Last time: Covering spaces:

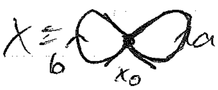
Def: $p: \tilde{X} \to X$ is a covering space if $\forall x \in X$, $\exists$ a nbhd U of x s.t. $p^{-1}(U) \cong U \times E$, where E is a discrete set, and ~~if $(u,e) \in p^{-1}(U)$, then~~ $p(u,e) = u$ $\forall (u,e) \in p^{-1}(U)$.

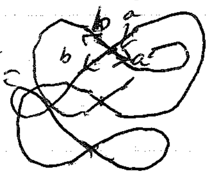
This is exactly the cond you need to lift —

Prop: If $p: (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ is a cover and $f: [0,1]^n \to X$ is a map s.t. $f(0) = x_0$, $\exists !$ lift $\tilde{f}: [0,1]^n \to \tilde{X}$ s.t. $\tilde{f}(0) = \tilde{x}_0$ and $f = p \circ \tilde{f}$.

Pf: Enough to ~~consider~~ show:
Lemma: ~~Let $A \subset [0,1]^n$ be~~ Let U be a nbhd s.t. $p^{-1}(U) \cong U \times E$. ~~Then~~ ~~Let~~ Suppose $f: Y \to U$, $A \subset Y$ is connected and $\tilde{f}: A \to p^{-1}(U)$ is a lift of f on A. Then $\tilde{f}$ extends uniquely to a lift on $[0,1]^n$.

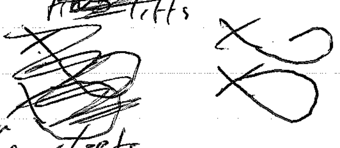
Pf: Since A is connected, $\tilde{f}(A) \subset p^{-1}(U) \cong U \times E$, $\exists e_0$ s.t. $\tilde{f}(a) = (f(a), e_0)$ $\forall a \in A$.
The unique extension is $\tilde{f}(y) = (f(y), e_0)$ //

Ex: $X =$  What are the three-sheeted covers?

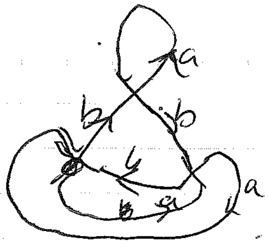
$\tilde{X} =$ 

x_0 has a nbhd that has a three sheet preimage. Further, ~~the proj on each of these determine~~ this comes w/ proj to X:

Now, what about the edges. Lift a — has to end up somewhere. And the other two ~~Why can't two of these end up at the~~
Why is this a permutation? 1) Make a topological argument
2) Lift $\bar{a}$ — ~~lifts~~ would violate uniqueness of lifts



So, lifting a permutes the sheets.
Likewise, lifting b permutes the sheets.
This ~~is complicated~~ diagram is complicated, really,



(So there are generally a lot ($\approx \frac{1}{2}(n!)^2$) of n-sheeted covers of ∞)

In fact, close relation betw covers, subgroups of π_1. Further, ~~Prop~~: injective, advice, groups of q.l.s.b.

Other spaces?

Thm (Classification ~~Fundamental Theorem~~ of Covering Spaces): Let X be ~~locally path connected~~ path-connected, ~~semi-locally simply connected~~ locally path connected, locally simply connected (e.g. a manifold or a complex) (i.e. $\forall$ nbhd U, $\exists$ smaller ~~connected~~ s.c. nbhd). ~~Then the~~
Then there is a bijection

$$\left\{\text{based covers of }(X,x_0)\right\}/\text{isomorphism} \longleftrightarrow \left\{\text{subgroups of } \pi_1(X,x_0)\right\}$$

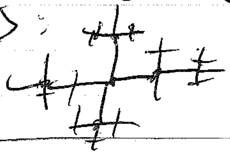
$$(\tilde{X},\tilde{x}_0) \longmapsto p_*(\pi_1(\tilde{X},\tilde{x}_0))$$

(We say $\begin{matrix}\tilde{X}_1\\ \downarrow p_1\\ X\end{matrix}$ and $\begin{matrix}\tilde{X}_2\\ \downarrow p_2\\ X\end{matrix}$ are isomorphic if $\exists f:\tilde{X}_1 \xrightarrow{\cong} \tilde{X}_2$ s.t.
$$\begin{tikzcd} \tilde{X}_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & \tilde{X}_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$)

Ex: $\pi_1(S^1) \cong \mathbb{Z}$. Subgroups: $n\mathbb{Z}$, 0

$n\mathbb{Z}$:  $S^1 \to S^1$, $e^{i\theta} \mapsto e^{ni\theta}$ $\quad$ $0: \mathbb{R} \to S^1$, $\theta \mapsto e^{i\theta}$

