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1. Lecture 1

1.1. Introduction. This course is called "Asymptotics of Filling Problems", and basically what I'm going to talk about all stems from one classical problem, the isoperimetric problem. Describe.

References: Many of the ideas and examples in this series of talks are taken from or inspired by examples of Gromov in "Asymptotic invariants of infinite groups" and other papers.

- Isoperimetry in the plane
- Generalizing
  - plane $\to \mathbb{R}^n$, Heisenberg group, hyperbolic plane, etc.
  - area enclosed $\to$ filling area/volume
  - curve of length $\ell \to$ cycle of mass V
  - Almgren: An n-cycle of mass V in $\mathbb{R}^N$ has filling volume at most $c_n V^{\frac{n+1}{n}}$.
- GMT and GGT have both looked for similar results in different spaces, but with different flavors.

These two directions have split apart a lot. Analytic vs. combinatorial. But in recent years, I think there's been a growing amount of overlap.

My main goal in this course is to talk about some avenues where the two questions meet.

Goals:

- Introduction to geometric group theory
- Applying geometric measure theory to geometric group theory
  - Hyperbolic and non-positively curved spaces
  - Nilpotent groups
  - Arithmetic groups and lattices in symmetric spaces (common thread in a lot of these is the asymptotic cone – we end up proving bounds by doing geometric measure theory in the asymptotic cone, then passing back to the original space)

I want to start by covering some of the geometric group theory background first: the Dehn function and how its related to hyperbolic groups and negative curvature, and then how this relates to asymptotic cones and trees.

1.2. Geometric group theory background.

- The basic idea of geometric group theory is to study the geometry of finitely-generated groups.
- Cayley graphs and word metric
- QIs.
- Examples: finite groups. Nets vs. whole space (so, for example, a finite-index subgroup of G is q.i. to G.
- Lemma: Let X be a proper geodesic metric space and let G act on X geometrically. Then G (with the word metric) is q.i. to X.
- So we can study groups through the spaces they act on!
- So: One major program in GGT: Classify groups up to qi. Subproject: Define and study quasi-isometry invariants.

- Examples of GGT invariants: growth rate (so you can tell whether a group is (virtually) nilpotent by looking at its large-scale geometry , number of ends.
- The Dehn function
  - Definition: let $\delta(\alpha)$ be the filling area of a closed curve α, let $\delta_X(\ell)$ be the Dehn function of X (supremal filling area over all curves of length at most ℓ).
  - QI invariance: If G_1 and G_2 are q.i. and act on X_1 and X_2 respectively, then δ_{X_1} has the same asymptotic growth as δ_{X_2}.

2. Lecture 2

2.1. **Dehn functions.** References:

- Bridson, "The geometry of the word problem"

Outline:

- The Dehn function of a group
  - The word problem: deciding whether a product of generators represents the identity.
  - Dehn function as a "measure of difficulty" – how many applications of relations to reduce a word to the trivial word?
  - $\delta(w)$ – filling area, $\delta_G(\ell)$ – maximum filling area for words of length $\leq \ell$.
- QI equivalence – Example: $\mathbb{Z}^2$ and $\mathbb{R}^2$.
- Examples:
  - $\mathbb{R}^n$ – quadratic
  - hyperbolic plane – linear
  - Sol_3 – exponential

3. Lecture 3

3.1. **Asymptotic cones.** One of the most powerful, yet most complicated QI invariants of a group is its asymptotic cone. This is a construction by Gromov; he said it was like looking at the group "from infinity". Imagine taking this finitely-generated group, this discrete lattice, and zooming out until all the separate points blur into a continuous space.

- Definition
  - (X, d) be a metric space, $\{x_n\}$ be a sequence of points in X (scaling centers), d_n a sequence of scaling factors that go to ∞, and ω be an ultrafilter.
  - Ultralimits – linear, limit points.
  - Sequence $(X, d/d_1, x_1), \dots$.
  - Limit of these metric spaces is a quotient of the set of bounded sequences $X_b^{\mathbb{N}}$.
- Examples:
  - $X = \mathbb{R}^n$ – $X_\infty = \mathbb{R}^n$
  - X is the hyperbolic plane – X_∞ is an $\mathbb{R}$-tree
    - No shortcuts in the hyperbolic plane
    - Constructing the cone with geodesics
- Properties:

  - Symmetry: If G acts on X geometrically, then X^∞ has a transitive group of symmetries (in fact, G_∞).
  - QI invariance.
  - Possibly many asymptotic cones (bouquets of circles)
- One last example: Sol_3
  - Fibered by two sets of hyperbolic planes
  - Subset of $Hyp \times Hyp$.
  - Fillings don't pass to the cone – the cone is not simply connected, even locally.

