2021-10-28

Last time: ~~B~~ A map $f:(X,x_0)\to(Y,y_0)$ induces a map $f_*:\pi_1(X,x_0)\to\pi_1(Y,y_0)$.

Functorial: $f_*\circ g_* = (f\circ g)_*$ Homotopy invariant: if $f\simeq g$ by htpy (fixing basepoint) then $f_*=g_*$.

Diagrams of spaces $\Longrightarrow$ diagrams of fundamental groups.

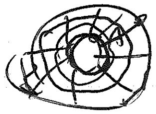
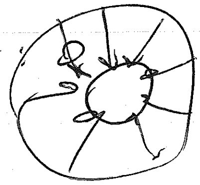
Prop: Let $A\subset X$, let $r:X\to A$ be a retraction (i.e. $r(a)=a\ \forall a\in A$), ~~and let~~ and let $a_0\in A$. Then $i:A\hookrightarrow X$ induces an ~~inclusion~~ injective map $\pi_1(A,a_0)\to\pi_1(X,a_0)$ (i.e., $\pi_1(A)$ is a subgroup of $\pi_1(X)$)

Pf:
$$\begin{tikzcd} A \arrow[rr, "id"] \arrow[dr, "i"'] & & A \\ & X \arrow[ur, "r"'] & \end{tikzcd} \qquad \begin{tikzcd} \pi_1(A,a_0) \arrow[rr, "id_* = id_{\pi_1(A)}"] \arrow[dr, "i_*"'] & & \pi_1(A,a_0) \\ & \pi_1(X,a_0) \arrow[ur, "r_*"'] & \end{tikzcd}$$

~~So r_*~~ //

(if $\gamma\in\pi_1(A)$ maps to 0, then r_* takes it to zero.)

Stronger: Def: ~~If~~ $r:X\to A$ is a ~~retraction~~ deformation retraction ~~if~~ if $r(a)=a\ \forall a\in A$, and ~~∃ a htpy h_t~~ $i\circ r\simeq id_X$ by a htpy fixing A pointwise. I.e. $h_t:X\to X$ s.t. $h_0=id_X$, $h_1=r$ and $h_t(a)=a\ \forall a\in A$.

Ex:  But not: 

~~But also~~

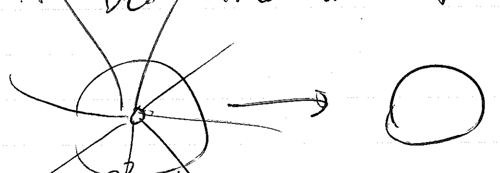
Prop: If $r:X\to A$ is a def. ret, then r_*, i_* ~~are~~ are isomorphisms.

Pf: ~~As~~ We know r_* is injective — in fact, $r_*\circ i_* = id_{\pi_1(A)}$.

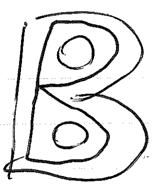
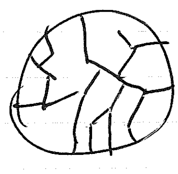
~~Since~~ OTOH, $i\circ r\simeq id_X$, so $i_*\circ r_*\simeq id_{\pi_1(X)}$ —

so r_*, i_* ~~is a~~ isomorphisms //.

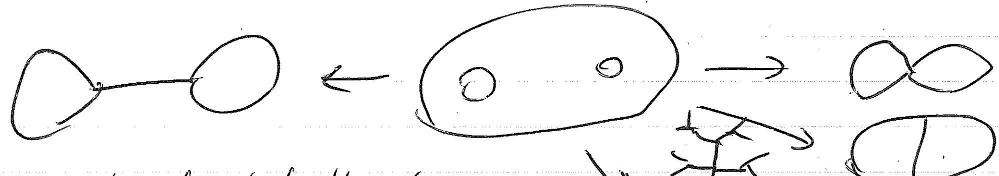
~~And~~ Def retracts are fairly broad:

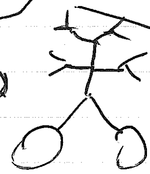


$\mathbb{R}^2\setminus 0 \longrightarrow S^1$

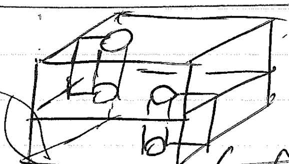
But also: ~~A space the~~ The equiv relation generated by def. retracts is ~~not~~ very broad..





and lot and allot these have isomorphic f.g.

