Today: Filling invariants of $H_3$ and relationship to the sub-R metric

Notation: $(H_3,d_0)$ - left-invariant Riemannian metric on $H_3$.
$(H_3,d_c)$ - left-invariant sub-R metric on $H_3$.
$H_3$ - integer Heisenberg group.

Then $H_3$ acts geometrically on $(H_3,d_0)$, so $S_{H_3} \sim F_{0}(H_3,d_0)$, and

Last time, we said that $S_{H_3} \sim n^3$. Why?

Upper bound isn't hard:

$H_3 = \langle X,Y,Z | [X,Y] = Z, [X,Z] = [Y,Z] = 1 \rangle$

So commuting $X$ and $Y$ produces $Z$ or $Z^{-1}$, depending on direction.

And a word $w = X^iY^jX^kY^lX^nY^mX^{-i}X^{-j}X^{-k}X^{-l}X^{-n}X^{-m}$ can be reduced to a word of the form $X^\alpha Y^\beta Z^\gamma$ by swapping $X$'s and $Y$'s and collecting $Z$'s - this takes cubically many steps, so $S_{H_3} \le n^3$.

Lower bound takes more argument. Idea: $[X^n,Y^n] \rightarrow [X^n, Z^n]$ takes $n^3$ steps and that's essentially optimal, hence we can't think of how to do better. So, for instance, let's that you can't replace it without commuting $X$'s or $Y$'s $n^3$ times. Can we prove this?

I want to show this.

Two ways, one algebraic, one geometric.

Algebraic: There is a central extension of $H_3$ that counts uses of $[X,Z]$ and $[Y,Z]$.

Central extension $G$ is a group $H$ with a projection $\pi : H \rightarrow G$ s.t.

ker $\pi$ is central in $H_3$ - std example $1 \rightarrow Z \rightarrow H_3 \rightarrow G \rightarrow 1$.

This is part of a sequence - let $F_2$ be the free nilpotent

group of rank 2 and step k: $F_{2,k} = \langle F_{x,0} F_{y,0} \rangle$, where

$F_2 = \langle x, y | [x,y] = 1 \rangle.

F_{2,0} = \langle x, y | [x,y] = 1 \rangle,

F_{2,1} = \langle x, y | [x,y] = 1, [x,[x,y]] = 1 \rangle,

F_{2,2} = \langle x, y | [x,y] = 1, [x,[x,y]] = 1, [x,[x,[x,y]]] = 1 \rangle,

\ldots

F_{2,k} = \langle x, y | [x,y] = 1, [x,[x,y]] = 1, \ldots, [x,[x,[x,[x,y]]]] = 1 \rangle.

H_3 = \langle x, y, z | [x,y] = 1, [x,z] = 1, [y,z] = 1 \rangle.
Then
\[ F_{2,3} = \langle \langle X, Y | [X, [X, [Y]]] \rangle \rangle \] is a central extension
\[ 1 \rightarrow \mathbb{Z} \rightarrow F_{2,3} \rightarrow \mathbb{Z} \rightarrow 1. \]

What does it mean that this connects applications of relations?

Central extensions let you lift curves. Let \( 1 \rightarrow \mathbb{Z} \rightarrow H \xrightarrow{\pi} G \rightarrow 1 \) be a central extension and \( a_1, \ldots, a_n \) generate \( G \), let \( A_1, \ldots, A_k \in H \) be elements such that \( \pi(A_i) = a_i \). Given a word \( w = w(a_1, \ldots, a_k) \), let \( \tilde{w} = w(A_1, \ldots, A_k) \). We say \( \tilde{w} \) is a lift of \( w \).

To construct this, we choose \( A_1, \ldots, A_k \), what if we change them? Let \( A' = A, A_1 = A_1 a_i \) be another choice and let \( \tilde{w}' = \tilde{w}(A', \ldots, A_k) \). Then the \( z_i \)'s are central, so
\[ \tilde{w}' = \tilde{w}(A_1, \ldots, A_k) \pi(z_1, \ldots, z_k) = \tilde{w} \cdot \pi(z_1, \ldots, z_k) \text{ where } \pi(z_i) = \# \text{ of } \mathbb{Z} a_i \text{'s in } w. \]

In particular, if \( a_i = 0 \forall i \), then \( \tilde{w} = H \tilde{w} \).

Ex: \( 1 \rightarrow \mathbb{Z} \rightarrow H \rightarrow \mathbb{Z}^2 \rightarrow 1 \)

If \( w = w(X, Y) = X Y X^{-1} Y^{-1} \) then \( n_X = n_Y = 0 \) and \( \tilde{w} \) is in \( \mathbb{Z} \) iff \( w \) is. Write \( \tilde{w} = \prod_{i=1}^k g_i r_i g_i^{-1} \). Then \( \tilde{w} = \tilde{w} \prod_{i=1}^k g_i r_i g_i^{-1} \), but \( \tilde{w} \in \mathbb{Z} \).

So \( \tilde{w} = \pi^{-1}(g_i) \) where \( \tilde{w}_i = \sqrt{X Y \cdots} = \mathbb{Z}^\pm1 \) and \( \tilde{w} = \mathbb{Z} \mathbb{Z}^\pm1 \).

