Proposition: Suppose that \( \mathcal{E} \subset (\mathbb{H}, d_{\mathbb{H}}) \) and that \( \gamma : [0, 1] \to \mathbb{H} \) is a curve such that intersects \( \mathcal{E} \) transversally in the following sense: \( \exists \varepsilon > 0 \text{ s.t. } \gamma' : [0, 1] \to \mathbb{H}, d(\gamma'(t), \gamma(t)) > \varepsilon \) for all \( t \in [0, 1] \), then \( \gamma \) also intersects \( \mathcal{E} \). Then \( \dim(\mathcal{E}) \geq 3 \)

Proof: We can approximate \( \gamma \) by a unit curve that has unit speed w.r.t. \( dc \). This is still transverse to \( \mathcal{E} \).

For \( x \in \mathbb{H} \), \( r > 0 \), let \( W(x, r) = \{ y \in \mathbb{H} \mid d(x, y) < r \} \). Let us bound the volume measure of \( W(x, r) \).

Let \( v_i = \chi(\frac{i}{r^3}) \) for \( i = 0, \ldots, r^3 \). If \( h \in W(x, r) \), then \( h \in B(x, r) \).

Let \( I = \{ i \} \) for \( 0 \leq i \leq r^3 \).

Then \( v(W(x, r)) \leq \sum v_i(B(x, r)) \leq \frac{r^3}{r^3} \cdot \sum v_i(B(x, r)) = r^3 \).

By transversality, \( \mathcal{E} \) intersects \( B(0,r) \).

By transversality, \( B_1(0,r) \) intersects \( \mathcal{E} \).

By transversality, \( B_2(0,r) \) intersects \( \mathcal{E} \).

Thus, \( \mathcal{E} \) intersects \( B(0,r) \).

So, what do we know?

- Any filling of \( \gamma \) has area \( \geq r^3 \).
- Any filling of \( \gamma \) has area \( \geq r^3 \).

Q: How are these facts related?

\( \gamma \) is a curve that intersects \( \mathcal{E} \) transversally.

The answer is that there is no single curve satisfying the transversality condition.

Nevertheless, ...

Idea: \( (\mathbb{H}, d_{\mathbb{H}}) \) is a limit of scaling of \( (\mathbb{H}, d_{\mathbb{H}}) \).

What does this mean? Neal asked.

Def: If \( X \) is a metric space, \( A, B \subset X \) are compact sets, the squared distance between \( A \) and \( B \) is

\[ d^2_{\text{Haus}}(A, B) = \inf_{\rho \in \mathcal{R}(A, B)} \sum d_{\mathbb{H}}(a, b)^2 \text{ for } a \in A, b \in B. \]

Def: Let \( X, Y \) be compact metric spaces. The Gromov-

Hausdorff distance is \( d_{\text{GH}}(X, Y) = \inf d_{\text{Haus}}(X, Y, M) \),

where \( M \) varies over metric spaces containing \( X, Y \) isometrically.
Ex: Let $D = \max \{ \delta(x, y) : x, y \in Y \}$, let $M = (X \times Y, d_{GH})$, where $d_{GH} = \delta + d_{X}$. Since there is no correct choice of such a space $Y$. Let $M = X \times Y$ contains both $X \subseteq X \times Y$ and $Y \subseteq X \times Y$. 

Another way to define dist between metric spaces — how close we can find a map that’s $\varepsilon$-close to an isometry. Let $D = \delta(x, y)$. There is a map $\psi: X \to Y$ such that $d_{GH}(\psi(x), \psi(y)) - d_{X}(x, y) \leq \varepsilon$. 

Proof: Embed $X \times Y$ in $M$, let $d_{GH} = D$. 

For $x \in X$, let $\psi(x)$ be the $\varepsilon$ closest to $x$. 

Then, for all $x$, $\psi(x)$ is $D$ apart from $x$, so the lemma follows by $\Delta$ reg.

Exercise: Prove the converse: $\forall \varepsilon$, $d_{GH}(X, \psi(X)) \leq \max \{ d_{X}(x, \psi(x)) \}$. 

So, small $GH$ distance = “almost isometric.”

