2018-09-27

Last time: Equivalence of geometric/combinatorial def of Dehn fn.
– gives a reduction, ~~for~~ a word to ε, construct a htpy from λ_w to $*$.

Lemma: Let K_G be the Cayley complex of G, let $w =_G 1$. Then $\exists c > 0$ s.t.

$$\delta(w) \leq c\, FA(\lambda_w).$$

(~~geometric~~ geom $\Rightarrow$ ~~combinatorial~~ comb)

(homotopy $\Rightarrow$ reduction)

Problem: 

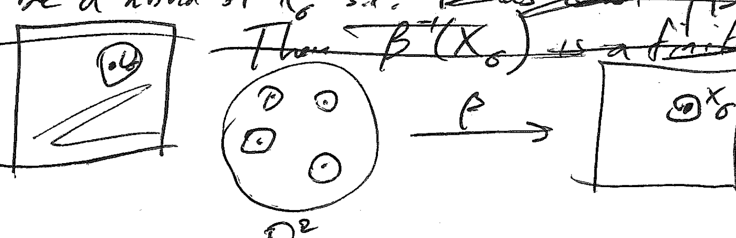
Pf: Let $\beta : D^2 \to K_G$ ~~be a disc~~ ~~$\partial\beta$~~ be piecewise smooth, $\partial\beta = \lambda_w$. Then

$$Area(\beta) = \int_{K_G} |\beta^{-1}(x)|\, dx = \sum_{\sigma \in F^2(K_G)} \int_\sigma |\beta^{-1}(x)|\, dx.$$

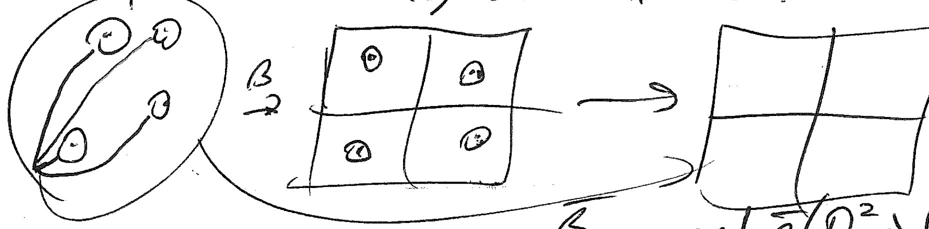
$\forall \sigma$, choose x_σ s.t. x_σ is a regular value of β and

$$\frac{|\beta^{-1}(x_\sigma)|}{\cancel{Area(\sigma)}} \leq \frac{\int_\sigma |\beta^{-1}(x)|\, dx}{Area(\sigma)}.$$

~~Let U_σ be a nbhd of x_σ s.t. β has const $|\beta^{-1}(x)|$ is constant on U_σ.~~ ~~Then $\beta^{-1}(X_\sigma)$ is a finite~~ $\beta^{-1}(x_\sigma)$ is finite. ~~derivative~~ β is a local homeo near each pt in $\beta^{-1}(x_\sigma)$.

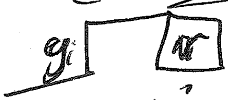


So if we take a small enough nbhd, β is locally a covering map.
Compose β with a map sending ~~x_σ~~ U_σ to σ. ~~This decreases Area(β).~~
Call the result $\bar{\beta}$. Then $Area(\bar{\beta}) \leq Area(\beta)$.
Now $\bar{\beta}$ has constant degree on each face. But better:



D^2 contains finitely many nbhds N_i that are each sent to a 2-cell of K_G, and $\bar{\beta}(D^2 \setminus \bigcup_i N_i) \subset K_G^{(1)}$.

~~Construct a "bundle of lollipops" L~~
~~$\bar{\beta}$ — The boundary~~ $\bar{\beta}$ sends the boundary of a lollipop to a ~~curve~~ $g r g^{-1}$.



~~∂D^2~~ $\beta(\partial D^2)$ is htpic to ~~β~~ $\bar{\beta}(\partial L)$ in K_G

$$\lambda_w \sim \lambda_{\prod g_i r_i g_i^{-1}}.$$

$$\Rightarrow w =_F \prod_{i=1}^n g_i r_i g_i^{-1}.$$

where $n \lesssim Area(\beta)$

So that connects comb to geom. ~~The other part: This should work~~ for any cplex, not just Cayley.

