THE GEOMETRY OF SURFACES AND 3-MANIFOLDS

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Note: Most of the illustrations in these notes are omitted. Please draw your own!

2. Geometric structures on surfaces

We know now what surfaces have flat structures, and it's not very many – plane, cylinder, Mobius strip, torus, Klein bottle. What about the rest?

Theorem 2.1. *Every compact surface is homeomorpic to either a flat surface, a spherical surface, or a hyperbolic surface.*

Recall that a flat surface is one which is locally isometric to the plane (small balls in the surface are isometric to small balls in the plane). A spherical surface is one which is locally isometric to a sphere, and a hyperbolic surface is locally isometric to the hyperbolic plane (which we'll talk about tomorrow).

These are examples of *geometric structures* on surfaces; a surface with a geometric structure is a surface which is locally isometric to some homogeneous *model space*. So in two dimensions, we can describe any compact surface using three model spaces: the plane, the sphere, and the hyperbolic plane. We've already seen the surfaces modeled on the plane, so let's look at the other two.

2.1. The sphere.

If you've done some air travel, geometry on the sphere should be a little familiar.

On the small scale, geometry on the sphere is arbitrarily close to geometry on the plane. Geodesics are great circles, so if we look at them close up, they're basically lines, and if I have some picture in the plane, I can draw something very close on the sphere.

You only really start to see the differences with the plane when you zoom out. So what changes? Geodesics are great circles, and any two great circles intersect, so there aren't any parallel lines. Triangles have larger angles than usual. (For example, an equilateral triangle with sides of length $\pi/2$ (i.e., 1/4th of a great circle) has three right angles.) In fact, there's a formula: in the plane, you have $\alpha+\beta+\gamma=\pi$, but in the sphere, you have $\alpha+\beta+\gamma=\pi+\operatorname{area}\Delta$.

Furthermore, circles are a little shorter than you'd expect: in the plane, a circle of radius r has circumference $2\pi r$. In the sphere, you can calculate, and it's $2\pi\sin(r)$.

2.2. Surfaces based on the sphere.

Suppose S is a complete surface with a geometric structure modeled on the sphere. Then what is S?

Just like before, if S is modeled on the sphere, its universal cover is a sphere, and we can view S as a quotient of the sphere by some fixed-point-free group of symmetries. Any symmetry of the sphere is either a rotation or a reflection plus a rotation. But any rotation has an axis (if you've done some linear algebra, try to prove this!) so the group of symmetries can't

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have any rotations except the identity. The square of any reflection or any reflection + rotation is a rotation, so any symmetry in the group has to have square equal to the identity. In fact there are only two possibilities: the group is trivial (so the surface is the whole sphere) or the group is made up of two symmetries, the identity and the map that sends every point of the sphere to the point exactly opposite (the *antipodal map*).

In this case, any hemisphere is a fundamental domain for the action. If we glue a hemisphere by gluing each point to its opposite point, we get a surface called the *projective plane*.