We define $\varepsilon$-GH-convergent for compact spaces is usual way, and define convergence for pointed non-compact spaces as follows:

Def: If $(X, x_0)$, $(X, x_1)$, ... and $(Y, y_0)$ are proper metric spaces, we say that $(X, x_0) \xrightarrow{\varepsilon} (Y, y_0)$ if $\forall \varepsilon > 0$, $B_R(x_0, \varepsilon) \xrightarrow{\varepsilon} B_R(y_0, \varepsilon)$.

Ex: grids in $IR^2$, $B_{R}(x_0, \varepsilon) \xrightarrow{\varepsilon} B_{R}(y_0, \varepsilon)$.

But also: $(H, d_{H}) \xrightarrow{\varepsilon} (L, d_{L})$.

In fact, we can construct almost-isometries: Let $\varepsilon(x, y) = d_{X}(x, y)$.

Claim: $\forall u, v, \forall \varepsilon \lim_{n \to \infty} \frac{d_{X}(u, v n)}{d_{X}(u, v)} \to d_{X}(u, v)$.

Why? Let's make the metric: Recall, $d_{X}$ is left-invert $R_{\varepsilon}$, so the left metric fields $X = (1, 0, 0)$, $Y = (0, 1, x)$ are left-invert so $\varepsilon$ fields $Z = (0, 0, 1)$.
\[
g = dx^2 + dy^2 + (dz - x dy)^2.
\]
If we pull back under \( s_n \),
\[
s_n^*g = \frac{1}{n^2} dx^2 + n^2 dy^2 + (n^2 dz - nx ndy)^2
\]
horizontal vectors have odd length, \( Z \) has length going to \( 
\infty \).
So, in an intrinsically sense, convergence, Distance \( d_n \) takes over.
Ex: Prove the claim about distances \( d_1 \) vs. \( d_n \).

This is a remarkable generalization of this idea.
Okay: These are limits of metric spaces. But there's a remarkable generalization of this idea; ultralimits.

Ultralimits:

Thm: If a linear functional \( L : l_\infty \to \mathbb{R} \) s.t. \( \forall \xi \in \ell_\infty (k_i) \exists \xi, L(k_i) \) is a point of accumulation of \( \xi \).
In particular, if \( (k_i) \) is convergent, then \( L((k_i)) = \lim k_i \).

This is tricky even just for \( 0-1 \) sequences:
\[
L(0, 1, 0, 1, \ldots) = 0
\]
\[
L(1, 1, 0, \ldots) = 1
\]
but they can't be the same because \( L((1, 1, \ldots)) = 1 = L((0, 1, \ldots)) + L((1, 0, \ldots)) \).

Once you define \( L \) on \( 0-1 \) seqs, done, And we can do that \( \forall A_0 \).

Def: A filter is a subset \( w \subset \mathbb{N} \) s.t.:
- \( w \) is closed under finite intersections.
- If \( A \in w \) and \( B \supset A \) then \( B \in w \).
- \( \emptyset \notin w \) and \( \mathbb{N} \notin w \).

A filter is an ultrafilter if \( \forall A \subset \mathbb{N}, \text{either } A \in w \text{ or } B \subset \mathbb{N} \setminus A \notin w \).

Idea: Finite, additively, measure \( w \), values \( w(A) \) in \( 0-1 \) valued measure:
- if \( A, B \subset \mathbb{N} \) and \( A \wedge B = \emptyset \), then
- \( w(A) + w(B) = w(A \cup B) \).
- \( 1 \) if \( A \wedge B = \emptyset \)\( \Rightarrow A \in w \),
- \( 0 \) if \( A \wedge B = \emptyset \) \( \Rightarrow B \in w \).

There's an obvious \( 0-1 \) valued measure - point measure.
A principal ultralimit is a filter of the form
\( w = \{ A \in \mathbb{N} \wedge A \cap A = \emptyset \} \).
Prop: \( \exists \) a non-principal ultralimit.
Prop: \( \forall \alpha \), \( \alpha = \text{ a \text{ cardinal of Choice}.} \)
Let \( \omega \) be a non-principal ultrafilter. Note that \( \omega \) contains no finite sets (if \( x_0, \ldots, x_n \in \omega \), then \( \{x_0, \ldots, x_n\} \not\in \omega \) for some \( i \), and then \( x_i = x_0 \).