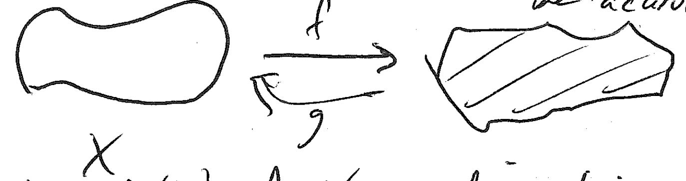
Now let's take this one step further — remember, we wanted to show
~~So Thm:~~ this for any ~~complex~~ simply connected complex on which G acts. By S-M, all those complexes are QI, so it suffices to show:

Prop: If X, Y are simply-connected ~~manifolds~~ / simplicial complexes, $X \sim_{QI} Y$, and $FA_X(n) < \infty$, $FA_Y(n) < \infty$ $\forall n$, then $FA_X \sim FA_Y$.

Pf: ~~Idea~~

~~Let $f: X \to Y$ g:~~
~~Suffices to consider~~
Let $f: X \to Y$, $g: Y \to X$ be quasi-inverse (k,c) QIs. Idea: let $\delta: S^1 \to X$ be a curve. Map to $f(\delta)$, fill $f(\delta)$, map back by g to get $g(\delta)$.

X



What are the possible problems?

X

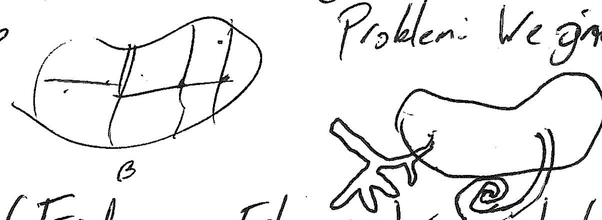
1 - ~~$f(\delta)$~~ $f \circ \delta$ is discontinuous
— Connect the dots (we saw this in ~~S~~ S-M)
2 - $g \circ B$ is discontinuous
— This takes a little more work ~~spec~~
3 - $\delta(g \circ B) = f(g(\delta))$, so we need to connect $\delta\delta$ to $\delta f(g(\delta))$.

So let's do this. Let $\delta: ~~S~~ [0, L] \to X$ be a closed curve with speed ≤ 1. ~~unit speed~~ Let $x_0, \ldots, x_L \in X$, $x_i = \delta(i)$.
~~Let~~ $\forall i$, let y_i be nearest vertex to $f(x_i)$.

~~Then $d(y_i, y_{i+1}) \leq d(y_i, f(x_i)) + d(f(x_i), f($~~ $\exists c > 0$ s.t.
~~Then~~ $d(y_i, y_{i+1}) \leq c$. Let $\bar{\delta}: [0, L] \to Y^{(1)}$ be an edge path connecting the y_i's with $\ell(\bar{\delta}) \leq cL$.

~~Let $\beta: D^2 \to Y$ be a filling of $\bar{\delta}$ with area$(\beta) \leq FA(cL)+1$.~~
Let $\beta: D^2 \to Y$ be a filling of $\bar{\delta}$ with area$(\beta) \leq FA_Y(cL)+1$.
We want to construct a $\bar{\beta}$ approximating $g(\beta)$. First idea: try to do the same as before: discretize β, map over, connect dots.

~~Persp~~



Problem: We only have an area bound, so it's hard to discretize!
We need a theorem:

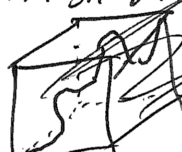
Thm (Federer-Fleming): Let X be an n-simplicial complex. Let $(M, \partial M)$ be a compact k-manifold w/ boundary (possibly $\partial M = \emptyset$). Let $\alpha: (M, \partial M) \to (X, X^{(k-1)})$ be a Lipschitz map. Then $\exists c(n) > 0$ s.t. there is an approximation $\hat{\alpha}: (M, \partial M) \to (X^{(k)}, X^{(k-1)})$ s.t. $\mathrm{vol}^k(\hat{\alpha}) \leq c \,\mathrm{vol}^k(\alpha)$, $\partial \alpha = \partial \hat{\alpha}$ and $\hat{\alpha}$ is htpic to α.

