2018-10-18

Last time: sub-Riemannian metric on $\mathbb{H}$, ~~sta~~ Hölder maps.

Today: Filling invariants of $\mathbb{H}$ and relationship to the sub-R. metric.

Notation: $(\mathbb{H}, d_0)$ - left-invariant Riemannian metric on $\mathbb{H}$.
$(\mathbb{H}, d_c)$ - left-invariant sub-R metric on $\mathbb{H}$.
$\mathbb{H}_\mathbb{Z}$ - integer Heisenberg group.
Then $\mathbb{H}_\mathbb{Z}$ acts geometrically on $(\mathbb{H}, d_0)$, so $\delta_{\mathbb{H}_\mathbb{Z}} \sim FA_{(\mathbb{H}, d_0)}$, and
Last time, we said that $\delta_{\mathbb{H}_\mathbb{Z}} \sim n^3$. Why?

Upper bound isn't hard:
$$\mathbb{H}_\mathbb{Z} = \langle X, Y, Z \mid [X,Y] = Z, [X,Z] = [Y,Z] = 1 \rangle$$
So commuting X and Y produces Z or Z^{-1}, depending on direction.
And a word $w = XYXYX^{-1}X^{-1}Y^{-1}Y^{-1}$ can be reduced to a word of the form $X^a Y^b Z^c$ by swapping X's and Y's and collecting Z's
- this takes cubically many steps, so $\delta_{\mathbb{H}_\mathbb{Z}} \lesssim n^3$.

Lower bound takes more argument. Idea: $[X^n, [X^n, Y^n]] \rightsquigarrow [X^n, Z^{n^2}]$
~~$\rightsquigarrow$~~ ε takes n^3 steps ~~and that's essentially optimal, beca~~
~~to commute n X's past n^2 Z's and that's essentially optimal.~~
~~and we can't think of how to do better.~~
So, for instance, ets that you can't reduce it without commuting X & Z n^3 times. Can we prove that?

I want to ~~use~~ show this
Two ways, one algebraic, one geometric:
<u>Algebraic</u>: There is a central extension of $\mathbb{H}_\mathbb{Z}$ that counts uses of $[X,Z]$ and $[Y,Z]$.
A central extension of G is a group H with a projection $\pi: H \twoheadrightarrow G$ s.t. $\ker \pi$ is central in H: std example
$$1 \to \mathbb{Z} \to \mathbb{H}_\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}^2$$
$\mathbb{Z} = \langle Z \rangle$; $X \mapsto x$, $Y \mapsto y$, $Z \mapsto 0$

This is part of a sequence. Let $F_{2,k}$ be the free nilpotent group of rank 2 and step k: $F_{2,k} = F_2 / F_2^{(k+1)}$, where
$G^{(0)} = G$, $G^{(i+1)} = [G, G^{(i)}]$.
So $F_{2,1} = F_2/[F_2,F_2] = \mathbb{Z}^2$, $F_{2,2} = F_2/[[F_2,F_2],F_2]$ $= \langle X, Y \mid [X,[X,Y]], [Y,[X,Y]] \rangle$
$= \langle X, Y, Z \mid Z = [X,Y], [X,Z] = [Y,Z] = 1 \rangle$
$= \mathbb{H}_\mathbb{Z}$.

Then

$$F_{2,3} = F_2 / [F_2,[F_2,[F_2,F_2]]] \text{ is a central extension}$$

$$1 \to \mathbb{Z}^2 \to F_{2,3} \to \mathbb{H}_{\mathbb{Z}} \to 1.$$

What does it mean that this counts applications of relations?
Central extensions let you lift curves.
Let $1 \to Z \to H \xrightarrow{\pi} G \to 1$, ~~and~~ be a central ext and $a_1, \dots, a_k$ generate G, let $A_1, \dots, A_k \in H$ be elements s.t. $\pi(A_i) = a_i$. Given a word $w = w(a_1, \dots, a_k)$, let $\tilde{w} = w(A_1, \dots, A_k)$. We say $\tilde{w}$ is a lift of w.