So now I can try to make it clear what I want to accomplish with this class: I want to discuss the connections between the geometry of a space (especially its filling inequalities) and the geometry and filling inequalities of its asymptotic cone.

4. Lecture 4

4.1. **Nilpotent groups.** Nilpotent groups are interesting because they are spaces with simple group theory, but complicated geometry. They're just above abelian groups in terms of complexity, and there are easy formulas for calculation in nilpotent groups, but there's a lot that's still not known about their geometry.

Here I want to give a brief introduction to the geometry of nilpotent groups and then discuss some aspects of filling problems in them.

4.1.1. *Introduction to nilpotent groups.* References:

- Pansu, "Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un"
- Gromov, "Asymptotic invariants of infinite groups"
- Ol'shanskii, Sapir, "Quadratic isoperimetric functions for Heisenberg groups: a combinatorial proof"
- Allcock, "An isoperimetric inequality for the Heisenberg groups"
- Magnani, "Spherical Hausdorff measure of submanifolds in Heisenberg groups"

Outline:

- Definition (lower central series, nilpotency class)
- As lattices in nilpotent Lie groups:
  - Every finitely-generated torsion-free nilpotent group is a lattice in a (unique) simply-connected nilpotent Lie group (its Mal'cev completion)
  - Every simply-connected nilpotent Lie group with rational structure coefficients has a cocompact lattice.
- Example: Heisenberg group (as 3×3 matrices, as Lie algebra, as finitely-generated group).
  - Distorted center
  - Scaling automorphism
  - Asymptotic cone is a Carnot-Carathéodory (CC) space.
    * CC-space: manifold with sub-riemannian metric.
    * Illustrate via pullbacks: metric is like $dm^2 = dx^2 + dy^2 + t^2\ dz^2$.

  * Ultimately, if we parameterize like $\begin{bmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{bmatrix}$, then the horizontal planes at (x, y, z) are spanned by $(1, 0, 0)$ and $(0, 1, x)$. These are non-integrable! (Try to keep CC-metric visible on board.)

- More generally:
  - A torsion-free class-k nilpotent group has distortion on the order of n^k.
  - Homogeneous nilpotent groups have families of scaling automorphisms. These distort terms in lower central series.
  - Thm (Pansu): The asymptotic cone of a nilpotent group is a CC-space; indeed, it is bilipschitz-equivalent to a left-invariant CC-metric on a homogeneous nilpotent group. If the group was originally homogeneous, the horizontal vectors can be taken to be the lowest level of the grading.
- Filling problems in Heisenberg groups: How difficult is it to fill a curve in (H_3, d) with a disc? Can we fill a Lipschitz curve in (H_3, d_c) with a Lipschitz disc?
  - $H_3 = \mathbb{R}^3$ with multiplication

$$(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + x_1 y_2)$$

  - d left-invariant Riemannian, $dx^2 + dy^2 + (dz - y\,dx)^2$
  - d_c left-invariant sub-Riemannian, horizontal vectors are spanned (orthonormally) by $(1, 0, 0)$ and $(0, 1, x)$.
  - Lifting of curves: $\gamma \mapsto (\gamma_1(t), \gamma_2(t), \int_0^t \dot{\gamma_2}(t)\gamma_1(t)\,dx)$.
  - Compare metrics, construct boxes ($d(x, y, z) \sim \max\{|x|, |y|, \min\{|z|, \sqrt{|z|}\}\}$, etc.)
  - Hausdorff dimension 4.
  - Figure-eight curve says that the answer to both questions is no. (Use cohomology.)
  - Remarkably, in H^5, the answer to both is yes! We'll see a couple different ways to prove this, and there are a couple of proofs in the references.

