~~Combinatorial~~ $\longleftrightarrow$ ~~geometric~~ 2022-02-15

~~Last time: group presentations~~ $\longleftrightarrow$ ~~fundamental groups~~

~~word problem~~ $\longleftrightarrow$

Last time: Discretization:

geometry		combinatorics
curves	$\longleftrightarrow$	words
discs	$\longleftrightarrow$	van Kampen diagrams.
$\delta_k(L) > e^{e^{e^L}}$	$\longleftrightarrow$	unsolvability of the word problem
	$\longleftrightarrow$	~~unsolvability of the isomorphism problem~~ (is $G \cong \{1\}$?)

Q: if the groups w/ unsolvable word problem have complicated presentations why would we ever encounter them?

1 — This makes other problems unsolvable:
Embedding lemma: Given $G = \langle g_1, \dots, g_n \mid r_1, \dots, r_s \rangle$, $w \in F(g_1, \dots, g_n)$, we can add generators, relations to G to get G_w s.t. $G_w \cong \{1\} \Leftrightarrow w =_G 1$.

Thus, triviality problem is unsolvable $\Leftrightarrow$ simple connectedness is unsolvable. $\Rightarrow$

~~Prop: Further, let G have unsolvable WP, enumerate the words $w_1, w_2, \dots$ Then $X_{G_{w_i}}$ has $\leq i$ cells.~~

there's no computable fn. f s.t. if X has $\leq N$ simplices then $\delta_X(10) < f(N)$. (otherwise, we could check whether $G \cong 1$ by checking whether every generator of $\pi_1(X_G)$ is null-htpic.)

$\Rightarrow$ classification of n-manifolds is impossible for $n \geq 4$
Given G, embed $X_G \subset \mathbb{R}^5$. Let U be a nbhd of X_G s.t. U def. retracts to X_G.
Then ∂U is a mfld and $\pi_1(\partial U) \cong \pi_1(X_G) = G$.
$\Rightarrow$ (Novikov) homeomorphism problem is unsolvable for $n \geq 5$
$\Rightarrow$ determining whether a complex is a manifold is unsolvable for $n \geq 6$.

2 — If it's impossible to recognize S^6, there must be very complicated manifolds homeo to S^6.

Thm (Nabutovsky–Weinberger): Let $\Omega(S^6) = \{$Riemannian metrics on S^6 with $|K| \leq 1\}$, Consider diam: $\Omega(S^6) \to \mathbb{R}$.
Then D has infinitely many local minima. In fact, if f is computable, then $\exists\, f(D)$ many local minima X s.t. $\operatorname{diam}(X) = D$

~~and any path from X to~~ and s.t. if $Y \in \mathbb{R}$, $\dim(Y) < \dim(X)$
then any path from X to Y goes through some Z with $\dim(Z) > \dim(X)$

~~But I want to go in a different direction.~~

~~Higher orders~~ Higher-order filling functions:

Q: Given an n-dimensional surface in X, what's the $(n+1)$-dimensional filling volume?

A few ways to formalize this, but we'll use a homological formulation:

$$FV^{n+1}(\alpha) = \inf_{\partial B = \alpha} \operatorname{mass}(B). \qquad FV^{n+1}_{X,cell}(V) = \sup FV^{n+1}(\alpha)$$

where $\alpha \in B_n(X)$, $\operatorname{mass}(\alpha) \le V$.

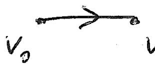
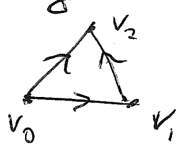
—but what's α? We can do this in a couple ways:

Simplicial: Let X be a simplicial complex, $\mathcal{F}^n(X) = \{n\text{-simplices}\}$.

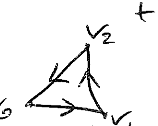
$$C_n(X) = \{\text{formal sums of simplices}\}$$
$$= \{ \sum_{i=1}^{k} a_i [\delta_i] \mid a_i \in \mathbb{R},\ \delta_i \in \mathcal{F}^n(X) \}.$$

~~Fix a total order on $\mathcal{F}^0(X) = V(X)$ — then each we can write each $\delta \in \mathcal{F}^n(X)$~~
Fix a total order on $\mathcal{F}^0(X) = V(X)$. Then we can write $\delta \in \mathcal{F}^n(X)$ uniquely as $\langle v_0, \dots, v_n \rangle$ where the v_i's are in order.