Define \( \lim x_i \) using \( \omega \). We can use this to define an \( \omega \)-limit.

Roughly, we want to define \( \lim x_i = \int x_i \, d\omega \).

(And I think that's a serious definition, but let me give the full theorem.)

If \( (x_i)_i \in \mathcal{A}_\omega \), \( y \in \mathbb{R} \), we say \( \lim x_i = y \) if \( \forall \varepsilon > 0 \),

\[ \exists i \in \mathbb{N} : |x_i - y| < \varepsilon \in \omega. \]

Then:

- \( \lim x_i \) is unique if it exists.
- \( \lim x_i \) is an accumulation point.
- \( \lim x_i \) is linear.
- \( \lim x_i \) exists \( \forall (x_i)_i \in \mathcal{A}_\omega \).

\[ \text{Def. Bifurcation} \]

We also define ultralimits of metric spaces.

Let \( (X; x_i) \) be a sequence of \( \mathcal{A}_\omega \) metric spaces.

Define \[ \ell^\omega (x_i) = \left\{ \text{bounded sequences} \, v \, : \, x_i(v) \right\} \] as \[ \ell^\omega (x_i) = \left\{ \text{bounded sequences} \, v \, : \, x_i(v) \right\} \] and \[ \ell^\omega (d(x_i, x_j)) = \ell^\omega (d(x_i, x_j)) \]

Given \( (a_i, b_i)_i \in \ell^\omega (x_i), \,(d(a_i, b_i)) = \ell^\omega (d(R)). \]

Define \( X^\omega = \left( \ell^\omega (x_i), \, d^\omega \right) \)

where \( d^\omega (a_i, b_i) = \lim d(x_i, a_i, b_i) \), and \( (a_i) \sim (b_i) \Leftrightarrow d^\omega ((a_i, b_i)) = 0. \)

Check: Then \( X^\omega \) is a metric space which we call the ultralimit of the \( x_i \).

Ex: \( \ell^\omega (X_i) = (\mathbb{R}, \delta) \xrightarrow{w} \mathbb{R} \) (obviously it should, but...)

\[ \ell^\omega (x_i) \xrightarrow{\ell^\omega} \lim x_i \xrightarrow{w} \mathbb{R} \]

\[ (x_i) \sim (y_i) \Leftrightarrow \lim (x_i - y_i) = 0 \Rightarrow \lim x_i = \lim y_i, \]

so \[ d^\omega ((x_i, y_i)) = \lim |x_i - y_i| = |\lim x_i - \lim y_i| \]

so \( (x_i)_i \xrightarrow{w} X \) if \( \ell^\omega \) is a well-defined con.

More generally if \( \ell^\omega \) is well-defined con.
Last time: Ultralimits of metric spaces:
- \((X_i, x_i)\) a seq of metric spaces.
- \(\lambda_w(X_i) = \{a \in X_i : \exists C > 0 s.t. \ d(a, x_i) < C \forall i\} \)
- \(\lim_{w} X_i = \{a \in \lambda_w(X_i) : \lim_{w} d(a, x_i) = 0\}\)

where \(d_w((a_i), (b_i)) = \lim_w d(a_i, b_i) \) and \((a_i) \sim (b_i) \iff d_w((a_i), (b_i)) = 0\).

Write \([a_i]\) for the equivalence class of \((a_i)\).

Today: Asymptotic cones: Let \((X, d)\) be a metric space, \(x \in \mathbb{R}\) seq of basepoints, \(w\) an ultralimit.

Let \(\text{cone}_w(X, x) = \lim_{w} (X, d, x)\).

This depends on \(w\) and on \(x\). Sometimes you'll see \(\text{Core}_w(X, x, d) = \lim_{w} (X, d, x)\) where \(d \to 0\),

but you can get the same 'effect' by changing \(w\).

Further dependence on basepoint is often unimportant. The only place the basepoint appears is in defining \(\lambda_w(X)\).