5. Lecture 5

- So, basic questions: How do we solve filling problems in these groups and metric spaces, how is the GGT of the nilpotent group related to the GMT of these CC spaces?
- Thm (Gromov): When there is a microflexible sheaf of horizontal maps $\mathbb{R}^k \to G$, then c-Lipschitz maps $S^k \to G$ have $O(c)$-Lipschitz extensions $D^{k+1} \to G$.
- Thm (Young): When G is a Carnot group and there are sufficiently many horizontal maps $\Delta^k \to G$, then $\mathrm{FV}^k(V) \preceq V^{k/(k-1)}$.
- Thm (Wenger): Converse to the above. If (G, d) is a Carnot group with $\mathrm{FV}^k(V) \preceq V^{k/(k-1)}$, then (G, d_c) satisfies filling inequalities for Lipschitz $(k-1)$-currents.

5.0.2. *Microflexibility and infinitesimal invertibility.* References:

- Gromov, “Carnot-Carathéodory spaces from the inside”
- Gromov, *Partial Differential Relations*

Outline:

- Thm (Gromov): When there is a microflexible sheaf of horizontal maps $\mathbb{R}^k \to G$, then c-Lipschitz maps $S^k \to G$ have $O(c)$-Lipschitz extensions $D^{k+1} \to G$.
- Microflexibility
  - Microflexibility: the existence of many horizontal surfaces and continuous families of horizontal surfaces.
  - Microflexibility implies flexibility under certain circumstances. Ex: $\gamma : [0,1] \to \mathbb{R}^2$ with γ making a small angle with $y = x$.
  - Extension from boundaries of simplices.
  - Scaling argument shows that a L-Lipschitz map on the boundary of a simplex can be extended to a cL-Lipschitz map on the simplex.
  - Extend maps from spheres to balls.
- Proving microflexibility using infinitesimal invertibility:
  - Horizontal planes are kernels of some 1-forms η_i.
  - If $f : \mathbb{R}^2 \to G$ is smooth, then it is horizontal iff $f^*(\eta_i) = 0$.
  - If f isn't horizontal, can we perturb it to make it horizontal?
  - Infinitesimal invertibility asks the same question infinitesimally – if f is infinitesimally non-horizontal, can we perturb it to be horizontal?
  - So let I be the differential operator $f \mapsto (f^*(\eta_i))_i$, and consider a perturbation f_t such that $\frac{d}{dt} f_t|_{t=0} = h$. Then consider $\frac{d}{dt} I(f_t)(v)|_{t=0}$.
  - A priori, this is a degree 1 differential operator.
  - But if we restrict to horizontal perturbations, the derivatives drop out, and $h \mapsto \frac{d}{dt} I(f_t)$ is a degree-0 differential operator on h (sending $\mathfrak{g}_1 \to Hom(\mathbb{R}^k, \mathfrak{g}_2)$ at each point). and if f satisfies some conditions, then that operator is surjective. So I is infinitesimally invertible for appropriate f; Gromov calls these f *regular*, and the Gromov-Nash Implicit Function Theorem implies microflexibility.
  - Examples: H^5, other Heisenberg groups, generic groups satisfying an inequality.

5.0.3. *Filling cycles by approximations.* Here's a completely different method of producing fillings of spheres:

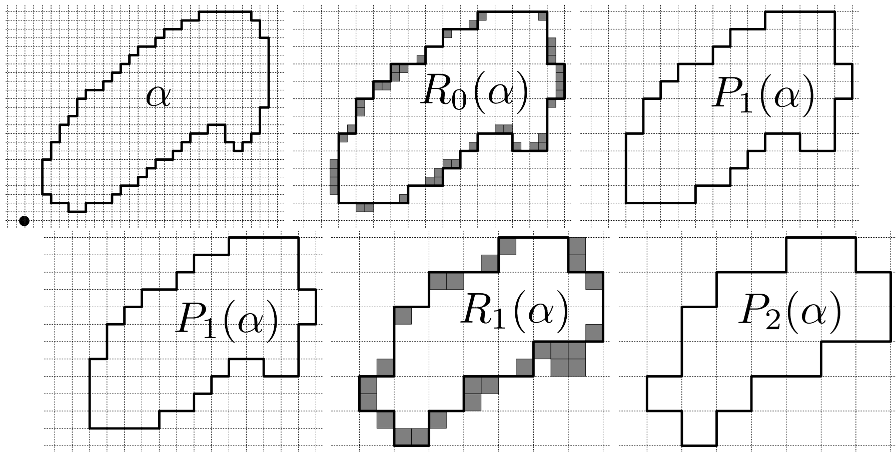
Thm (Young): When G is a Carnot group and there are sufficiently many horizontal maps $\Delta^k \to G$, then $\mathrm{FV}^k(V) \preceq V^{k/(k-1)}$.

