Thm: Every compact manifold ~~of dimension~~ with $d \neq 4$ is homeomorphic to a cell complex
(Radó) $d=2$, (Moise, Bing) $d=3$,
(Kirby–Siebenmann) $d \geq 5$.
Open in $d=4$. ~~There~~

~~(break)~~

But: Thm (Milnor): Every manifold is homotopy equivalent to a CW complex.

(break) | 2021-10-14

~~Last time: Complexes; Ops up~~
Homotopy: Said before: One of the main q's of topology:
Q: ~~How can you tell if~~ when are X, Y homeomorphic? / classify spaces in lots/complexes ~~Describe the set of spaces~~ up to $\cong$
Here's another, possibly more important:
Q: When is there a cts map $X \to Y$ with given properties.
Q(alt): ~~What is the space~~
Describe the set of cts maps $X \to Y$. (Given X, Y, classify the maps)
But there are too many! We need an equiv relation — homotopy.

Def: If $f, g: X \to Y$ cts (all maps will be cts), we say that f and g are homotopic if $\exists$ a family of maps $h_t: X \to Y$, $t \in [0,1]$, s.t. $h_0 = f$, $h_1 = g$, $H(t,x) = h_t(x)$ is a cts map $X \times [0,1] \to Y$.
We call this family (interchangeable, H) a homotopy from f to g, and write $f \simeq g$.

Prop: This is an equivalence relation (Exer).

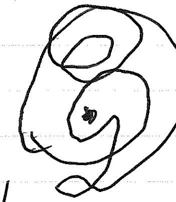
Q: What are the classes of this relation? Start by looking at simple cases: $X = S^1$

Idea:

Ex: If ~~$Y = \mathbb{R}^2$~~, $f, g: X \to \mathbb{R}$ then ~~$f \simeq g$~~

Pf: ~~Let~~ $h_t(x) = (1-t)f(x) + t\,g(x)$. is a htpy from f to g.

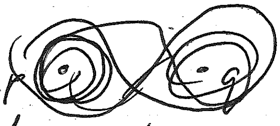
Ex: ~~$Y=\mathbb{R}$~~ $X = S^1$, $Y = \mathbb{R}^2 \setminus \{0\}$.



$f \simeq g \iff f$ and g have same winding number.

~~i.e. every map is htpic to~~

Ex: $X=S^1$, $Y=\mathbb{R}^2\setminus\{p,q\}$

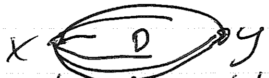


So, even when X is just ~~the~~ a circle, this can be complex.
~~But it turns out that~~
But one reason to focus on this case is that it turns out that the classes form a group.

The Fundamental Group:

Def: A *path* from x to y is a map $f:[0,1]\to X$ s.t. $f(0)=x$, $f(1)=y$.
A *homotopy of paths* is a family of paths $h_t:[0,1]\to X$ s.t. $h_t(0)=x$, $h_t(1)=y$, $H(t,s)=h_t(s)$ is continuous.
~~We say If~~ We say h_0 and h_1 are homotopic, $h_0\simeq h_1$.

[Slightly different from before because endpts fixed
Otherwise,



Let $[\gamma]$ = set of paths htpic to γ.

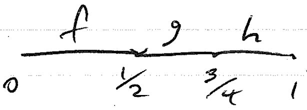
If $f:[0,1]\to X$ is a path and $p:[0,1]\to[0,1]$ is a path from 0 to 1, we call $f\circ p$ a *reparametrization* of f. Then $f\simeq f\circ p$

(Pf: $h_t(s)=f((1-t)s+tp(s))$)

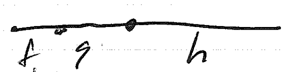
If f is a path from x to y,
g " " " y to z, let $f\cdot g$ be the concatenation $f\cdot g(s)=\begin{cases} f(2s) & \text{if } s\le \frac12\\ g(2s-1) & \text{if } s\ge \frac12.\end{cases}$

This is not associative:

$f(g\cdot h) =$

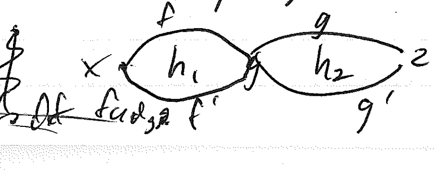


$(f\cdot g)\cdot h =$



But it's assoc up to homotopy! $(f\cdot g)h$ is a reparam of $f\cdot(g\cdot h)$.

~~So~~ And if $f\simeq f'$, $g\simeq g'$, then $f\cdot g\simeq f'\cdot g'$



~~This · is a~~ well-defined ~~assoc operation~~ on htpy classes.

We ~~will use it to Of~~ fudge

$[\lambda]\cdot[\gamma] = [\lambda\cdot\gamma]$ is an

Thus: ~~it's~~ an assoc. operation on homotopy classes.

~~We thus def~~

Def: Let X be a top. space, $x_0 \in X$. The fundamental group of X with basepoint x_0 is

$$\pi_1(X, x_0) = \{\text{htpy classes of paths from } x_0 \text{ to } x_0\}$$
(loops based at x_0)

with group operation concatenation.

Thm: This is a group.
Assoc? Ident? Inverses? $[\gamma][\bar\gamma] = \cancel{[id]}\ 1$

And this is a ~~good~~ way to tell spaces apart:

Ex: ~~$\mathbb{R}^2 \setminus \{0\}$~~ $\pi_1(\mathbb{R}^2 \setminus \{0\}, x_0) \cong \mathbb{Z}$,
$\pi_1(\mathbb{R}^2 \setminus \{a,b\}, x_0) \cong$ free group of rank 2.