Can we generalize?

Classic example: House with two rooms:  later.

~~Homotopy~~ Def: If $f: X \to Y$, $g: Y \to X$, we say g is a homotopy inverse of f if $f\circ g \simeq id_Y$, $g\circ f \simeq id_X$. If f has a homotopy inverse, we say f is a homotopy equivalence and that $X \simeq Y$. (X is homotopy equivalent to Y).

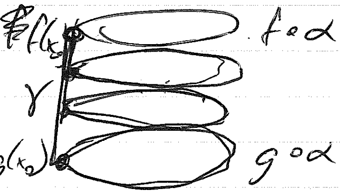
Ex: If r is a def. retract, then r, i are htpy inverses.

Thm: This is an equivalence relation. Pf: Exercise.

Thm: If f is a htpy equiv, then $\forall x_0 \in X$, $\pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is an isom. for any $x_0 \in X$.

Lemma: Suppose $f, g: X \to Y$ and $f \simeq g$ via h_t. Let $x_0 \in X$. Then $\exists$ a path γ from $f(x_0)$ to $g(x_0)$ s.t. $f_* = b_\gamma \circ g_*$ — differ by base change map. — specifically, $\gamma(t) = h_t(x_0)$.

Pf: Let α be a loop based at x_0. — homotope $f\circ\alpha$ to $g\circ\alpha$ and add a tail to keep it based at $f(x_0)$. Ends at $b_\gamma(g_*(\alpha)) = \gamma \cdot (g\circ\alpha)\cdot\bar{\gamma}$.



Pf of thm: If $f: X \to Y$, $g: Y \to X$ are htpy inverses, consider

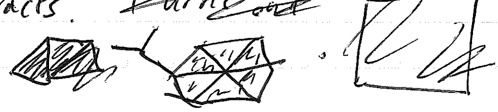
$$\pi_1(X, x_0) \xrightarrow{f_*} \pi_1(Y, f(x_0)) \xrightarrow{g_*} \pi_1(X, g(f(x_0))) \longrightarrow \pi_1(Y, f(g(f(x_0))))$$

$\cong$ (COB) $\cong$ (COB)

~~So f_* is injective, g_* is surjective.~~

So f_* is injective. Likewise, g_* is injective, so f_* is surjective //

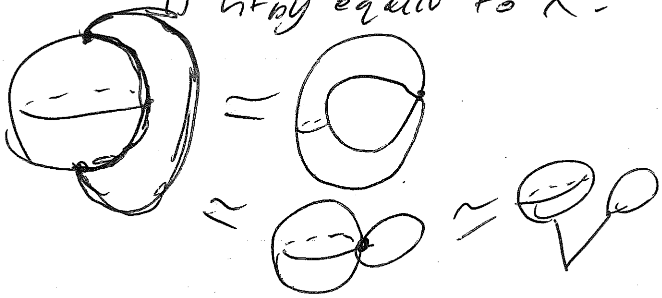
What things are htpy equiv? Def retracts, chains of def retracts. ~~Furnace~~

Ex: Collapse:  ~~But there are more complicated examples: House w/ two rooms.~~

~~Def: A space is contractible if it's htpy equiv to a point.~~
Def: A space is <u>contractible</u> if its htpy equiv to a point.

~~Another way to produce htpy equivalences.~~
Def: A <u>subcomplex</u> of X is a set $A \subset X$ which is closed an a union of cells of X.
~~(mostly does what it says~~

Thm: If $A \subset X$ is a subcomplex and A is contractible, then X/A (i.e., $X/\sim$, where $a \sim a'$ $\forall a, a' \in A$) is htpy equiv to X.



(break)

Back to fundamental groups: First, spheres:
Thm: $\pi_1(S^n) = 0$ when $n \geq 2$.
Pf: ~~Note that~~ ~~$S^n \sim \mathbb{R}^{n+1} \setminus \{0\}$, so suffices to~~
Let $x_0 \in S^n$. Let ~~$[\gamma] \in \pi_1(S^n$, $\gamma : [0,1] \to (S^n, x_0)$~~.
γ be a loop based at x_0.

If γ misses a point $y_0 \in S^n$ then $S^n \setminus \{y_0\} \cong$ open n-ball, so $\gamma \simeq 0$. But what if γ is space-filling?