References:

- Young, “Filling inequalities for nilpotent groups”

Outline:

- The basis of this idea is producing fillings using approximations. $\mathbb{R}^n$ is a good example: Federer and Fleming initially proved a bound on the higher filling functions of $\mathbb{R}^n$ using approximations, and this is a variant of their approach:

- 

- Calculating area:

Approximations	Homotopies	Area
α : ℓ segments of length 1		
$P_1(\alpha)$: $\sim$ $\ell/2$ segments of length 2	$R_1(\alpha)$: $\sim \ell$ 1×1 squares	$\sim \ell$
$\vdots$	$R_2(\alpha)$: $\sim \ell/2$ 2×2 squares	$\sim 2\ell$
$P_k(\alpha)$: $\sim$ $\ell/2^k$ segments of length 2^k	$R_k(\alpha)$: $\sim$ $\ell/2^k$ $2^k \times 2^k$ squares	$\sim 2^k\ell$

All this stops when $P_k(\alpha)$ is nothing – i.e., when $\ell \sim 2^k$. The total area is then $\sim 2^k\ell \sim \ell^2$.

- Generalize to n-cycles.

Approximations	Homotopies	Volume
α : V n-cubes of side 1	$R_1(\alpha)$: $\sim V$ $(n+1)$-cubes of side 1	$\sim V$
$P_1(\alpha)$: $\sim$ $V/2^n$ n-cubes of side 2	$R_2(\alpha)$: $\sim V/2^n$ $(n+1)$-cubes of side 2	$\sim 2V$
$\vdots$		
$P_k(\alpha)$: $\sim$ $V/2^{kn}$ n-cubes of side 2^k	$R_k(\alpha)$: $\sim V/2^{kn}$ $(n+1)$-cubes of side 2^k	$\sim 2^k V$

All this stops when $P_k(\alpha)$ is nothing – i.e., when $V \sim 2^{kn}$. The total volume is then $\sim 2^k V \sim V^{\frac{n+1}{n}}$.

- When is this possible in nilpotent groups? When we can approximate by simplicial cycles and when we can go from one approximation to another.

  Thm (Young): If G is a Carnot group with a lattice Γ and if there are "enough" horizontal maps $\mathbb{R}^k \to G$, then $FV^k(V) \preceq V^{k/(k-1)}$.
- Basic tool: Federer-Fleming Deformation Lemma: (a simplified version)

Theorem 1. *If τ is a simplicial complex and if α is a singular Lipschitz k-chain in τ such that $\partial\alpha$ is a simplicial $k-1$-cycle in τ, then there is a simplicial k-chain $P_\tau(\alpha)$ which approximates α in the sense that:*
- $\operatorname{mass} P_\tau(\alpha) \leq c \operatorname{mass} \alpha$
- $\partial P_\tau(\alpha) = \partial\alpha$
- $FV(\alpha - P_\tau(\alpha)) \leq c \operatorname{mass} \alpha.$

So if $f : \tau \to G$ is a map which is horizontal on k-cells of τ, then $f_\sharp(P_\tau(\alpha))$ is horizontal. So part of "enough" is that such a map f exists. Let $P(\alpha) = f_\sharp(P_\tau(\alpha))$.