~~Claim~~ To construct this, we chose $A_1, \dots, A_k$ — what if we change them? let ~~$A_1', \dots, A_n'$~~ $A_i' = A_i z_i$ be another choice, let $\tilde{w}' =_H w(A_1', \dots, A_k')$. Then the z_i's ~~are~~ are central, so

$=_H w(A_1 z_1, \dots, A_k z_k) = w(A_1, \dots, A_n) w(z_1, \dots, z_k)$

$=_H \tilde{w} \cdot \prod z_i^{\kappa_i}$ where $\kappa_i = \#$ of a_i's in w.

In particular, if $\kappa_i = 0 \ \forall i$, then $\tilde{w} =_H \tilde{w}'$.

Ex: $1 \to \mathbb{Z} \to \mathbb{H}_{\mathbb{Z}} \to \underset{\langle X, Y \rangle}{\mathbb{Z}^2} \to 1$.

what else? If $w = w(X,Y) =_{\mathbb{Z}^2} 1$ then $n_X = n_Y = 0$ and $\tilde{w} \in \ker \pi = \mathbb{Z}$. Write $w =_{F_2} \prod g_i r_i g_i^{-1}$. Then $\tilde{w} =_{\mathbb{H}_{\mathbb{Z}}} \prod \tilde{g}_i \tilde{r}_i \tilde{g}_i^{-1}$, but $\tilde{r}_i \in \mathbb{Z}$,

so $\tilde{w} = \prod \tilde{r}_i$ where $\tilde{r}_i = [X,Y]^{\pm 1} = Z^{\pm 1}$. So $\tilde{w} = Z^k$, where $k = \#$ of appls of $[X,Y]$ to reduce w, counted w/ sign.

And the same for the next extension: $F_{2,3} = F_2 / F_2^{(4)}$

$= \langle X, Y \mid [X,[X,[X,Y]]], \dots \rangle$

$$1 \to K \to F_{2,3} \to \mathbb{H}_{\mathbb{Z}} \to 1$$

$K \cong$ ~~things that are nonzero in $F_{2,3}$, zero in $\mathbb{H}_{\mathbb{Z}}$~~ $\mathbb{Z}^2$

$= \langle \underset{A}{[X,[X,Y]]}, \underset{B}{[Y,[X,Y]]} \rangle$ (relations in $\mathbb{H}_{\mathbb{Z}}$).

If $w(X,Y) =_{\mathbb{H}_{\mathbb{Z}}} 1$, then $n_X = n_Y$, and

$\tilde{w} = \prod \tilde{g}_i \tilde{r}_i \tilde{g}_i^{-1} = \prod \tilde{r}_i$, where $r_i = [X,[X,Y]]^{\pm 1}$ (and $\tilde{r}_i = A^{\pm 1}$)

or $r_i = [Y,[X,Y]]^{\pm 1}$ (and $\tilde{r}_i = B^{\pm 1}$)

$\Rightarrow \tilde{w} = A^{\#\text{ of apps of }[X,[X,Y]]} B^{\#\text{ of apps of }[Y,[X,Y]]}$

So # of apps of $[X,[X,Y]]$ or $[Y,[X,Y]]$, with sign, doesn't depend on choice of reduction.

In particular, $[X^n,[X^n,Y^n]] =_{E_{2,3}} A^{n^3}$, so $\delta([X^n,[X^n,Y^n]]) \geq n$

Exercise. The theorem behind this is the following:

Thm: The set of central extensions of G by A is in bijective correspondence with the set $H^2(G,A)$.

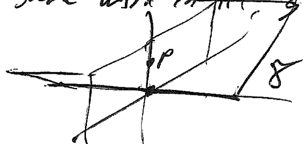
(Here, $H^2(G,A) \cong H^2(X,A)$ for any CW-complex X s.t. $\pi_1(X)=G$, $\pi_n(X)=0 \ \forall n>1$).

<u>Geometric</u>: ~~To let let γ be the lift of a figure~~

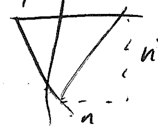
Last time, closed curves in the plane lift to closed curves in $\mathbb{H}$ iff signed area zero. Let γ be the curve

let $\tilde{\gamma}$ be its lift. We claim that $\tilde{\gamma}$ has cubic FA. First let's calc the lift. Here, $\gamma = X^n Y^n X^{-n} Y^{-2n} \ldots$

If we take the same word ~~in $\mathbb{H}$~~, get.. Suppose Σ fills $\tilde{\gamma}$ — how can we bound area Σ from below?