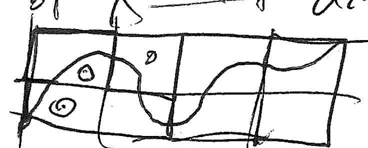
by a htpy h_t ~~$(M,\partial M) \to (X, X^{(k-1)})$~~ $M \to X$ that fixes ∂M ptwise and satisfies $vol^{k+1}(h) \leq c\, vol^k(a)$.

Further, we may suppose that $\hat{a}$ is admissible: $\exists$ disj open balls $B_1, \ldots, B_m$ st. each B_i is a k-cell and $\hat{a}(M \setminus B_i) \subset X^{(k-1)}$

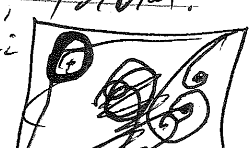
~~Pf~~ Sketch: Similar to the ~~lemma~~ construction from before

~~Induction on dimension of X: if $\dim X = k$, trivial.~~



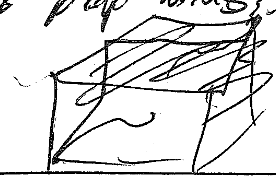


Problem:

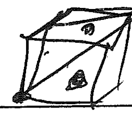


~~So~~ But if you take a point at random, you can calculate that this map works. ~~So that. In gene~~

So



What about higher dimensions?



Repeat for each dimension.

So: let $B : D^2 \to Y$ be a filling of $\gamma : S^1 \to Y^{(1)}$

We approximate by $\beta : D^2 \to Y^{(2)}$ with same boundary.

~~Define~~ Now we map the discrete version back:

~~Let $h : Y^{(2)} \to X$ be a ... map~~

Define $h : Y^{(2)} \to X$ as ~~follows~~:

~~$\forall v \in V(Y)$, let $h(v)$ = nearest vertex to $g(v)$~~

~~$\forall e \in E(Y)$, let $h(e)$ = edge path ...~~ geodesic

~~$\forall \delta \in F(Y)$, let~~

So that $h(v) = g(v)$ $\forall v \in V(Y)$, $h(e)$ is a geodesic $\forall$ edge,

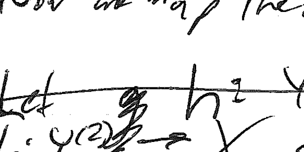
For each $\Delta \in F(Y)$, $\ell(h(\partial \Delta)) \leq C_0$; ~~let~~ define h so that $Area(h(\Delta)) \leq FA(c) + 1$.

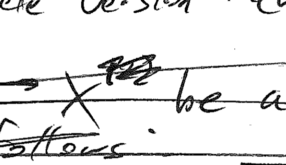
Let $\bar{B} = h \circ \beta$

Then: $Area(\bar{B}) \leq c\, area(B) \leq c \cdot FA(c\ell)$.

Last: $\partial \bar{\beta} = h(\bar{\gamma})$. Need to connect γ to $h(\bar{\gamma})$.

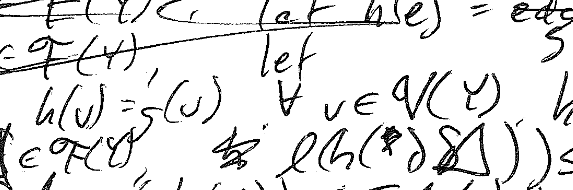
Recall





~~Draw~~ Draw geodesics from x_i to $h(y_i)$

Each curve has length $\leq c$, so there are fillings w/ area $\leq FA_X(c)$. Combining all of these, we



get a filling of γ with area $c \cdot FA(c\lceil \ell \rceil) + c\lceil \ell \rceil \le cFA(c\ell + c) + c\ell + c$ //

Time permitting:
~~Thm (Federer-Fleming): $0 < k < n$.~~

~~Higher-dimensional isoperimetric inequalities: We can do similar for spheres/surfaces. The main diff is that it's helpful to instead fill~~
~~Let X be k-connected complex / Riem mfld~~
~~A smooth singular chain It's helpful to work homologically instead of htpically.~~ We can define filling ineqs in higher dims too. Instead of filling curves by disks, its gen. better to fill surfaces by surfaces or cycles by chains. ~~Recall homology for a topology:~~ ~~1-cycles~~

~~Recall singular~~ Two ways: cellular or singular.