- Scalings give us the rest of the P_i: let
$$P_i(\alpha) = s_{2^i}(P(s_{2^{-i}} \circ \alpha)).$$
- With some extra horizontal maps, we can connect those approximations to get the $R_i(\alpha)$.
- Example: H_5
- Example: Central products of free nilpotent groups.
- Example: Higher-order Dehn functions of Heisenberg groups H_{2n+1} – Lots of horizontal maps for $k \le n$, none for $k > n$
- Example: Jet groups
- Open question: what happens when there aren't as many horizontal maps?
  - Example: Higher-dimensional Heisenberg groups:
  - Chains are non-horizontal and scale differently.
  - Say we fix a $k-1$-sphere α: then the volume grows like $t^{d_{k-1}}$, and the filling volume is at most t^{d_k}; so you might guess that $FV^k(V) \sim V^{d_k/d_{k-1}}$. This turns out to be correct (conjecture of Gromov). Does it generalize?
  - So, sample conjecture: If G is a Carnot group, is there a cycle α so that $s_t(\alpha)$ has near-maximal filling volume?

5.0.4. *Lower bounds using the asymptotic cone.* References:

- Wenger, "Nilpotent groups without exactly polynomial Dehn function"

Outline:

- Examples using central products:
  - Quotients by commutators have quadratic Dehn function.
    Thm: Let G be a class-2 nilpotent Lie group with Lie algebra $\mathfrak{g} = V_1 \oplus V_2$ and such that $FV^2(\ell) \sim \ell^2$. If $u, v \in V_1$, then $\mathfrak{g}' = \mathfrak{g}/\langle [u,v] \rangle$ is the Lie algebra of a nilpotent Lie group G' such that $FV^2_{G'}(\ell) \preceq \ell^2$.
    Proof: We identify Lie groups and their Lie algebras by the exponential map. Recall horizontal lifting – if $\mathfrak{h} = W_1 \oplus W_2$ is the lie algebra of a Carnot group H and $\gamma : [0,1] \to W_1$ is Lipschitz, we define
    $$\tilde{\gamma}(t) = (\gamma(t), \int_0^t \frac{1}{2}[\gamma(t), \dot{\gamma(t)}]\, dt).$$
    This is horizontal, and if $p : \mathfrak{h} \to W_1$ is the projection and $\alpha : [0,1] \to \mathfrak{h}$ is a horizontal map such that $\alpha(0) = 0$, then α is the lift of $p \circ \alpha$.
    Let α' be a closed curve in G'; without loss of generality, we can assume that α' is horizontal and $\alpha'(0) = 0$. Let $\beta : [0,1] \to V_1$ be the projection of α' to V_1 and let α' be the lift of β to G. This is no longer closed, but the endpoint only differs by a multiple of $[u,v]$. Fill it in with the commutator $[tu, tv]$, fill in G, then project to G'. After adding in a square filling $[tu, tv]$, we get a filling of α.
  - Quotients by anything have $n^2 \log n$ Dehn function.
    Thm: Let G be a class-2 nilpotent Lie group with Lie algebra $\mathfrak{g} = V_1 \oplus V_2$ and such that $FV^2(\ell) \sim \ell^2$. If $w \in V_2$, then $\mathfrak{g}' = \mathfrak{g}/\langle w \rangle$ is the Lie algebra of a nilpotent Lie group G' such that $FV^2_{G'}(\ell) \preceq \ell^2 \log \ell$.

Proof:
As before start with α' and lift it to G'. This is no longer closed, but the endpoint only differs by a multiple of w. Close up the curve with a geodesic γ, then scalings of γ.
  - Is this ever sharp?
- Wenger proved a result in this direction – that if w is not a commutator, then the Dehn function is not quadratic.
- Technique: Take fillings of a scaled curve. These converge to a Lipschitz current, using the same techniques as before (trimming to make it uniformly compact, etc.) But analysis of the asymptotic cone makes it impossible – fill in details if necessary.

5.0.5. *Open questions.*

- Given a nilpotent group (even a class-2 nilpotent group), can we calculate its Dehn function? Higher-order Dehn functions?
  - If $\dim \mathfrak{g}_1 \geq 2 \dim \mathfrak{g}_2$, then quadratic for generic G.
  - OTOH, if $\binom{\dim \mathfrak{g}_1}{3} < (\dim \mathfrak{g}_1)(\dim \mathfrak{g}_2)$, then cubic. What happens inbetween?
- Given a nilpotent group with a Carnot metric, what are the possible Hausdorff dimensions of closed, compact submanifolds of different dimensions?
- Filling questions involving Hausdorff dimensions.