(we'll have to prove this)

Further, if X is path-connected, we can talk about the f.g of a space: write $\pi_1(X)$

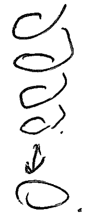
Prop: Suppose γ is a path from x_0 to y_0. Then $\pi_1(X, y_0) \cong \pi_1(X, x_0)$.
by the iso $i_\gamma: [\alpha] \longmapsto [\gamma\cdot\alpha\cdot\bar\gamma]$

Pf: ~~The~~ ~~$i_\gamma([\alpha])$~~ $[\gamma\cdot\alpha\cdot\bar\gamma][\gamma\cdot\beta\cdot\bar\gamma] = [\gamma\cdot\alpha\cdot\beta\cdot\bar\gamma]$,
so this is a homomorphism.
Further, $i_{\bar\gamma}: \pi_1(X, x_0) \to \pi_1(X, y_0)$ is the inverse,
so it's an isomorphism //

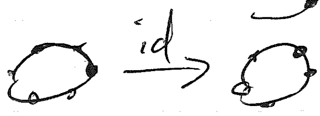
Let's try computations:

Thm: $\pi_1(S^1) \cong \mathbb{Z}$.

Pf: Let $p: \mathbb{R} \to S^1$, $p(t) = (\cos 2\pi t, \sin 2\pi t)$



We say that a map $f: X \to S^1$ ~~lifts~~ has a lift ~~to a map~~
$\tilde f: X \to \mathbb{R}$ if $f = p \circ \tilde f$. (and $\tilde f$ is continuous).

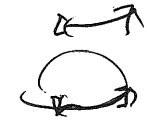
Not every map lifts: 

But: ~~Le~~

Lemma: Let $x_0 = (1,0)$. If $f: [0,1] \to S^1$, $f(0) = x_0$,
then $\exists!$ lift $\tilde f: [0,1] \to \mathbb{R}$ s.t. $\tilde f(0) = 0$.

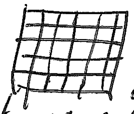
Pf: ~~Choose n s.t. $f([\frac{i}{n},\frac{i+1}{n}]) \subset S^1 \setminus \{(1,0)\}$ or~~
Let $n \in \mathbb{N}$, ~~If n is large enough~~, let $I_k = [\frac{k}{n}, \frac{k+1}{n}]$
~~I choose~~ then n is large enough so that $f(I_k) \subset J_k$, where J_n is an interval of length 1 in S^1.

~~For each n~~ Construct the lift inductively. Suppose $\tilde{f}$ is defined on $[0, \frac{k}{n}]$. Then $\tilde{f}(\frac{k}{n}) \in p^{-1}(J_k)$ and $p^{-1}(J_k) \cong$ ~~$J_k \times \mathbb{Z}$~~ ~~say $\tilde{f}(\frac{k}{n}) = (f(\frac{k}{n}), 1)$~~
~~Define~~ is infinitely many copies of J_k:
~~Send Send Send I_k to its lift in~~
Send I_k to the appropriate copy of J_k.

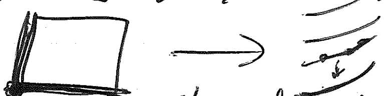


~~Conversely~~ And this is unique: If $\tilde{g}$ is another lift, ~~then $\tilde{g}(I_k) \subset f^{-1}(J_k) = J_k \times \mathbb{Z}$. Since I_k is connected,~~
~~Say~~ $\tilde{g} = \tilde{f}$ on $[0, \frac{k}{n}]$. Then $\tilde{g}(I_k) \subset p^{-1}(J_k) \cong J_k \times \mathbb{Z}$.
Since I_k is connected, $\tilde{g}(I_k)$ lies in ~~one of the copies of J_k~~ the same copy of J_k as $\tilde{g}(\frac{k}{n}) = \tilde{f}(\frac{k}{n})$. So $\tilde{f} = \tilde{g}$ on I_k. //

Likewise: Lemma: If $f : [0,1]^2 \to S^1$, $f(0,0) = x_0$, then $\exists!$ $\tilde{f} : [0,1]^2 \to S^1$ s.t. $\tilde{f}(0,0) = 0$.
Pf: Same idea:



s.t. each square maps to an interval. Lift square by square, starting at $(0,0)$.
What does that mean? Zoom in



Lemma: If $f : [0,1]^2 \to J \subsetneq S^1$ and $A \subset [0,1]^2$ is a nonempty connected subset, and $\tilde{f}_0 : A \to \mathbb{R}$ is a lift of $f|_A$, then $\tilde{f}_0$ extends uniquely to a lift of f.

Ex: Use the lemma to prove the ~~to prove~~
Thm: ~~$\pi_1(S^1)$~~ $\pi_1(S^1, x_0) \cong \mathbb{Z}$.
Pf: Let's construct ~~the~~ map: Let $\gamma : [0,1] \to S^1$ be a loop based at x_0. ~~Define $w([\gamma]) = \tilde{\gamma}$~~ Let $\tilde{\gamma}$ be the lift of γ s.t. $\tilde{\gamma}(0) = 0$. Define $w([\gamma]) = \tilde{\gamma}(1)$.

Claim: This is a well-defined isomorphism from $\pi_1(S^1, x_0)$ to $\mathbb{Z}$.
Pf: Well-defined, homomorphism, surjective, injective.