The Dehn function of $SL(n;\mathbb{Z})$

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Dehn functions of $SL(n;\mathbb{Z})$

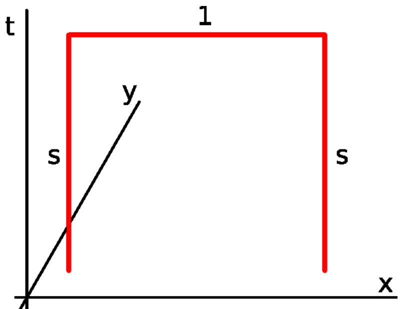
- $SL(2;\mathbb{Z})$ is virtually free – linear Dehn function
- $SL(3;\mathbb{Z})$ – exponential Dehn function (Thurston-Epstein)
- $SL(4;\mathbb{Z})$ – conjectured to be quadratic (Thurston)
- $SL(5+;\mathbb{Z})$ – quadratic (Y)

Sol_3: Exponential distortion

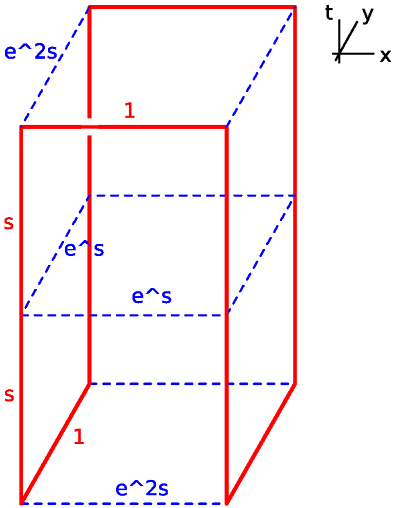
$$\mathrm{Sol}_3 = \begin{pmatrix} e^t & 0 & x \\ 0 & e^{-t} & y \\ 0 & 0 & 1 \end{pmatrix} = \mathbb{R} \ltimes \mathbb{R}^2 = \langle t \rangle \ltimes \langle x, y \rangle$$

$\mathbb{R}^2$ is exponentially distorted:

$$\underbrace{\begin{pmatrix} e^s & 0 & 0 \\ 0 & e^{-s} & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{length } s} \underbrace{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{length } 1} \underbrace{\begin{pmatrix} e^{-s} & 0 & 0 \\ 0 & e^s & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{length } s} = \begin{pmatrix} 1 & 0 & e^s \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



Sol_3: Exponential Dehn function



Length: $\sim s$
Area: $\sim e^{2s}$
$\rightsquigarrow \delta_{\mathrm{Sol}_3}(n) \sim e^n$
BUT

$$\mathrm{Sol}_5 = \begin{pmatrix} e^{t_1} & & & x \\ & e^{t_2} & & y \\ & & e^{t_3} & z \\ & & & 1 \end{pmatrix},$$

$t_1 + t_2 + t_3 = 0$ has quadratic Dehn function.

The Steinberg presentation

Generators:

$$e_{ij}(x) = \begin{pmatrix} 1 & & x & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} = [e_{ij}(1)]^x = e_{ij}^x$$

Relations:

$$\begin{aligned} [e_{ij}, e_{k\ell}] &= I \text{ if } i \neq \ell, j \neq k \\ [e_{ij}, e_{jk}] &= e_{ik} \text{ if } i \neq k \\ (e_{12} e_{21}^{-1} e_{12})^4 &= I \end{aligned}$$

The Dehn function of $SL(2;\mathbb{Z})$

$SL(2;\mathbb{Z})$ is virtually free $\Rightarrow$ linear Dehn function.
$SL(2;\mathbb{Z})$ is virtually free $\Rightarrow$ powers of generators are far from I:

$$d_{SL(2;\mathbb{Z})}(I, e_{ij}(x)) \sim x,$$

BUT

$$d_{SL(3;\mathbb{Z})}(I, e_{ij}(x)) \sim \log |x|.$$

Why?