So, if we move basepoints by a bounded dist, no change.

Frequently note: This is particularly useful if \(G\) acts.

- If \(G\) acts geometrically on \(X\), then this is independent of basepoint.
- If \(f : X \to Y\), then \(f([a_i]) = [f(a_i)]\) is a well-defined, bi-Lipschitz homeomorphism. (Exercise)

So \(\text{cone}_w(G)\) is a group invariant.

In general, cone can be complicated. Calculate:

**Ex.** First, exercise (Lemma):

Lemma: \(I \iff (X, x) \to Y\) and \(Y\) is proper then
\(\lim_{w} X_i = Y\).

Proof: Since \(Y\) is proper, bounded seq in \(Y\) is \(\text{ultralimit}\).

Let \(w : \mathcal{U} \to X \to Y\) be a sequence almost-isometries

so that \((p_i(x_i)) \in \lambda_w(Y). \) Then \(\lambda_w(x_i) \in \lambda_w(Y)\).

Ex. Then \(\lim_{w} \lambda_w(x_i) \to \lambda_w(Y)

Exercise. \(\lim_{w} X_i \to Y\)

\(\lambda_{wa}(a_i) \to \lambda_{w}(b(a_i))\) is an isometry.

So: \(\text{cone}_w(\mathbb{R}^n) = \mathbb{R}^n\)
\(\text{cone}_w(\mathbb{H}, d_0) = (\mathbb{H}, d_0)\).
More generally, Thm (Gromov): If G is torsion-free, nilpotent, then $(G, d)$ acts on $X$ where $X$ is a sub-Riemannian manifold.

In fact, he origin of asymptotic cones is a sort of converse to this statement - they originated Gromov's proof of the polynomial growth theorem.

Suppose $G$ is finitely generated.
Growth of $G$ is $\Gamma(r) = \# B_r(1)$ with respect to the word metric.

Thm (Gromov): If $\Gamma(r)$ is bounded by a polynomial, then $G$ has a finite-index nilpotent subgroup.

While proof is too complicated for now but let me show you one ingredient:

Prop: If $\Gamma(r)$ is bounded by a polynomial, then $\mathcal{C}$ is proper metric space with a transitive isometry group.

Proof: Let $n > 0$, let $e_n$. Suppose $\Gamma(\epsilon) \leq e_n$. Claim: $\forall n \exists B_n(n) > 0$ s.t. $B_n(B_n(1))$ can be covered by $B_n(n)$ balls of radius $\frac{1}{n}$.

Let $x$ be a $\frac{1}{n}$-net in $B_n(1)$. Then $B_n(\alpha_i)$ cover $B(1)$ and $B_n(\alpha_i)$ are disjoint balls.

So $B_n \not\subseteq B_r(1 + \frac{1}{n}) \Rightarrow \Gamma(r(\frac{1}{2n})) \geq m \frac{r^2}{\frac{1}{2n}} \Rightarrow m \leq \frac{c r^2 n}{c \frac{1}{2n}}$. 

Hence, let $\alpha_0, \alpha_1, \ldots, \alpha_k$ be pts s.t. $B_n(\alpha_i)$ covers $B(1)$.
Then $\mathcal{B} \subseteq B_1([1]) \subset \text{Con}_{\epsilon_0} G$.

Suppose $[b_i] \subset B_1([1]) \subset \text{Con}_{\epsilon_0} G$.
Then $\mathcal{B} \subseteq \mathcal{B}_i([1]) \subset \text{Con}_{\epsilon_0} G$.

From $\mathcal{B} \subseteq \mathcal{B}_i([1]) \subset \text{Con}_{\epsilon_0} G$, $b_i \subset B(1)$ for $i = 1, 2, \ldots, m$. 

Let \( R_k = \frac{1}{\ln k} \). Then \( d_{h}(\mathcal{B}_k, \ell_{\infty}(k)) \leq \frac{1}{k} \).

\[ d_{h}(\mathcal{B}_k, \ell_{\infty}(k)) \leq \frac{1}{k} \]

As a compact subset, \( \mathcal{B}_k \) is similarly for any ball.