Ex: Covers of ∞ $\pi_1(\infty) = \langle a,b\rangle$ — three-sheeted cover corresp to
$G = \langle b^3, bab^{-1}, ab, a^2\rangle$

$0 \subset \langle a,b\rangle$:  - infinite tree

~~Let's stop prove this~~. (break)

~~Let $(\tilde{X},\tilde{x}_0)$~~
~~Prop: If $(\tilde{X},\tilde{x}_0)$ Key to the proof~~ To prove this, we need the following extension of lifting:
Fund Thm of Covering Spaces: If $p:(\tilde{X},\tilde{x}_0)\to(X,x_0)$ is a cover, $f:(Y,y_0)\to(X,x_0)$, and X, $\tilde{X}$, Y are path-connected, ~~$\tilde{X}$~~ Y is locally p.c. then $\exists$ a lift $\tilde{f}:(Y,y_0)\to(\tilde{X},\tilde{x}_0)$ iff $f_*(\pi_1(Y))\subset p_*(\pi_1(\tilde{X}))$.
If $\tilde{f}$ exists, it is unique.

Pf: ($\Rightarrow$) $f = p\circ\tilde{f} \Rightarrow f_*(\pi_1(Y)) = p_*(\tilde{f}_*(\pi_1(Y)))\subset p_*(\pi_1(\tilde{X}))$.

($\Leftarrow$) Suppose $f_*(\pi_1(Y))\subset p_*(\pi_1(\tilde{X}))$. For all $y\in Y$ let γ_y be a path from y_0 to y. Then $f\circ\gamma_y$ is a path from x_0 to $f(y)$. ~~Let~~
Therefore, we must have $\tilde{f}(y) = \widetilde{f\circ\gamma_y}(1)$. ~~Claim: this is well-def'd, cts, and~~
So, if this is well-def'd, cts, it's the unique lift ~~unique~~ we want.

~~Well defined:~~ Well def'd:

~~Lemma: Let γ_y, γ_y' be two paths.~~

~~Key point:~~

Lemma: If γ is a based loop in X, then $\tilde{\gamma}$ is a loop $\iff [\gamma] \in p_*(\tilde{X})$.

Pf: ($\Rightarrow$) $[\gamma] = p_*[\tilde{\gamma}]$

($\Leftarrow$) Use lifts: Suppose $\gamma \simeq p \circ \lambda$ for some $\lambda \in \pi_1(\tilde{X})$.

Then there's a htpy h from γ to $p \circ \lambda$.

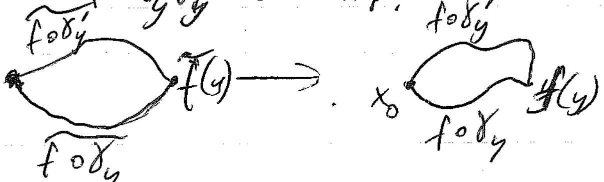
Let $\tilde{h}$ be the lift of h. Then there is a htpy from $\tilde{\gamma}$ to λ –

~~so~~ $\tilde{h}$ fixes endpts, so $\tilde{\gamma}(1) = \lambda(1) = \tilde{x}_0$ //

Well-def'd: Let γ_y, γ_y' be two paths from y_0 to y. Then

$$[\gamma_y \overline{\gamma_y'}] \in \pi_1(Y) \Rightarrow [f \circ \gamma_y \overline{\gamma_y'}] \in p_*(\pi_1(\tilde{X}))$$

$$\Rightarrow \widetilde{f \circ \gamma_y \overline{\gamma_y'}} \text{ is a loop, so } \widetilde{f \circ \gamma_y}(1) = \widetilde{f \circ \gamma_y'}(1).$$



Continuity: Let $y \in Y$, let $U \subset X$ a nbhd of $f(y)$ s.t. $p^{-1}(U) \cong U \times E$. Let $\tilde{U} \cong U \times e_0 \subset \tilde{X}$ be the copy of U containing $\tilde{f}(y)$.

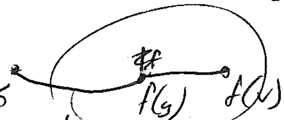
First coord. of lift is cts – need to describe second coord.

~~By lpc path-connected nbhd~~

~~Then $\exists$ a nbhd $V \ni y$ s.t. $f(V) \subset U$.~~

~~Any path λ from $\tilde{f}(y)$~~

Then IZTS that $\exists$ a nbhd $y \in V$ s.t. $\tilde{f}(V) \subset \tilde{U}$.