And the same for the next extension: \( F_{2,3} = F_2 \times F_2 \)
\[ \rightarrow F_{2,3} = \langle \langle X, Y | [X, [X, [Y]]] \rangle \rangle \]
\[ 1 \rightarrow \mathbb{Z} \rightarrow F_{2,3} \rightarrow \mathbb{Z} \rightarrow 1. \]

If \( w(X, Y) = 1 \), then \( n_X = n_Y = 0 \) and
\[ \tilde{w} = \prod_{i=1}^k g_i r_i g_i^{-1} = \prod_{i=1}^k \tilde{w}_i, \text{ where } r_i = [X, [X, Y]]^{r_i} \quad \text{and } \tilde{w}_i = \mathbb{Z}^{r_i}. \]

If \( w(X, Y) = 1 \), then \( n_X = n_Y = 0 \) and
\[ \tilde{w} = \prod_{i=1}^k g_i r_i g_i^{-1} = \prod_{i=1}^k \tilde{w}_i, \text{ where } r_i = [Y, [X, Y]]^{r_i} \quad \text{and } \tilde{w}_i = \mathbb{Z}^{r_i}. \]
\[ \Rightarrow \text{# of apps of } \mathbb{L}_x, [x,y] \]

\[ \text{# of apps of } [x,y] \text{ or } [y,[x,y]], \text{ with } s(n), \]

\[ \text{depends on choice of reduction.} \]

\[ \text{In particular, } [x^n, [x^n, y^n]] = A^n, \text{ so } S([x^n, [x^n, y^n]]) = 1. \]

**Exercise:** The theorem behind this is the following:

**Thm:** The set of central extensions of \( G \) by \( A \) is in bijective correspondence with the set \( H^2(G,A) \).

(Here, \( H^2(G,A) \cong H^2(X,A) \) for any CW-complex \( X \) s.t. \( \pi_n(X) = G, \pi_n(X) = 0 \) \( \forall n > 1 \).)

**Geometric:** Let \( \tilde{G} \) be the lift of a figure last time closed curves in the plane lift to closed curves in \( H \)

if signed area zero. Let \( \gamma \) be the curve \( \square \)

let \( \tilde{\gamma} \) be its lift. We claim that \( \tilde{\gamma} \) has \( \gamma \)

cubic \( PA \). First lets cal the lift. Here, \( \gamma \sim x^n y^n x^{-n} y^{-n} \).

If we take the same unknot, yet...

Suppose \( \Sigma \) fills \( \gamma \) —

How can we bound area \( \Sigma \) from

below?

We claim that \( \Sigma \) intersects a lot of different curves.

Spec: take a point \( p \) like \( 50 \), draw the line through \( \perp p \) to \( x \) — this intersects \( \Sigma \). In fact, if we rotate this \( 90^\circ \),

we get \( \square \) a triangle of area \( n^3 \). Every line through

this triangle intersects \( \Sigma \).

This doesn't immediately imply an area bound, but:

Prop: Any filling of \( \tilde{\gamma} \) cannot be covered by fewer than \( n^3 \) 100

unit balls in \( (H, d_0) \).

Pt: Check: \( B_0 \preceq [-1,1] \times [-1,1] \times [-10,10] \)

Any translation \( B_0(x,y,z) = (x,y,z) \cdot B_0(0) \)

\( = (x,y,z) + [-1,1] \times [-1,1] \times [-10,10] \).

So the projection to \( yz \)-plane is contained in a

parallelogram \( \square \) of area \( 40 \). If \( \Sigma \) is covered by \( k \) such balls

then their projections have area \( \geq n^3 \),

so \( 40k \geq n^3 \). \( \square \)
Recall: Let $E \subseteq \mathbb{R}^k$ be a metric space, $k \geq 0$. Then
\[ H^k(E) = \lim_{\epsilon \to 0} \int_{E} \sqrt{1 + \epsilon^2} \frac{d\lambda}{\epsilon}, \quad \text{where } \lambda \text{ is Lebesgue measure}. \]
(i.e., $E$ is covered by a countable family of disjoint sets $E_i$ such that $\lambda(E_i) < \epsilon$)
\[
\dim_H(E) = \sup \{ k | H^k(E) > 0 \}.
\]

Here, $B_r(x) = \{ y | d(x, y) < r \}$, $\mathbb{R}^k = \bigcup_r B_r(0)$, $\mathbb{R}^k = \bigcup_r B_r(x)$.

So if $E$ is covered by balls of radius $r_1, \ldots, r_k$, then
\[
\frac{1}{k} \sum_{i=1}^{k} r_i^3 \geq \frac{1}{n^3} \implies H^3(E) \geq \frac{n^3}{k} \implies \dim_H(E) \geq 3.
\]

And this is powerful. We can do a lot by looking at intersections with a family of curves. Here's an example.

**Theorem (Vapnik-Chervonenkis)** Let $X$ be a finite set, $A \subseteq \mathbb{R}^3$.

**Proof:** Let $G$ be the Cayley graph of $H_3$. It suffices to show that, if $U \subseteq V(N)$, then $|U| < 2^N$.

Let $U = \{ v \in V(N) | v \notin A \}$.

Suppose that $g \in H_3$ is s.t. $|U \cap Ug| \leq \frac{1}{2} |U|$.

Let $g$ be a geodesic from $1$ to $g$. Then $U \cap Ug$ is a geodesic from $1$ to $g$.

Suppose we pick $g$ randomly from $B_r(0)$.

Each edge $e \in U$ is on at most $2^{l(x)}$ such curves (first, second, $\epsilon$-t, etc.), so $|U| \leq \frac{2^{l(x)}}{2^{l(x)}}$. It remains to find a short $g$.

Suppose we pick $g$ randomly from $B_r(0)$.

For $U \subseteq V(N)$, $\frac{1}{2} |U| \leq \frac{1}{2^{l(x)}} |U|$, so $\frac{|U \cap Ug|}{|U|} \leq \frac{1}{2^{l(x)}} |U|.

\[ E \subseteq \mathbb{R}^3, \quad \exists c \text{ s.t. } |B(c, r)| \geq cr^3, \quad \text{so if } r = \frac{|U|}{2c}, \text{ then } \exists g \in B_r(0) \text{ s.t. } |U \cap Ug| \leq \frac{|U|}{2}, \]