THE GEOMETRY OF SURFACES AND 3-MANIFOLDS

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Note: Most of the illustrations in these notes are omitted. Please draw your own!

4. A 3-manifold bestiary

How does all of this generalize to three-dimensional manifolds? In general, the picture is a lot more complicated, because 3-manifolds are a lot more complicated. So let's start with some examples.

A surface (or 2-manifold) is a space where every point has a neighborhood which looks like a plane locally — every point has a neighborhood topologically equivalent to part of the plane. A 3-manifold is a space where every point has a neighborhood that looks like part of 3-space. So what are some examples?

First, easy examples:

- 3-space
- The 3-torus (a cube with opposite faces glued together)
- The 3-sphere (two balls with surface glued together, or a sphere in 4-space, or $\mathbb{R}^3$ plus a point at infinity)
- 3-dimensional hyperbolic space

Intermediate examples:

- Quotients of the above by group actions.

  For example, while every rotation of the 2-sphere has an axis, there are "rotations" of the 3-sphere without fixed points. Recall that rotations in the plane have matrices like:

  $$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

  If we combine a rotation by θ in the xy-plane with a rotation by ϕ in the zw-plane, we get:

  $$\begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\phi & \sin\phi \\ 0 & 0 & -\sin\phi & \cos\phi \end{pmatrix}$$

  If $\theta \neq 0$ and $\phi \neq 0$, this has no fixed points. Furthermore, if we take $\phi = \theta = \frac{2\pi}{n}$, then repeating the rotation n times takes us back to the identity. The quotient by this action is a *lens space*.
- There are weirder examples too. For example, there's a manifold called the Poincaré homology sphere which is the quotient of S^3 by a group which is isomorphic to the group of symmetries of the icosahedron.

Combinations of manifolds:

- Products: If we have a surface Σ, we can make a 3-manifold out of it in a couple different ways. First, consider the space consisting of

Date: August 17, 2012.

pairs of points, one in the surface and one in the real line. This is a 3-manifold called $\Sigma \times \mathbb{R}$.

There are a couple ways to think about this – either it's a collection of lines, one for each point in the surface, or a collection of surfaces, one for each point in $\mathbb{R}$. This is just one example of a *bundle*.

Similarly, we can take $\Sigma \times S^1$, the product of Σ with a circle.

- There are more complicated ways to construct bundles. For example, say I want to construct a bundle of surfaces over a circle. We've seen one of these – the product $\Sigma \times S^1$ has one surface for every point in S^1.

  Here's another: cut $\Sigma \times S^1$ along one circle, then glue it back along some different map. There are a *lot* of different maps, for example:
  - Symmetries of surfaces: for example, you can embed the surface of genus 2 in $\mathbb{R}^3$ so that a rotation by $2\pi/3$ is a symmetry.
  - Dehn twists: cut one of the handles, twist, and reglue.
  - Puncture-dragging: If you have a punctured surface, you can "drag" the puncture around the surface to get a map from the surface to itself. You can drag handles, too.

  So there are a lot of manifolds that come from this construction.
- The previous bundle had one surface for every point on the circle. Here's an example of a bundle with one circle for every point on a surface: Consider the space of directions on a surface. Then every point is associated with the circle of directions at that point, so there's one circle for each point. We can check that this is a 3-manifold. This is called the *unit tangent bundle*.

Advanced examples: The techniques above only construct some 3-manifolds, but there are a few different constructions that lead to every 3-manifold.

- Any 3-manifold can be triangulated, so we can construct manifolds out of polyhedra and gluings.

