Name:___________________________

Differential Geometry II — Midterm Exam — March 19, 2025

Solve any **five** of the following **six** problems. You may use one (3" by 5") index card of notes.

1. Suppose that M is homogeneous, i.e., for every $p, q \in M$, there is an isometry $h_{p,q}$ of M such that $h(p) = q$. Show that M is complete. (Be sure to name any theorems/lemmas that you use in your proof.)

2. Let γ be a geodesic and let J be a Jacobi field on γ. Show (without using theorems or results from the problem sets) that if $a \neq b$ and $\langle J(a) \mid \gamma'(a)\rangle = \langle J(b) \mid \gamma'(b)\rangle = 0$, then $J(t)$ is orthogonal to $\gamma'(t)$ for all t.

3. Equip $M = \mathbb{R}^2$ with the standard coordinate system (x, y) and let g be the metric

$$dg^2 = dx^2 + f(x)^2 dy^2,$$

where $f\colon \mathbb{R} \to \mathbb{R}$ is a smooth function with $f(x) > 0$ for all x. (That is, for the standard basis ∂_x, ∂_y, we have $\langle \partial_x \mid \partial_x \rangle = 1$, $\langle \partial_x \mid \partial_y \rangle = 0$, $\langle \partial_y \mid \partial_y \rangle = f(x)^2$.)

Show that for any y_0, the map $\gamma(t) = (t, y_0)$ is a geodesic. (There is way to do this using symmetry if you remember a problem from a problem set.)

Use this fact to construct a pair of parallel orthonormal fields on γ and calculate $\nabla_x \partial_x$ and $\nabla_x \partial_y$. How can you check that your answer makes sense?

4. Let $M = \mathbb{R}^2$ with metric $dg^2 = dx^2 + f(x)^2 dy^2$ as in the previous problem. Let $x_0 \in \mathbb{R}$ and let λ be the curve $\lambda(t) = (x_0, t)$.

   Use the first variation formula to compute $D_t \lambda'$ and $\nabla_y \partial_y$. How can you check that your answer makes sense?

5. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a positive function and let $M \subset \mathbb{R}^3$ be the surface of revolution given by

$$x^2 + y^2 = f(z)^2.$$

Let $\gamma(t) = (\gamma_x(t), \gamma_y(t), \gamma_z(t)) \in M$ be a unit-speed geodesic on M and let

$$L(t) = \gamma_x(t)\gamma_y'(t) - \gamma_y(t)\gamma_x'(t).$$

Show that $L(t) = L(0)$ for all t and that

$$\gamma_x(t)^2 + \gamma_y(t)^2 \geq L(0)^2$$

for all t.

6. Let M and N be Riemannian manifolds. Suppose that $f\colon M \to N$ is a map such that for all $x, y \in M$, $d_N(f(x), f(y)) = d_M(x, y)$. Show that if $\gamma\colon I \to M$ is a geodesic, then $f \circ \gamma$ is a geodesic.

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