Cellular homology: Let X be a simplicial/CW-complex, let $n \ge 0$. $C_n^{cell}(X) = \{$integer linear combinations of n-cells of $X\}$.

$C_n^{cell}(X) \xrightarrow{\partial} C_{n-1}^{cell}(X)$, $Z_n(X) = \ker \partial$

If $\alpha = \sum_{\delta \in F^n(X)} a_\delta \delta$, $a_\delta \in \mathbb{Z}$, let $\|\alpha\|_1 = \sum_{\delta \in F(X)} |a_\delta|$.

Def: If $\alpha \in Z_n(X)$, define $FV_X^{n+1}(\alpha) = \min_{\beta \in C_{n+1}(X),\ \partial\beta = \alpha} \|\beta\|_1$

$\forall V > 0$, let $FV_X^{n+1}(V) = \max_{\alpha \in Z_n(X),\ \|\alpha\|_1 \le V} FV_X^n(\alpha)$

Singular: $C_n^{sing}(X) = \{$integer linear combinations of Lipschitz/piecewise smooth maps $\delta : \Delta^n \to X\}$

$C_n^{sing}(X) \to C_{n-1}^{sing}(X)$ If $\alpha = \sum_i a_i \delta_i$, let $mass(\alpha) = \sum |a_i| \operatorname{vol}^{n} \delta_i$.

(i.e., volume with multiplicity)
If $\alpha \in Z_n(X)$, define $FV_X^{n+1}(\alpha) = \inf_{\partial\beta = \alpha} mass(\beta)$

$FV_X^{n+1}(V) = \sup FV_X^{n+1}(\alpha)$

Thm: - ~~Under some conditions~~ Federer-Fleming: Singular and cellular defs are equivalent up to $\sim$
- ~~FV_X^n~~ If X, Y are n-connected, $FV_X^k(V) < \infty$, $FV_Y^k(V) < \infty$ $\forall k = 0, \dots, n+1$, $V > 0$,
and $X \sim_{QI} Y$, then $FV_X^{n+1} \sim FV_Y^{n+1}$.
~~I'm not going to prove these here, but the tools are~~ main tool is ~~Federer-Fleming~~.

2018-10-03

Last time: ~~Fillings~~ Dehn fns and filling curves by discs.
And we can generalize to more dimensions.
Given a surface  how much volume does it take to fill?

Like before, we can represent surfaces combin.

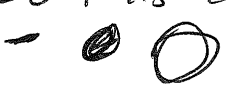
Combinatorics: Let K be a simplicial/~~cell~~ (cw) ~~complex~~ complex. 

~~A n-dim surface is a sum of n-cells~~
~~An n-cycle~~ ~~cellular~~ simplicial/cellular $\alpha =$
~~s.t. $\partial\alpha = 0$~~. An n-chain is a sum of n-cells, ~~and there's~~ ~~a boun~~ Let $C_n^{cell}(K) = \{n\text{-chains}\}$. There's a boundary op.
$\partial : C_n^{cell}(K) \to C_{n-1}^{cell}(K)$, and an n-cycle is an ~~n~~ n-chain w/ $\partial\alpha = 0$. A filling of an n-cycle is an $n+1$-chain β s.t. $\partial\beta = \alpha$, and we can define

$$FV_K^{n+1}(\alpha) = \min_{\partial\beta=\alpha} \|\beta\|_1 \qquad FV_K^{n+1}(V) = \max_{\|\alpha\|_1 \le V} FV^{n+1}(\alpha).$$

Or geometrically: Let X a manifold/simp. complex  A singular Lipschitz n-chain is a sum $\alpha = \sum a_i \delta_i$ where $\delta_i : \Delta^n \to X$ is a Lipschitz map. Let $C_n^{Lip}(X) = \{\text{Lipschitz } n\text{-chains}\}$
$Z_n^{Lip}(X) = \{\alpha \in C_n^{Lip}(X) \mid \partial\alpha = 0\}$.

