2$.
- Find $\operatorname{injrad}(p)$ and show that when $z < \frac{1}{2}$, there is a geodesic from p to itself of length $2\operatorname{injrad}(p)$.

4. Let $H = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$, equipped with the metric

$$dg^2 = \frac{1}{y^2}(dx^2 + dy^2)$$

That is, g is the Euclidean metric scaled by a factor of $\frac{1}{y}$.

Let ∂_x and ∂_y be the coordinate fields and let ∇ be the Levi–Civita connection. Let $\nabla_x = \nabla_{\partial_x}$, $\nabla_y = \nabla_{\partial y}$. Then one can calculate

$$\nabla_x \partial_x = \frac{1}{y}\partial_y \qquad \nabla_x \partial_y = -\frac{1}{y}\partial_x.$$

Calculate $\nabla_y \partial_y$ (you can use the Levi–Civita equations or compute $\partial_y \langle \partial_x \mid \partial_y \rangle$ and $\partial_y \langle \partial_y \mid \partial_y \rangle$) and calculate the Gaussian curvature of H.

5. A *symmetric space* M is a connected Riemannian manifold such that for every $p \in M$, there is an isometry $f_p \colon M \to M$ such that $(Df_p)_p \colon T_pM \to T_pM$ satisfies $(Df_p)_p(V) = -V$ for all $V \in T_pM$.

   Suppose M is a symmetric space.

   - Show that M is geodesically complete.
   - Show that M is homogeneous, i.e., for every $p, q \in M$, there is an isometry $h_{p,q}$ of M such that $h(p) = q$.

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6. Let $f\colon M \to \mathbb{R}$ be a smooth function.

   - Show that there is a unique vector field ∇f such that for any $X \in \mathcal{V}(M)$,

   $$Xf = \langle X \mid \nabla f \rangle.$$

     We call ∇f the *gradient* of f.

   - Show that if $\|\nabla f\| = 1$ on all of M and if γ is a smooth curve satisfying $\gamma'(t) = X_{\gamma(t)}$ for all t (an *integral curve* of X), then γ is a geodesic.

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