5.1. **Filling inequalties and asymptotic cones.** Last time: Gave a vague illustration of how to go back and forth between a space and its asymptotic cone: this time, I want to go into more detail about that transfer, with some theorems about how to go back and forth.

- Going back and forth. Lipschitz maps with the same Lipschitz constant work well.
- OTOH, a map to X_∞ is a sequence of maps to scaled copies – $f = \lim_\omega f_i$, where $f_i : M \to (X, d/d_i)$, but it's hard to control that sequence: $d(f_i(x), f_i(y))/d_i \to d(f(x), f(y))$ only on a subsequence of i's.
- Thm (Papasoglu): If X is a simply-connected, geodesic metric space with bounded local geometry (e.g., a manifold on which a group acts geometrically) such that all of its asymptotic cones have $FA(\ell; s) \leq \ell^p$ for all $\ell > 1$, then X has $\delta(\ell) \preceq \ell^{p+\epsilon}$.
- Thm (Gromov): If X is a simply-connected, geodesic metric space with bounded local geometry and all of its asymptotic cones are s.c., then there is a p such that $\delta(\ell) \preceq \ell^{p+\epsilon}$.
- Define $FA(\ell; s)$:
  - You can break down any curve of length ℓ into $FA(\ell; s)$ curves of length s.
  - If τ is a partition of a disc into polygons ρ_i and $f : \tau^{(0)} \to X$, then $Mesh(f, \tau) = \max perim(f(\rho_i))$.
  - $FA(\alpha; s) = \min_{Mesh(f,\tau) \leq s} \#\tau$
- Proof:
  - ETS that $FA^X_{n/2}(n) \leq c$ for some c.
  - Proceed by contradiction. Find a limiting curve. This has a filling, partition very finely by uniform continuity, etc.

Similarly, simply connected implies polynomial Dehn function. A different argument (due to ?) shows that a quadratic Dehn function implies a simply-connected asymptotic cone.

5.2. **Hyperbolic groups.** Okay, so there are connections between the Dehn function and the geometry of the asymptotic cone. Last time, one of our examples of an asymptotic cone was the hyperbolic plane – it had an $\mathbb{R}$-tree as its asymptotic cone. Now, that's really distinctive, and it turns out to have noticable consequences for the Dehn function. What I'd like to do is talk about similar spaces – spaces whose asymptotic cones are trees – and how that affects their Dehn function and talk about how that generalizes to higher-dimensional questions.

So, the asymptotic cone of the hyperbolic plane is an infinite tree, and this is basically because if you pick a direction and go out in that direction, then pick a different direction, and so on, then the best way back is basically to retrace your steps – there are not shortcuts. You can see this in the behavior of the random walk – if you take an n-step walk in the plane, you end up at a distance of $\sqrt{n}$. If you do that in the hyperbolic plane, you end up at a distance of n, because there are no shortcuts.

Gromov had the idea that this "no shortcuts" property ought to be a large-scale geometry phenomenon – that there ought to be a way of defining a QI-invariant notion of "negative curvature". The generalization he came up with is known as δ-hyperbolicity or Gromov hyperbolicity, and it turns out to be a really powerful notion; there are a lot of equivalent ways to define hyperbolicity, and it links a lot of different phenomena.

References:

- Alonso et al., H. Short (editor), Notes on word hyperbolic groups
- Bowditch, A short proof that a subquadratic isoperimetric inequality implies a linear one
- Väisälä, Gromov hyperbolic spaces

Outline:

- Negative curvature phenomena:
  - A unique geodesic between any pair of points
  - Geodesics diverge exponentially quickly.
  - Triangles are close to tripods.
- δ-hyperbolic spaces are spaces with thin triangles.
  - Neighborhoods of edges (Rips)
  - Gromov product: Given a base point $w \in X$, define

    $$(x, y)_w = \frac{1}{2}(d(x, w) + d(w, y) - d(x, y))$$

    In a tree, this represents the distance from w to the geodesic from x to y.