  (Warning: We have to be a little more careful here than when we were working with triangulations of surfaces. With surfaces, as long as every edge is glued to exactly one other edge, you have a surface, but not for 3-manifolds – you need to check that the neighborhood of each vertex is homeomorphic to part of $\mathbb{R}^3$. Example: dodecahedron with opposite faces glued.)
- Some of our constructions involved cutting up a manifold and then gluing it back together. This is sometimes called *surgery.*

  One type of surgery is *Dehn surgery*; if we have a curve in a manifold, we can thicken it into a solid torus. We can cut that torus out and glue it back in differently. Some gluings don't change the manifold, but some gluings change it a lot – in fact, you can construct any closed 3-manifold by starting with a sphere and then applying repeated Dehn surgeries to it.
- Heegaard splittings: A genus-g handlebody is a solid ball with g handles glued on – imagine a genus-g surface embedded in $\mathbb{R}^3$ plus its inside. This is a 3-manifold with boundary, and its boundary is a surface of genus g. It's a theorem that any 3-manifold can be cut into two handlebodies: for example, you can cut the sphere S^3 into two balls (handlebodies of genus 0) by cutting along an "equator", or into two solid tori (handlebodies of genus 1) (try it!) A decomposition like this is called a Heegaard splitting.

So that means that we can make any 3-manifold by gluing the surfaces of two handlebodies! This gives us another way to turn a homeomorphism from a surface to itself into a 3-manifold.

Even if not all these constructions are clear, it's clear that there are a lot of 3-manifolds – certainly, that there are a lot more 3-manifolds than surfaces. Surfaces were simple enough that we could construct geometric structures essentially "by hand", but 3-manifolds seem a lot more complicated.

That's why the Perelman-Thurston Theorem (which we'll talk about tomorrow) is so remarkable; it says that any 3-manifold can be cut into pieces, each of which has a geometric structure!

4.1. **Exercises.** In class, we discussed the lens space which comes from the action of

$$\begin{pmatrix} \cos\frac{2\pi}{n} & \sin\frac{2\pi}{n} & 0 & 0 \\ -\sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} & 0 & 0 \\ 0 & 0 & \cos\frac{2\pi}{n} & \sin\frac{2\pi}{n} \\ 0 & 0 & -\sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} \end{pmatrix}$$

on the 3-sphere. This is sometimes called $L_{1,n}$. These problems will help you visualize this space. (It may help to have a computer available to plot some of the things we'll describe.)

(1) First, let's try to visualize rotations of the 3-sphere. Remember that we can think of the 3-sphere in two ways: as a subset of $\mathbb{R}^4$, namely
$$S^3 = \{(x, y, z, w) \mid x^2 + y^2 + z^2 + w^2 = 1\}$$
or as $\mathbb{R}^3$ plus a "point at infinity". Show that the map
$$f(x, y, z) = \frac{(2x, 2y, 2z, 1 - x^2 - y^2 - z^2)}{1 + x^2 + y^2 + z^2}$$
maps $\mathbb{R}^3$ to S^3. What happens to the point at infinity? (This map is called a *stereographic projection*, and its inverse is the map:
$$f^{-1}(x, y, z, w) = \frac{(x, y, z)}{1 - w}$$
We'll use this map to visualize different parts of S^3.)

(2) The rotation
$$M = \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\phi & \sin\phi \\ 0 & 0 & -\sin\phi & \cos\phi \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\phi & \sin\phi \\ 0 & 0 & -\sin\phi & \cos\phi \end{pmatrix}$$
can be broken down into two rotations. Each of these is a rotation "around" a plane in $\mathbb{R}^4$. The first one is a rotation around the zw-plane and the second is a rotation around the xy-plane.

Which points in S^3 are fixed by each rotation? What are their images in $\mathbb{R}^3$? We can think of these as two axes of rotation in S^3.

(3) What do rotations around these axes look like? For example:
- The orbit of a point under a rotation is a circle. What do these circles look like in $\mathbb{R}^3$?
- The rotation by angle $\theta = 2\pi/n$ around the zw-plane generates an action of $\mathbb{Z}/p$ on S^3. What's a fundamental domain for this action?
- The rotation by angle $\phi = 2\pi/n$ around the xy-plane generates an action of $\mathbb{Z}/p$ on S^3. What's a fundamental domain for this action?

(4) If you choose the fundamental domains in the previous question right, you get fundamental domains for the action of M on S^3. Use them to describe $L_{1,n}$ as a polyhedron with some of its faces glued together.

Can you see where the name *lens space* comes from?