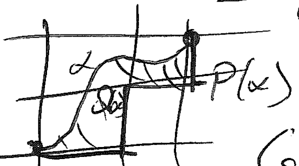
$$FV^{n+1}(\alpha) = \inf_{\partial\beta=\alpha} \text{mass}\,\beta^{n}, \text{ where}$$
$$\text{mass}\left(\sum b_i \delta_i\right) = \sum |b_i| \operatorname{vol} \delta_i.$$
$$FV_X^{n+1}(V) = \sup_{\text{mass}\,\alpha \le V} FV^{n+1}(\alpha)$$

(I, suppose I haven't defined volume. Let $f : \mathbb{R}^m \to \mathbb{R}^n$, $m < n$, be differentiable. Let $Df_x : \mathbb{R}^m \to \mathbb{R}^n$ be the derivative of f at x.
– this is an $m\times n$ matrix. If ~~we were working in~~ $m = n$, define Jacobian as the determinant, but here, non-square matrix.
Let $J(x) = \underline{\text{product of singular}}$ values of Df_x.
$= \sqrt{\det (Df_x)^T (Df_x)}$. This measures the distortion of m-vol.
Then $\forall U \in \mathbb{R}^m$ $\mathcal{H}^m(Df_x(U)) = |J(x)|\,\mathcal{H}^m(U)$
and we define $\operatorname{vol}(f|_U) = \int_U |J(x)|\, d\mathcal{H}^m(U)$
So this counts size of the image and multiplicity:
 winding around a circle twice, or going forward then back counts ≥ 1 time.)

And, like before, we can approximate geometric by combinatorial.

Deformation Theorem (Federer–Fleming): Let X be ~~a~~ a finite-dimensional simplicial complex such that ~~for every simplex~~ $\forall$ ~~δ~~ $\forall$ simplex δ,

$\exists$ a bilipschitz homeomorphism from δ to $\Delta^{\dim \delta}$ (unit simplex).
(~~There are~~ Then, $\forall \alpha \in C_n^{cell}(X)$, $\|\alpha\|_1 \approx \mathrm{mass}(\alpha)$).
$\exists\, C>0$ s.t.: $\forall \alpha \in C_n^{Lip}(X)$ s.t. $\partial\alpha$ is ~~a lin~~ a simplicial chain, $\exists$ ~~an app~~ $P(\alpha) \in C_n^{cell}(X)$, $Q(\alpha) \in C_{n+1}^{Lip}(X)$ s.t.
- $\partial P(\alpha) = \partial \alpha$. — $\|P(\alpha)\|_1 \le C\,\mathrm{mass}(\alpha)$
- $\partial Q(\alpha) = \alpha - P(\alpha)$, — $\mathrm{mass}(Q(\alpha)) \le C\,\mathrm{mass}(\alpha)$.

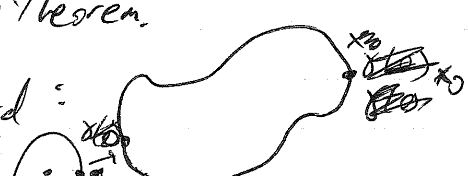


Cor: Simplicial $FV_X^n \sim$ Lipschitz FV_X^n.
Cor: If X, Y are complexes as in the theorem, $X \sim_{QI} Y$, and if $FV_X^k(V) < \infty$, $FV_Y^k(V) < \infty$ $\forall\, 2 \le k \le n$, $V \ge 0$,
then $FV_X^n(V) \sim FV_Y^n(V)$.

Phew — I feel like ~~we've spent we~~ it's taken forever to get through all these definitions. Why did I want to spend time on these defs and equivs? Well, one of the big reasons is this theorem. It seems small — we can approx geom by combin — but it's really powerful, ~~especially when we go from single-scale questions to many scale qu~~ especially combined with scaling ~~ar~~ arguments.

Ex (and this was actually one of the first uses of this) we can prove the isop. ineq for $\mathbb{R}^n$, with just ~~the~~ the Deformation Theorem.