    If this satisfies a coarse ultrametric inequality:

    $$(x, y)_w \geq \min\{(x, z)_w, (y, z)_w\} - \delta$$

    then the space is δ-hyperbolic. (Independent of base point up to a change in δ.)
  - Geodesic stability: Geodesics are isometric embeddings; quasi-geodesics are QI-embeddings. If X is δ-hyperbolic, then all (k, c)-quasi-geodesics

between a and b are within $C(k, c, \delta)$ of one another, and vice versa. Note – this property is QI-invariant.

- Examples: trees are 0-hyperbolic. Hyperbolic space. Not $\mathbb{R}^n$.
- Hyperbolicity can also be characterized using the Dehn function: G is a group which is hyperbolic wrt the word metric iff G has a linear Dehn function (even subquadratic).
- Remark: This means that there's a gap in the isoperimetric spectrum
- Why?
  - One possibility: if X is δ-hyperbolic, then its asymptotic cone is an $\mathbb{R}$-tree. By Papasoglu, X has a subquadratic DF.
  - Conversely, if the DF is linear, then there aren't any thick triangles

The beauty of hyperbolicity is that there are so many equivalent definitions – hyperbolic spaces are so distinctive that these three ways of looking at them (metric inequalities, asymptotic cones, Dehn functions) give the same definition.

Are spaces which have 2-dimensional asymptotic cones or strong bounds on the filling volumes of spheres equally distinctive?

5.3. **Generalizing hyperbolicity.** References:

- Gromov, "Filling Riemannian manifolds"
- Wenger, "A short proof of Gromov's filling inequality"
- Wenger, "The asymptotic rank of metric spaces"
- Kleiner, "The local structure of length spaces with curvature bounded above"

Summary:

- One candidate: G acts on a rank k Hadamard space.
- Hadamard spaces - generalization of a manifold with nonpositive sectional curvature.
  - Are complete CAT(0) spaces.
  - CAT(0) – geodesic metric space with non-positive curvature. Triangles are "thinner" than those in Euclidean space.
  - Example: non-positively curved manifolds, but also trees, products of trees, etc.
  - Consequences:
    - Unique geodesics.
    - Distance function is convex – if γ_1, γ_2 are geodesics, then $d(\gamma_1(t), \gamma_2(t))$ is convex.
    - Hadamard spaces have $\delta(\ell) \preceq \ell^2$.
    - Hadamard spaces satisfy cone-type inequalities: $\mathrm{FV}^k(\alpha) \leq diam(\alpha) mass(\alpha)$
- Euclidean rank: dimension of the maximal isometric copy of $\mathbb{R}^k$ in a space.
- Examples:
  - Hyperbolic space is HR_1
  - Trees
  - $H^2 \times H^2$ is HR_2
  - product of two trees has rank 2.
- Define: X is HR_k if it is a Hadamard space with euclidean rank k and which has a proper, cocompact group action.
- Gromov's aphorism: Rank k spaces should show hyperbolic behavior in dimensions k and higher.

- Recent research has studied how to generalize notions of hyperbolicity:
  - Asymptotic cones: HR_k implies that asymptotic cone has geometric dimension k (asymptotic rank is k).
  - Dehn function: HR_k spaces have small higher-order Dehn functions (i.e. spheres are easy to fill).
  - Metric inequalities: Coarse uniqueness of quasi-qeodesics vs. Kleiner and Lang's work on quasi-minimizers

The proof of the second thing above (that HR_k spaces have small higher-order Dehn functions) is a particularly good illustration of how the geometry of the asymptotic cone affects the geometry of the space; the next section will be a sketch of the proof.

5.4. **Higher-order filling functions of HR_k spaces.**

- References:
  - Gromov, "Filling Riemannian manifolds"
  - Wenger, "A short proof of Gromov's filling inequality"
  - Wenger, "The asymptotic rank of metric spaces"
- Higher-order filling functions:
  - Define FV_k^X geometrically for $(k-1)$-connected spaces X (using Lipschitz singular chains)
  - Define FV_k^G geometrically for groups G which act geometrically on $(k-1)$-connected spaces.
- Ex: Almgren: $\mathrm{FV}_k^{\mathbb{R}^n}(V) \sim V^{k/(k-1)}$.

The main difference between Dehn functions and higher-order Dehn functions is that closed curves can be parameterized by length, but spheres can't be. This has a lot of consequences.