Define $\partial : C_n(X) \to C_{n-1}(X)$ to be the linear map
$$\partial \underbrace{\langle v_0, \dots, v_n \rangle}_{\delta} = \sum_i (-1)^i \langle v_0, \dots, \hat{v}_i, \dots, v_n \rangle$$
(faces of δ with appropriate signs)

We draw:  

$$\partial \langle v_0, v_1, v_2 \rangle = \langle v_1, v_2 \rangle - \langle v_0, v_2 \rangle + \langle v_0, v_1 \rangle$$

Then $\partial \circ \partial = 0$ 

$$B_n(X) = \{ \partial \alpha \mid \alpha \in C_{n+1}(X) \}$$
$$Z_n(X) = \{ \alpha \in C_n(X) \mid \partial \alpha = 0 \} \supset B_n(X).$$
$$H_n(X) = Z_n / B_n \qquad \operatorname{mass}\left(\sum_{i=1}^{k} a_i [\delta_i]\right) = \sum |a_i|.$$

And we can define $FV^{n+1}(\alpha) = \inf \operatorname{mass}(B)$, $FV^{n+1}_X(V) = \sup FV^{n+1}(\alpha)$

~~Problem~~ Q: Take $\mathbb{R}^n$, ~~equip with std~~ ~~Take unit grid~~ Take $\mathbb{R}^n$, equip w/ std structure. What is $FV^n(V)$?

~~We need~~ Higher-dim isoperimetry in $\mathbb{R}^n$:

We expect ~~$FV^{k+1}_{\mathbb{R}^n}(V) \sim V^{\frac{k+1}{k}}$ (round sphere). Why?~~
~~Ex~~ a ~~sphere~~ ~~sub~~ closed k-surface of volume V in $\mathbb{R}^n$ bounds a $k+1$)-surface of volume $\lesssim V^{\frac{k+1}{k}}$. Why? Scaling – ~~suppose every suppose~~ ~~given a surface of~~ rescale by $V^{1/k}$ to get a surf of volume 1. Fill, scale back up.

So: Why is there a uniform bound on filling volume for surfaces of vol 1?
First idea: Coning again – but doesn't work b.c. diameter
First: We need some new defs – that argument doesn't work for simplices

So: Lipschitz singular chains are formal sums of Lipschitz maps ~~maps~~ of simplices

$$C_n^{Lip}(X) = \{ \sum_{i=1}^{k} a_i [\sigma_i] \mid a_i \in \mathbb{Z},\ \sigma_i : \Delta^n \to X \text{ Lipschitz} \}.$$

Write $\Delta^n = \langle e_0, \dots, e_n \rangle$

$$\partial : C_n^{Lip}(X) \to C_{n-1}^{Lip}(X).$$

$$\partial [\sigma] = \sum_{i=0}^{n} (-1)^i [\sigma|_{\langle e_0, \dots, \hat{e}_i, \dots, e_n \rangle}].$$

Again, $\partial\partial = 0$. Thm: Let X be a simplicial complex. ~~or a CW complex with nice metric~~ Then $H_n^{Lip}(X) \cong H_n^{cell}(X)$

If $f : X \to Y$ is Lipschitz, it induces a map

$$f_\# : C_n^{Lip}(X) \to C_n^{Lip}(Y)$$

$$f_\#([\sigma]) = [f \circ \sigma].$$

Define $\operatorname{mass}(\sum_{i=1}^{k} a_i [\sigma_i]) = \sum_i |a_i| \operatorname{vol}(\sigma_i)$.

Then, as before, define $FV^{n+1}_{X, Lip}(\alpha) = \inf_{\beta \in C_{n+1}^{Lip}(X)} \partial\beta = \alpha$.

$$FV^{n+1}_{X, Lip}(V) = \sup_\alpha FV(\alpha).$$

Q: How are these related to simplicial? Clearly, $C_n^{\Delta}(X) \subset C_n^{Lip}(X)$.

~~Prop: Let X~~ Def: We say $A, B \in C_n^{Lip}(X)$ are <u>thin-equivalent</u> if $\exists\ C \in C_{n+1}^{Lip}(X)$ s.t. $\partial C = A - B$ and $\operatorname{mass} C = 0$.