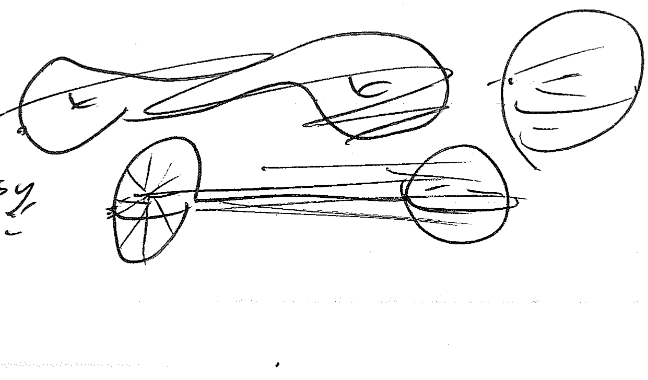
Isop. ineq for curves isn't hard:
coning argument: $\gamma: S^1 \to \mathbb{R}^n$



$\beta(r,\theta) = x_0 + r(\gamma(\theta) - x_0)$
Can ~~then~~ calculate $\mathrm{vol}(\beta) \le c\,\ell(\gamma)^2$.
$\Rightarrow FA_{\mathbb{R}^n}(\ell) \le c\ell^2$
word problem: $\mathbb{Z}^n$ acts geometrically on $\mathbb{R}^n$, so $\delta_{\mathbb{Z}^n} \sim FA_{\mathbb{R}^n}$.

$\mathbb{Z}^n = \langle x_1, \dots, x_n \mid [x_i, x_j] = 1 \rangle$. If $w \in \{x_1, \dots, x_n\}^*$ and $w =_{\mathbb{Z}^n} 1$, then we can reduce w by collecting x_1's, then x_2's, etc. in $\ell(w)^2$ steps.

But surfaces are harder:



coning a sphere is easy
but what about:

But! Thm (Federer–Fleming): ~~If~~ $\forall$ $n \geq k$, ~~then~~ $FV^{k+1}_{\mathbb{R}^n}(V) \sim V^{\frac{k+1}{k}}$

Pf: Let $\alpha \in C_k^{Lip}(\mathbb{R}^n)$ be a cycle of mass V.
For $r>0$, let τ_r be the ~~$r\times r$ grid in $\mathbb{R}^n$ with each~~ barycentric subdivision of the $r \times r$ grid in $\mathbb{R}^n$.
We want to apply Deformation Theorem, but the thm only applies to complexes whose cells are close to unit simplices.

For $t>0$, let $s_t : \mathbb{R} \to \mathbb{R}$, $s_t(x) = tx$ (scaling map). Define
$P_r(\alpha) = s_r(P(s_{r^{-1}}(\alpha)))$, where $P(\beta)$ is the ~~approx of~~ ~~β by a~~ simplicial chain in τ_1 approximating β.
Then $\exists\, C>0$ s.t.

— $P_r(\alpha)$ is a simplicial cycle in τ_r.

— $\|P_r(\alpha)\|_1 \leq C\,\mathrm{mass}(s_{r^{-1}}(\alpha)) = C r^{-k}\,\mathrm{mass}(\alpha) = C r^{-k} V$.

— $\mathrm{mass}(P_r(\alpha)) \leq \|P_r(\alpha)\|_1 \cdot C r^k \leq C^2 V$.

— Let $Q_r(\alpha) = s_r(Q(s_{r^{-1}}(\alpha)))$.
Then $\partial Q_r(\alpha) = \alpha - P_r(\alpha)$ and

$$\mathrm{mass}(Q_r(\alpha)) = r^{k+1}\,\mathrm{mass}(Q(s_{r^{-1}}(\alpha))) \leq r^{k+1}\cdot C \cdot r^{-k}\cdot \mathrm{mass}(\alpha) = CrV.$$

Choose $r = (2CV)^{1/k}$. Then $\|P_r(\alpha)\|_1 \leq \frac{CV}{2CV} = \frac{1}{2}$
$\Rightarrow P_r(\alpha) = 0$.

