2021-09-26

Last time: Examples of topological spaces.

To finish, two more: Let X be a set

- Def: A <u>sub-basis</u> of a topology on X is a set $\mathcal{S} \subset P(X)$ s.t. $\forall x \in X, \exists S \in \mathcal{S}$ s.t. $x \in S$.

~~The topology generated by $\mathcal{S}$ is the set~~ ~~$T = \{$arbitrary unions of~~

If $\mathcal{S}$ is a sub-basis, then $\mathcal{B} = \{$finite intersections of elements of $\mathcal{S}\}$ is a basis. We call $T = \{$arbitrary unions of finite intersections of elements of $\mathcal{S}\}$ the <u>topology generated</u> by $\mathcal{S}$.

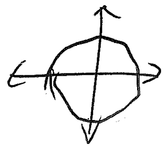
Ex: $\mathcal{S} = \{(-\infty, a) \mid a \in \mathbb{R}\} \cup \{(a, \infty) \mid a \in \mathbb{R}\}$ generates the std top on $\mathbb{R}$.

$\mathcal{S} = \{X \setminus \{x\} \mid x \in X\}$ generates the cofinite topology on X

- <u>Subset topology</u>: Let X be a topological space with topology T. If $Y \subset X$, then $T' = \{Y \cap U \mid U \in T\}$ is called the <u>subset topology</u> on Y.

Ex: Gives us a way to define ~~the~~ e.g. circles and spheres:

$S^n = \{v \in \mathbb{R}^{n+1} \mid \|v\| = 1\} \subset \mathbb{R}^{n+1}$ with the subset topology.



~~Start working w/ top. spaces.~~

Today: ~~Start working with topologies~~ - maps between top. spaces, idea of a topological property

- First some terminology: If X is a top. space, a set $S \subset X$ is <u>closed</u> $\iff$ $X \setminus S$ is open ($A \setminus B = \{a \in A \mid a \notin B\}$)

(Standard note: A set may be open, closed, both, or neither)

Then: - $\emptyset, X$ are closed.

~~- arbitrary intersections of closed set~~

- finite unions of closed sets are closed:

~~Because $(X \setminus S) \cup (X \setminus T) = X \setminus (S \cap T)$~~

~~- Arbitrary unions, intersections of closed~~

Pf: If S, T are closed, then $X \setminus S$, $X \setminus T$ are open. And $X \setminus (S \cup T) = (X \setminus S) \cap (X \setminus T)$ is open, so $S \cup T$ is closed.

- arbitrary intersections of closed sets are closed.

Pf: If S_α is closed for all α, then $X \setminus S_\alpha$ is open.

Claim: $\bigcap_{\alpha \in A} S_\alpha$ is closed

$X \setminus (S \cap T) = (X \setminus S) \cup (X \setminus T)$

Check: $X \setminus \bigcap_{\alpha \in A} S_\alpha = \bigcup_{\alpha \in A} (X \setminus S_\alpha)$ is open.

- If S is closed, then S is closed under limits. I.e., if $s_i \in S$ for all i and $s_i \to x$, then $x \in S$ (Exercise).

Given a set $S \subset X$, we define the closure of S to be the smallest closed set containing S. What does that mean? How do we know there's a smallest one? Define:

$$\overline{S} = \bigcap_{\substack{T \text{ closed} \\ S \subset T}} T$$

This is closed (arb. intersection) and minimal (every closed set containing S also contains $\overline{S}$)

How do we tell what's in the closure? Def: Let $x \in X$. A neighborhood of x is an open set U s.t. $x \in U$.

Prop: If $S \subset X$ and $x \in X$, then:

* $x \in \overline{S} \iff$ every neighborhood of x intersects S

$\iff$ every basis element containing x intersects S.

~~Two~~ facts:

~~Pf~~ 1. $x \in \overline{S} \iff$ every closed set containing S contains x.

2. $S \subset \overline{S}$.

~~$(\Rightarrow)$ Pf: If U is a nbhd of x, $S \cap U = \emptyset$, then $X \setminus U$ is closed, $S \subset X \setminus U$, and $x \notin X \setminus U$~~ is a closed set ~~containing S~~.

~~$x \notin X \setminus U \Rightarrow x \notin \overline{S}$.~~

~~$(\Leftarrow)$~~ $(\Rightarrow)$ ~~Suppose $x \notin \overline{S}$. Then there is a closed set K s.t.~~

~~Then $X \setminus \overline{S} \ni x$ and $X \setminus \overline{S}$ is open, so $X \setminus \overline{S}$ is a nbhd of x s.t. $S \cap X \setminus \overline{S}$~~

~~Pf?~~

~~$(\Rightarrow)$ Suppose $x \notin \overline{S}$. Then $X \setminus \overline{S}$ is a nbhd of x s.t. $S \cap (X \setminus \overline{S}) = \emptyset$~~

~~$(\Leftarrow)$ Suppose every nbhd of x intersects S. Let $T \subset X$ be a closed set containing S. Then $X \setminus T$ is an open set. If $x \in X \setminus T$, then $U = X \setminus T$ is a nbhd of x and $(X \setminus T) \cap S = \emptyset$~~ that doesn't intersect

~~Pf: Suppose $x \in \overline{S}$ and let U be a neighborhood of x.~~

~~Pf: Suppose $x \in \overline{S}$. Let K be a closed set s.t. $S \subset K$. Then $x \in K$~~

Pf: Suppose $x \in \overline{S}$. Let U be a nbhd of x. If $U \cap S = \emptyset$, then $X \setminus U$ is a closed set s.t. $S \subset X \setminus U$ and $x \notin X \setminus U$. $\times$ Therefore, $U \cap S \neq \emptyset$.