- Higher-order difficulty: The diameter of a curve is bounded by its length; not so for spheres. Even simple facts become difficult (example: filling spheres in $\mathbb{R}^n$).
- Gromov solved this by a cutting procedure (Wenger has given a very nice simplified and generalized version of this procedure, whose exposition I'll follow).
- Lemma: There is a $d > 1$ such that for almost every $x \in spt(\alpha)$ and all $0 < \epsilon \ll 1$, there is an $r > 0$ such that $\alpha \cap B(x,r)$ is a chain $\alpha_{x,r}$ such that:
  (1) Roundness: $M(\alpha_{x,r}) \geq \frac{r^k \epsilon^{k-1}}{d}$
  (2) Small caps: $\mathrm{FV}(\partial \alpha_{x,r}) \leq d\epsilon M(\alpha_{x,r})$
  (3) Maximality: $M(\alpha_{x,5r}) \leq dM(\alpha_{x,r})$

Proof. Let $r_0 = \max\{r \geq 0 \mid M(\alpha_{x,r}) \geq \epsilon^{k-1} r^k\}$. This is > 0 for almost every $x \in spt(\alpha)$. If $r_0 \leq r \leq 2r_0$, then properties 1 and 3 are clear. It remains to check 2.

Since $M(r_0) = \epsilon^{k-1} r_0^k$ and $M(2r_0) < \epsilon^{k-1}(2r_0)^k$, there is an $r_0 \leq r \leq 2r_0$ such that $M'(r) \leq 2^k \epsilon^{k-1} r_0^{k-1}$. By induction,

$$\begin{aligned} \mathrm{FV}(\partial \alpha_{x,r}) &\leq cM(\partial \alpha_{x,r})^{k/(k-1)} \\ &\leq d\epsilon^k r_0^k \\ &\leq d\epsilon M(\alpha_{x,r_0}). \end{aligned}$$

□

- By Vitali's covering lemma, we can find enough disjoint $\alpha_{x,r}$'s to make up a positive fraction of α. Take $\hat{\alpha}_{x,r}$ their sealed version, and look at $\alpha - \sum \hat{\alpha}_{x,r}$. If ϵ is small, this is a smaller cycle – repeat until satisfied.
- Wenger showed that for an HR_k space, we can do slightly better:
- Thm (Wenger): If G is HR_k, then
$$\mathrm{FV}_i^G(V) \sim V^{(i+1)/i} \quad \text{when } i \leq k$$
and
$$\mathrm{FV}_i^G(V) \prec V^{(i+1)/i} \quad \text{when } i > k.$$
In particular, you can determine the rank of G from its filling functions.
- Sketch:
  - If G is HR_k, then G_∞ has geometric dimension k. In particular, there are no nonzero $k+1$-currents in G_∞.
  - On the other hand, G_∞ is contractible, because G is Hadamard. So any k-cycle in G_∞ is the boundary of a $k+1$-current – so any k-cycles must be 0 too.
  - We claim that if there is a sequence of cycles which is hard to fill, then there is a non-zero k-cycle in G_∞ – this is a contradiction.
  - How do we show this? Using similar methods to the ones we used before, we can show that, in a CAT(0) space with $\mathrm{FV}_{k+1}^G(V) \sim V^{(k+1)/k}$, we can find a sequence of tendril-free cycles α_n with mass V_n such that $V_n \to \infty$ and $\mathrm{FV}_{k+1}^G(\alpha_n) \sim V_n^{(k+1)/k}$.
    That is, since X is a CAT(0) space, we can take a sequence of cycles with maximal filling volume and trim them to be uniformly compact.
  - Tendril-freeness means that the supports of (scalings of) these cycles are uniformly compact. The support converges to a subset of the asymptotic cone, and the cycles converge to a non-zero cycle in the asymptotic cone – this is a contradiction.
- Open questions:
  - Can we improve "strictly less than $V^{(k+1)/k}$" to linear?
  - Can we define spaces like this in some coarse way, as for δ-hyperbolicity?
  - Corollary of previous two: Can we generalize the existence of a "gap" in the Dehn function? What's the largest class of groups where there's a gap in the higher-order DFs?