Exer: Suppose $\sigma : \Delta^n \to X$, $f : \Delta^n \to \Delta^n$ ~~bi-Lipsch~~ Lipschitz map preserving $\partial\Delta^n$. Then $[\sigma]$ is thin equiv to $[\sigma \circ f]$.

In fact, X an n-dim simplicial complex.

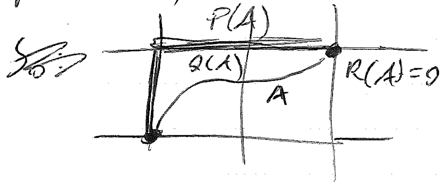
Prop: Suppose $A \in C_n^{Lip}(X)$ and $\partial A \in C_{n-1}^{\Delta}(X)$. Then $\exists B \in C_n^{\Delta}(X)$ s.t. ~~is thin-equivalent to~~ A is thin-equiv to B.

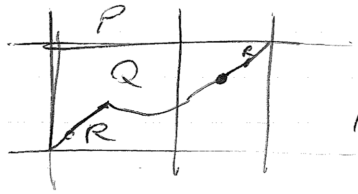
(Namely, ~~for each~~ the degree $\deg_\delta(A)$ is well-defined for every $\delta \in T^n(X)$. - let $B = \sum \deg_\delta(A)\delta$.)

Thm (Federer - Fleming): let X be a finite-dimensional simplicial complex (or a space bilipschitz equivalent to a simplicial complex). $\exists c > 0$ s.t. $\forall A \in C_n^{Lip}(X)$, $\exists P(A) \in C_n^{\Delta}(X)$, $Q(A) \in C_{n+1}^{cell}(X)$, $R(A) \in C_n^{cell}(X)$ such that:

- $A = P(A) + \partial Q(A) + R(A)$
- mass $P(A) \leq c$ mass(A) (approx not too big)
- mass $Q(A) \leq c$ mass (A) (approx not far from A)
- mass $R(A) \leq c$ mass(∂A).

~~In particular~~, If $\partial A \in C_{n-1}^{\Delta}(X)$, we can take $R(A) = 0$.





Pf later, but use now:

In particular: $\forall n, k$, $k \leq n-1$, $\exists \alpha$ s.t. $FV_{\mathbb{R}^n}^{k+1}(V) \leq \alpha V^{\frac{k+1}{k}}$. Consider cell struct on $\mathbb{R}^n$ w/ unit grid

~~Pf: Let c s.t. mass$(P(A)) \leq c$ mass(A) for all $A \in C_k^{Lip}(\mathbb{R}^n)$~~

Pf: Let c s.t. mass $(P(A)) \leq c$ mass(A) $\forall A \in C_k^{Lip}(\mathbb{R}^n)$. Let m be vol of smallest k-cell of $\mathbb{R}^n$.

Given $A \in C_k^{Lip}(\mathbb{R}^n)$, $\partial A = 0$, mass$(A) = V$, let

$$s: \mathbb{R}^n \to \mathbb{R}^n, \quad s(x) = \left(\frac{m}{2cV}\right)^{1/k} x.$$

Then mass $\hat{A} = s_\#(A) = \frac{\text{mass} A}{2cV} = \frac{m}{2c}$.

Therefore, mass $P(s_\#(A)) = P(\hat{A}) \leq \frac{m}{2} < m$. So $P(s_\#(A)) = 0$.

~~$s_\#(A) = P(s_\#(A)) + \partial Q(s_\#$~~

$$\hat{A} = \underset{=0}{P(\hat{A})} + \partial Q(\hat{A}) + \underset{=0}{R(\hat{A})}$$

$$\hat{A} = \partial Q(\hat{A}) \text{ and mass } Q(\hat{A}) \leq c \text{ mass}(\hat{A}) = \frac{m}{2}.$$

And $\partial(s^{-1}_\#(Q(\hat{A}))) = s^{-1}_\#(\hat{A}) = A$, with

$$\text{mass}(s^{-1}_\#(Q(\hat{A}))) \leq \frac{m}{2} \cdot \left(\frac{2cV}{m}\right)^{\frac{k+1}{k}} \approx V^{\frac{k+1}{k}}. \quad //$$

Overflow: sketch proof of Fed-Flem.