Furthermore, let \( \mathcal{B}(x, r) \) be a ball. Then \( \mathcal{B}(x, r) \subseteq \mathcal{B}(x, 2r) \). Hence, \( \mathcal{B}(x, r) \) is compact.

Let \( \mathcal{B}_n \) be a sequence of balls. Then \( \mathcal{B}_n \subseteq \mathcal{B}_{n+1} \). Hence, \( \mathcal{B}_n \) is compact.

\[ \mathcal{B}(x, r) \subseteq \mathcal{B}(x, 2r) \]

\[ \mathcal{B}_n \subseteq \mathcal{B}_{n+1} \]

On the other hand, if \( \mathcal{B}(x, r) \) is not compact, then \( \mathcal{B}(x, r) \) is not a compact subset.

Lemmas. If \( \mathcal{B}(x, r) \) is not compact, then \( \mathcal{B}(x, r) \) is not proper. Lemma: If \( \mathcal{B}(x, r) \) is compact, then \( \mathcal{B}(x, r) \) is proper.

Let \( \mathcal{B}(x, r) \) be a ball. Then \( \mathcal{B}(x, r) \) is compact.

\[ \mathcal{B}(x, r) \subseteq \mathcal{B}(x, 2r) \]

\[ \mathcal{B}_n \subseteq \mathcal{B}_{n+1} \]

Let \( \mathcal{B}_n \) be a maximal \( \mathcal{B}_n \)-net in \( \mathcal{B}(x, r) \). Then \( \mathcal{B}_n \) is compact.

Consider \( \mathcal{B}(x, r) \) and \( \mathcal{B} \). Then \( \mathcal{B}(x, r) \subseteq \mathcal{B} \). Hence, \( \mathcal{B}(x, r) \) is compact.

Since \( \mathcal{B}_n \) is compact, there are only finitely many \( \mathcal{B}_n \)-nets in \( \mathcal{B}(x, r) \). Then \( \mathcal{B}_n \) is compact.

\[ \mathcal{B}(x, r) \subseteq \mathcal{B} \]

\[ \mathcal{B}_n \subseteq \mathcal{B}_{n+1} \]

Since \( \mathcal{B}_n \) is compact, there are only finitely many \( \mathcal{B}_n \)-nets in \( \mathcal{B}(x, r) \). Then \( \mathcal{B}_n \) is compact.
In fact, it is more than that. Let $x \in X$. For $x \in [0,1]$, let

$$x = \sum_{i=1}^{\infty} a_i / i! \cdot \frac{1}{i!}.$$ 

If $x \notin B$, then $d(x, y)$ = 2 for any uncountably many points in $B$, separated by $\frac{1}{i}$. 

Ex: $X = \mathbb{R}$ 3-regular tree.

- Exponential growth.
- Note $\text{Core}_w(X)$ is the universal rooted tree.
- $\text{Core}_w(X)$ is a metric tree. For any two points, there are no simple closed curves.
- Lemma: $\text{Core}_w(X)$ contains no simple closed curves. That is, $\text{Core}_w(X)$ is a metric tree.

Proof:

1. Suppose $x \in X$, $\lambda : [0, \infty) \to T$ be the geodesic rays with $\lambda(0) = \lambda(0) = 0$.

Let $d_w(t) = d(\lambda(t), \lambda(0))$. These are geodesics.

2. Note $\text{Core}_w(T)$.

We have $d_w(t) = d(\lambda(t), \lambda(0)) = 2t$, so $d(X(1), X(1)) = 2(1-r)$.

3. There are uncountable many geodesics based at $V$, so we can uncountably many disjoint geodesics from $V$.

Further, $\text{Isom}(T)$ is transitive.

4. Further, $\text{Isom}(T)$ acts cocompactly on $T$.

$\text{Isom}(\text{Core}_w(X))$ acts transitively on $\text{Core}_w(T)$.

5. Uncountably many rays from every point.

Further:

- $\text{Isom}(T)$ acts cocompactly on $T$.

$\Rightarrow$ $\text{Isom}(\text{Core}_w(X))$ acts transitively on $\text{Core}_w(T)$.

- Uncountably many rays from every point.