~~With~~ We claim that Σ intersects a lot of different curves. Spec, take a point p like so, draw the line through p parallel to X — this intersects Σ. In fact, if we rotate this $90°$ horiz, we get



a triangle of area n^3. Every line through this triangle intersects Σ.

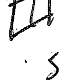
This doesn't immediately imply an area bound, but:

Prop: Any filling of γ cannot be covered by fewer than $n^3/100$ unit balls in $(\mathbb{H}, d_0)$.

Pf: Check: $B_1(0) \subset [-1,1] \times [-1,1] \times [-10,10]$.

Any translation $B_1(x,y,z) = (x,y,z) \cdot B_1(0)$
$\subset (x,y,z) + [-1,1]\cdot(1,0,0) + [-1,1]\cdot(0,1,x) + [-10,10](0,0,1)$

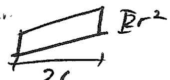
So the projection to the yz-plane is contained in a parallelogram



of area 40. If Σ is covered by k such balls then their projections have area $\geq n^3$, so $40k \geq n^3$. //

~~This works even better for~~
And if we apply this to the CC metric, we get bounds on $\dim_{Haus}$.

Recall: Let $E \subset X$ be a metric space, $k > 0$. Then
$$\mathcal{H}^k(E) = \lim_{\delta \to 0} \inf \{ \sum (\operatorname{diam} U_i)^k \mid E \subset \bigcup U_i,\ \operatorname{diam} U_i < \delta \}$$
(i.e.  into k^2 squares of diam $\approx \frac{1}{k}$)
$$\dim_{\mathcal{H}}(E) = \sup \{ k \mid \mathcal{H}^k(E) > 0 \}.$$

Here, $B_r(0; d_c) \subset [-r,r] \times [-r,r] \times [-r^2, r^2]$
$B_r(x,y,z; d_c) \subset [-r,r](1,0,0) + [-r,r](0,1,x) + [-r^2,r^2](0,0,1)$
projects to ~~a~~  of area $4r^3$.

So if Σ is covered by balls of radius $r_1, \dots, r_k$, then
$$\sum_{i=1}^{k} 4 r_i^3 \geq n^3 \Rightarrow \mathcal{H}^3(\Sigma) \gtrsim n^3$$
$$\Rightarrow \dim_{\mathcal{H}}(\Sigma) \geq 3.$$

And this is powerful — we can do a lot by looking at intersections with a family of curves. Discrete example now, cts later.

~~Ex~~ Thm (Varopoulos) Let ~~X be~~ Γ ~~be the Cayley graph~~
~~$FV^3(V) \leq$~~ $FV^3_{(\mathbb{H}, d_c)}(A) \lesssim A^{4/3}$

Pf: Let Γ be the Cayley graph of $\mathbb{H}_{\mathbb{Z}}$. It suffices to show that if $U \subset V(\Gamma)$, $|U| < \infty$, then $\exists c$ s.t.
$$|\partial U| \leq c |U|^{3/4}. \quad \text{(Exercise: Why?)}$$
where $\partial U = \{ e = (v,w) \in E(\Gamma) \mid v \in U, w \notin U$ or vice versa$\}$.

Suppose that $g \in \mathbb{H}_{\mathbb{Z}}$ is s.t. $|U \cap Ug| \leq \frac{1}{2}|U|$.
Let γ be a geodesic from 1 to g. Then $u\gamma$ is a geod from u to ug and $\exists$ $u_1, \dots, u_{\frac{1}{2}|U|} \in U$ s.t. $u_i \gamma$ crosses ∂U at least once, ~~Each $u\gamma$ crosses ∂U at most no more than~~

Each edge $e \in \partial U$ is part of $\leq \ell(\gamma)$ such curves (first, second, etc), so $|\partial U| \geq \frac{\frac{1}{2}|U|}{\ell(\gamma)}$. Remains to find a short g.
Suppose we pick g randomly from $B_r(0)$.
$\forall u \in U$, $P(ug \in U) \leq \frac{|U|}{|B_r(0)|}$, so $E(|U \cap Ug|) \leq \frac{|U|}{|B_r(0)|} |U|$.
$\exists c$ s.t. $|B_r(0)| \geq c r^4$, so if $r = \left(\frac{2|U|}{c}\right)^{1/4}$, then $\exists g \in B_r(0)$ s.t. $|U \cap gU| \leq \frac{1}{2}|U|$.