Conversely, suppose $x \notin \overline{S}$. Then $X \setminus \overline{S}$ is an open set and $x \in X \setminus \overline{S}$.
$\Rightarrow X \setminus \overline{S}$ is a nbhd of x and $S \cap (X \setminus \overline{S}) = \emptyset$. //

Ex: $x \in \overline{S} \iff$ every basis element containing x intersects S.

Def: x is a <u>boundary point</u> of S if any neighborhood of x intersects S and $X \setminus S$. Let $\partial S = \{$bdry pts of $S\}$.

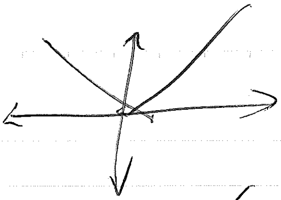
Prop 1. $\overline{S} = S \cup \partial S$. 2. S is closed $\iff \partial S \subset S$.
Pf: exercise.

(break)

Now we can define: <u>Continuous functions</u>:
Def: A function $f: X \to Y$ is <u>continuous</u> if $f^{-1}(U)$ is open for every open set $U \subset Y$. TFAE:
1 - f is continuous
2 - $f^{-1}(B)$ is open for every basis element of Y
3 - $f^{-1}(K)$ is closed for every closed set $K \subset K$.
4 - $\forall A \subset X$, $f(\overline{A}) \subset \overline{f(A)}$
5 - If $x \in X$ and $U \subset Y$ is a nbhd of $f(x)$, then $\exists$ a nbhd $T \subset X$ of x s.t. $f(T) \subset U$.

All of these are fairly simple — I'll do one, leave rest as exercise: 1 $\iff$ 5
$1 \Rightarrow 5$: If f is cts, then $T = f^{-1}(U)$ is open, $x \in T$, and $f(T) \subset U$
$5 \Rightarrow 1$: Suppose $U \subset Y$ is open. Claim that $f^{-1}(U)$ is open:
Enough to show that every $x \in f^{-1}(U)$ is contained in an open.
Let $x \in f^{-1}(U)$, let $y = f(x)$. Then U is a nbhd of y, so $\exists$ $T_x \subset X$ a nbhd of x s.t. $f(T_x) \subset U \Rightarrow T_x \subset f^{-1}(U)$.
Then $f^{-1}(U) = \bigcup_{x \in f^{-1}(U)} T_x$ is a union of open sets $\Rightarrow f^{-1}(U)$ is open //.

~~And one of the most important~~ Foll is helpful to construct:
Prop: Let X, Y, Z be topological spaces.
- If $A \subset X$ is a subspace (a subset w/ the subspace topology) ~~and $f: X \to Y$ is cts~~, then $i: A \to X$ is cts.
- If $f: X \to Y$, $g: Y \to Z$ is cts, then $g \circ f: X \to Z$ is cts.
- If $f: X \to Y$ is cts and $A \subset X$, the restriction $f|_A: A \to Y$ is cts.
- If $X = \bigcup_{\alpha \in A} U_\alpha$ and U_α is open $\forall \alpha$ and $f: X \to Y$ is a fn s.t. $f|_{U_\alpha}$ is cts $\forall \alpha$, then f is cts.
- If $X = K_1 \cup \cdots \cup K_n$ and K_i is closed and $f: X \to Y$ is s.t. $f|_{K_i}$ is cts $\forall i$, then f is cts.
(These let define piecewise fns.) Pf: exer.



$f(x)=|x|$, $\mathbb{R}=(-\infty,0]\cup[0,\infty)$

$f|_{(-\infty,0]}(x)=-x$ $\quad f|_{[0,\infty)}(x)=x$

So f is cts.

And we can use cts fns to discuss one of most important concepts in topology: Def: A homeomorphism from X to Y is a map $f: X\to Y$ which is continuous and has a continuous inverse. We say X and Y are homeomorphic if there is a homeomorphism from X to Y, and write $X\cong Y$.

Q: ~~When~~ How can you tell if $X\cong Y$? ~~Topological properties:~~
Topological properties – properties of a space that are preserved by homeomorphisms.

More next week, but one example: ~~one can limit~~
~~Hausdorff spaces. We saw in the PS that limits can~~
~~go to many different values.~~
Def: A ~~topol~~ space X is Hausdorff if $\forall x,y\in X$, ~~$\exists$ opens~~ if $x\neq y$, then $\exists$ open sets U, V s.t $x\in U$, $y\in V$ and $U\cap V=\emptyset$. (We say any two pts can be separated by open sets).

This is preserved by homeos (Exer: if $f: X \xrightarrow{\cong} Y$ is a homeo, and $U\subset X$ is an open set, then $f(U)\subset Y$ is an open set)

Thm: If X is Hausdorff, then any finite set is closed.
Pf: Let $x\in X$. For any $y\in X$ s.t $x\neq y$, $\exists$ an open $V_y\ni y$ s.t. $x\notin V_y$. Then $\bigcup_{\substack{y\in X\\ x\neq y}} V_y = X-\{x\}$.
So $\{x\}$ is closed for any $x\in X$. If $F=\{x_1,\dots,x_k\}$ is finite, then $F=\{x_1\}\cup\{x_2\}\cup\dots\cup\{x_k\}$ is a finite union of closed sets $\Rightarrow F$ is closed. //

Prop: If X is Hausdorff, then a sequence in X converges to at most one point.
Pf: Suppose $x_n\to x$ and $x_n\to y$ and $x\neq y$. Let U be a nbhd of x, V a nbhd of y s.t. $U\cap V=\emptyset$. Then $\exists N$ s.t. $\forall n\geq N$, $x_n\in U$ and $x_n\in V$ ※.
Overflow: Questions?