Let $n > 0$ s.t. $|\gamma(s) - \gamma(t)| < \frac{1}{10}$ whenever $|s - t| < \frac{1}{n}$.
~~Then~~ Let λ be the path consisting of geodesics ~~from $\gamma(0)$ to~~ connecting $\gamma(0), \gamma(\frac{1}{n}), \gamma(\frac{2}{n}) \cdots \gamma(1)$.
Then $\gamma \simeq \lambda$ (Ex: why?) and λ is not surjective, so $\lambda \simeq 0$.

~~With that: Borsuk~~ App:
Thm: Borsuk–Ulam: Let $f: S^2 \to \mathbb{R}^2$. Then $\exists$ $x \in S^2$ s.t. $f(x) = f(-x)$.
~~Pf: Suppose~~ (Note: Similar for $S^1 \to \mathbb{R}$:
define $g(x) = f(x) - f(-x)$. Then g is continuous
Say $g(x) \neq 0$, then $g(x) = -g(-x)$ — so then follows from intermediate value.

Suppose there is no x s.t. $f(x)=f(-x)$.

Pf: Let $g(x) = \dfrac{f(x)-f(-x)}{|f(x)-f(-x)|}$ ~~Then~~ Then $g: S^2 \to S^1$ and $g(x) = -g(-x)\ \forall x \in S^2$.

Consider the ~~equator~~ curve $\gamma(t) = (\cos 2\pi t, \sin 2\pi t, 0)$ around equator ~~Then $[\gamma] = 0 \in \pi_1(S^2)$~~, ~~Then~~ ~~and the image $g\circ\gamma$~~ and image $g\circ\gamma$.

On one hand $\gamma \simeq 0$, so $g\circ\gamma \simeq 0$. OTOH,

~~Let x_0 on the equa~~ ~~On one hand, $g(\gamma(0)) = g(\gamma(\tfrac12))$~~.

~~WLOG, say~~ let ~~$\gamma(0)$~~ $= x_0$ WLOG $g(\gamma(0)) = x_0$. Consider the lift $\widetilde{g\circ\gamma}$. Then $\widetilde{g\circ\gamma}(0) = 0$, $\widetilde{g\circ\gamma}(\tfrac12) = \tfrac{k}{2}$, where k is odd. ~~But $g\circ\gamma|_{[0,\frac12]} = g\circ\gamma$~~

But $g\circ\gamma(t) = -g\circ\gamma(t+\tfrac12)$ – so $\widetilde{g\circ\gamma}|_{[0,\frac12]}$ is a translation of $\widetilde{g\circ\gamma}|_{[\frac12,1]}$ and $\widetilde{g\circ\gamma}(1) = k \neq 0$. ※

Combining spaces. Simple:

Thm: $\pi_1(X\times Y, (x_0,y_0)) \cong \pi_1(X,x_0)\times\pi_1(Y,y_0)$.

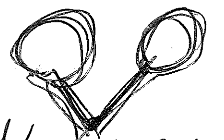
Pf: Exercise.

More complicated:

Def: If (X,x_0), (Y,y_0) are spaces, the wedge ~~product~~ sum of X and Y is $W = \dfrac{X \sqcup Y}{x_0 \sim y_0}$. If X, Y are complexes, x_0, y_0 are verts, this is h.e. to the bouquet  .

p_0

~~We expect~~ $\pi_1(W)$?

~~Well, the some elements include~~ Ex:  $S^1 \vee S^1$

It seems like elements involve going around loops in any orientation, any order

~~More yet, we expect that $\pi_1(X \vee Y)$ consists of products of~~ loops in X, Y ~~concat.~~

This is called "the free group of rank 2":

~~Formally~~: ~~Def~~. $F_2 = \{$ ~~words~~ strings of ~~a~~ $a^{\pm1}$'s, $b^{\pm1}$'s $\}/\sim$

$$w a a^{-1} w' \sim w w' \sim w a^{-1} a w'$$
$$w b^{\pm1} b^{\mp1} w' \sim w w'$$

under concatenation. So $(aba^{-1}b)^{-1} = b^{-1}ab^{-1}a^{-1}$

~~"Free" because~~

– Every element has a unique "reduced" form (no $a^{\pm1}a^{\mp1}$, $b^{\pm1}b^{\mp1}$)

– "Free" because if G is a group, $g, h \in G$, $\exists!$ homomorphism $f: F_2 \to G$ s.t. $f(a) = g$, $f(b) = h$.

More gen: $\pi_1(X \vee Y) = \pi_1(X) * \pi_1(Y)$. – special case of vKT.