Distortion in $SL(3;\mathbb{Z})$

Let

$$F = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then if

$$H = \left\{ \begin{pmatrix} F^t & & x \\ & & y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y, t \in \mathbb{Z} \right\} \subset SL(3;\mathbb{Z}),$$

then

$$H \cong \begin{pmatrix} \lambda^t & 0 & x \\ 0 & \lambda^{-t} & y \\ 0 & 0 & 1 \end{pmatrix} \subset \mathrm{Sol}_3$$

In particular,

$$\delta_H(n) \sim e^n \rightsquigarrow \delta_{SL(3;\mathbb{Z})}(n) \sim e^n$$

Why should $\delta_{SL(4;\mathbb{Z})}$ be quadratic?

1. Part of a larger conjecture by Gromov: Lattices in high-rank symmetric spaces should have polynomial Dehn functions, because there are enough flats.
2. By the analogy between $SL(n;\mathbb{Z})$ and Sol_{2n-3}.

$SL(n;\mathbb{Z})$ and Sol_{2n-3}

$$\mathrm{Sol}_{2n-3} = \left\{ \begin{pmatrix} a_1 & & & & z_1 \\ & a_2 & & & z_2 \\ & & \ddots & & \vdots \\ & & & a_{n-1} & z_{n-1} \\ & & & & 1 \end{pmatrix} \middle| \prod a_i = 1 \right\} \subset SL(n;\mathbb{R}).$$

There is a lattice $H \subset \mathrm{Sol}_{2n-3}$ such that $H \subset SL(n;\mathbb{Z})$.

$SL(3;\mathbb{Z})$:

- Contains copies of Sol_3.
- There are words $\widehat{e}_{ij}(x)$ of length $\sim \log|x|$ which represent e_{ij}^x. (exponential distortion)
- $\delta([\widehat{e}_{13}(x), \widehat{e}_{23}(x)]) \sim x^2$ (exponential Dehn function)

$SL(4;\mathbb{Z})$:

- Contains copies of Sol_5.
- There are words $\widehat{e}_{ij}(x)$ of length $\sim \log|x|$ which represent e_{ij}^x.
- $\delta([\widehat{e}_{14}(x), \widehat{e}_{24}(x)]) \sim (\log|x|)^2$ (quadratic fillings for some words)

$SL(5;\mathbb{Z})$:

- Contains overlapping solvable groups with quadratic Dehn functions.
- There are words $\widehat{e}_{ij}(x)$ of length $\sim \log|x|$ which represent e_{ij}^x.
- Different choices of $\widehat{e}_{ij}(x)$ can be connected by quadratic-area homotopies.
- "Simple" words have quadratic fillings.

"Simple" words have quadratic fillings

Theorem (Y)

In $SL(5,\mathbb{Z})$*, words of the forms*

- $[\widehat{e}_{ij}(x),\widehat{e}_{k\ell}(y)]$ *for* $i\neq\ell$*,* $j\neq k$*,*
- $[\widehat{e}_{ij}(x),\widehat{e}_{jk}(y)]\widehat{e}_{ik}(xy)^{-1}$ *for* $i\neq k$*,*
- $\widehat{e}_{ij}(x)\widehat{e}_{ij}(y)\widehat{e}_{ij}(x+y)^{-1}$.

have quadratic ($\sim(\log|x|+\log|y|)^2$*) filling areas.*

Breaking words into simple words

Let $\mathcal{E} = SL(n;\mathbb{R})/SO(n)$ (symmetric space).

1. Let w be a word in $SL(n;\mathbb{Z})$; this is also a curve in $\mathcal{E}$.
2. Fill w with a disc and triangulate that disc by triangles of side length ~ 1.
3. Map the triangulation into $SL(n;\mathbb{Z})$, one dimension at a time.

   Dim. 0 It suffices to have a map $\mathcal{E} \to SL(n;\mathbb{Z})$: fix a fundamental domain S and map any vertex contained in gS to g.

   Dim. 1 If x, y are the ends of an edge, then $d(x, y) \leq 1$. We can see where nearby points go by looking at $SL(n;\mathbb{Z})\backslash\mathcal{E}$.

   Dim. 2 We can replace the edges of the triangulation by "simple" words, then use the theorem above to fill each one with a quadratic filling.

n^2 triangles, each filled with area n^2, leads to a quartic filling inequality