So $\partial Q_r(\alpha) = \alpha - P_r(\alpha) = \alpha$, and $\mathrm{mass}\, Q_r(\alpha) \leq CrV = (2C^{k+1})^{\frac{1}{k}} V^{\frac{k+1}{k}}$ //

~~Let's~~ ~~Now let's~~ I'd like to apply this to something more complicated, but before we can do that, need to define something more complicated.

Ex: The Heisenberg group $\mathbb{H}$ is the Lie group

$$\mathbb{H} = \left\{ \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} \middle| x,y,z \in \mathbb{R} \right\}$$

This contains a ~~lattice~~ subgroup

$$\mathbb{H}_{\mathbb{Z}} = \left\{ \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} \middle| x,y,z \in \mathbb{Z} \right\} = \langle X, Y, Z \mid [X,Y] = Z \rangle,$$

where $X = \begin{pmatrix} 1 & 1 & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix}$, $Y = \begin{pmatrix} 1 & & \\ & 1 & 1 \\ & & 1 \end{pmatrix}$, $Z = \begin{pmatrix} 1 & & 1 \\ & 1 & \\ & & 1 \end{pmatrix}$, and $\mathbb{H}_{\mathbb{Z}}$ acts geometrically on $\mathbb{H}$.

The notable thing about $\mathbb{H}$ is that $[X^n, Y^n] = \left[\begin{pmatrix} 1 & n & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & 1 & n \\ & & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & & n^2 \\ & 1 & \\ & & 1 \end{pmatrix} = Z^{n^2}$

(we say that $\langle Z \rangle$ is quadratically distorted)
Since Z commutes with X, Y, $[X^n[X^n, Y^n]]$ represents the identity, and the naive way to reduce to the trivial word

~~goes takes at least~~ n^3 steps to commute X's past Z's
goes $[X^n,[X^n,Y^n]] \to [X^n, Z^{n^2}] \to \varepsilon$
is sharp up to constants - n^3 steps.
In fact, this $\delta_{\mathbb{H}}(n) \sim n^3$.

What's fascinating about this group is that if we change the group slightly, we get a different DF —

$$\mathbb{H}^5 = \left\{ \begin{pmatrix} 1 & x_1 & x_2 & z \\ & 1 & & y_1 \\ & & 1 & y_2 \\ & & & 1 \end{pmatrix} \middle| \ x_i, y_i, z \in \mathbb{R} \right\}$$

$$\mathbb{H}^5_{\mathbb{Z}} = \langle X_1, X_2, Y_1, Y_2, Z \mid [X_1, Y_1] = Z, [X_2, Y_2] = Z, \text{all other generators commute} \rangle$$

As before, Z is quadratically distorted:
$[X_1^n, Y_1^n] = [X_2^n, Y_2^n] = Z^{n^2}$. But

$\delta_{\mathbb{H}^5_{\mathbb{Z}}}(n) \sim n^2$. Want to try to explain why.

To do that, we need to understand some of the geometry involved.
Let's start with $\mathbb{H} = \mathbb{H}^3$
Remember, I said that I want to apply F-F, like for $\mathbb{R}^n$ — in order to do that, we need a notion of scaling. ~~Since there's~~ ~~a sp of upper~~

Let $s_t : \mathbb{H} \to \mathbb{H}$, $s_t\begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & tx & t^2 z \\ & 1 & ty \\ & & 1 \end{pmatrix}$
If $t \in \mathbb{Z}$ $s_t(X) = X^t$, $s_t(Y) = Y^t$, $s_t(Z) = Z^{t^2}$.

This is an automorphism of $\mathbb{H}$. ~~And if we're careful we can find~~ a Furthermore, $\mathbb{H}$ has a invariant metric compatible with the scaling: ~~at the group st~~ a sub-Riemannian metric

Roughly, a sub-Riemannian metric is a Riemannian metric where some directions have infinite length.

~~Let Let $\Lambda \subset T\mathbb{H}$ be the left invariant bundle s.t. $\Lambda(0) = xy$-plane.~~

The directions with finite length form a ~~distribution~~ ~~left~~ sub-bundle of the tangent bundle. Specifically, they'll form the horizontal bundle $\Lambda \subset T\mathbb{H}$, where
$\Lambda(x,y,z) = \langle dx, dy + x\,dz \rangle$.