Let $V \subset f^{-1}(U)$ be a path-connected nbhd of y. ~~Let $v \in V$.~~

For $v \in V$, let λ_v be a path from y to v. Then

$$\tilde{f}(v) = \widetilde{f \circ \gamma_v}(1) = \widetilde{f \circ \gamma_y \lambda_v}(1)$$

~~But~~ $\widetilde{f \circ \lambda_v} \subset \tilde{U} \Rightarrow \tilde{f}(V) \subset \tilde{U}$

$\widetilde{f\circ\gamma_y}$ $\widetilde{f\circ\lambda_v}$

$f\circ\gamma_y$ $f\circ\lambda_v$

~~But also~~

$$\begin{tikzcd} & U \times E \arrow[d] \\ f^{-1}(U) \arrow[ur, "\tilde{f}"] \arrow[r] \arrow[d, hook] & U \\ {} & \end{tikzcd}$$

~~Applications:~~ Applications:

Let $p_i : (\tilde{X}_i, \tilde{x}_0) \to (X, x_0)$ ~~be~~, $i = 1, 2$, be covers.

~~Then $\tilde{X}_1$ and $\tilde{X}_2$ are isomorphic~~

If $(p_1)_*(\pi_1(\tilde{X}_1)) = (p_2)_*(\pi_1(\tilde{X}_2))$, then $\tilde{X}_1$ and $\tilde{X}_2$ are isomorphic.

Pf:

Pf:
$$\begin{tikzcd}(\tilde X_1, \tilde x_0) \arrow[r, shift left, "f=\tilde p_1"] \arrow[dr, "p_1"'] & (\tilde X_2, \tilde x_0) \arrow[l, shift left, "g=\tilde p_2"] \arrow[d, "p_2"] \\ & (X, x_0)\end{tikzcd}$$

~~p_1~~ $p_2 \circ f = p_1$
$p_1 \circ g = p_2$.

~~$f\circ g$~~ $g \circ f : \tilde X_1 \to \tilde X_1$

$p_1 \circ g \circ f = p_2 \circ f = p_1$ So $p_1 \circ g \circ f = p_1$ ~~$\circ id_{\tilde X_1}$~~,

i.e.
$$\begin{tikzcd} & \tilde X_1 \arrow[d, "p_1"] \\ \tilde X_1 \arrow[ur, "g\circ f"] \arrow[r, "p_1"'] & X\end{tikzcd}$$
, i.e. $g \circ f$ is the lift of p_1.

But $p_1 \circ id_{\tilde X_1} = p_1$ too — by uniqueness, $id_{\tilde X_1} = g\circ f$. Likewise $id_{\tilde X_2} = f\circ g$ ///

Key point: If $f_1, f_2 : Y \to \tilde X$ and $p\circ f_1 = p \circ f_2$ ~~If Y is path-connected~~, lpc, an and $f_1(y_0) = f_2(y_0)$ then $f_1 = f_2$.

Therefore, if X is path-conn, locally path-conn then

$$\frac{\{\text{covers of } X\}}{\sim} \longrightarrow \{\text{subgroups of } \pi_1(X)\}$$
$$\tilde X \longmapsto p_*(\pi_1(\tilde X))$$

is injective.

~~Remains to show: this map is surjective.~~ Surjective?

When does X have a simply-connected cover?
Thm: If X is pc, lpc, lsc, it has a simply-connected cover.
By above, this cover is unique up to isomorphism.

Pf: Let $\tilde X = \{[\gamma] \mid \gamma$ is a path in X starting at $x_0\}$.
Let $p : \tilde X \to X$, $p[\gamma] = \gamma(1)$. We only need a topology:

Let $G = \{U \subset X \mid U$ is open, p.c., sc$\}$

If γ is a path in X, $U \in G$ is a nbhd of $\gamma(1)$, let

$U_{[\gamma]} = \{[\gamma \cdot \lambda] \mid \lambda$ is a path in $U\}$ (nearby should mean a nearby path.)

Then: ~~p is a bijection from $U_{[\gamma]}$ to U.~~

- The $U_{[\gamma]}$ form a basis for $\tilde X$. (because lpc, lsc)
- p is a homeo from $U_{[\gamma]}$ to U (ex)
- p is a cover: $p^{-1}(U) = \bigcup_{[\lambda] \in p^{-1}(\gamma(1))} U_{[\lambda]}$

Remains: ~~p is~~ $\tilde X$ is sc.