# Surfaces in Heisenberg groups and quantitative rectifiability

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cims.nyu.edu/~ryoung/slides/slidesGMT.pdf

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# The Heisenberg groups

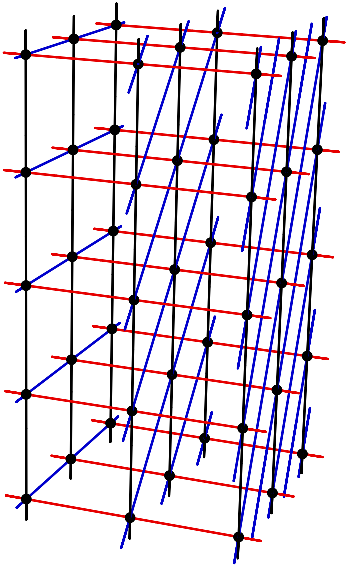
Let $n \geq 1$. Let $H_n \subset M_{n+2}$ be the $(2n+1)$–dimensional nilpotent Lie group

$$H_n = \left\{ \begin{pmatrix} 1 & x_1 & \dots & x_n & z \\ & 1 & & & y_1 \\ & & \ddots & & \vdots \\ & & & 1 & y_n \\ & & & & 1 \end{pmatrix} \ : \ x_i, y_i, z \in \mathbb{R} \right\}$$

with Lie algebra

$$\mathfrak{h}_n = \langle X_1, \dots, X_n, Y_1, \dots, Y_n, Z : \\ [X_i, Y_i] = Z, \text{ all other pairs commute} \rangle.$$

# A lattice in $H$



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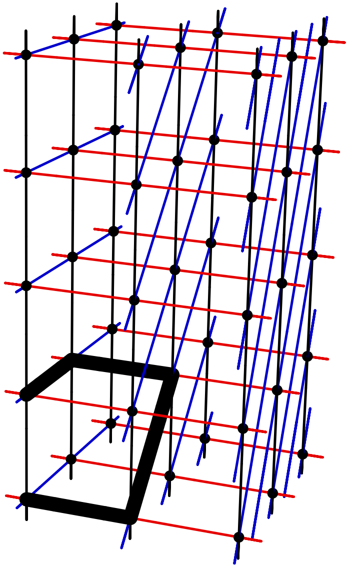

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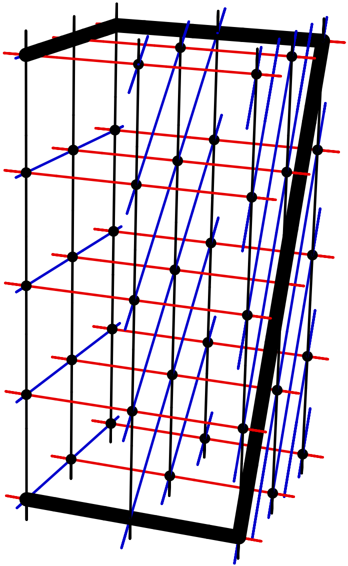

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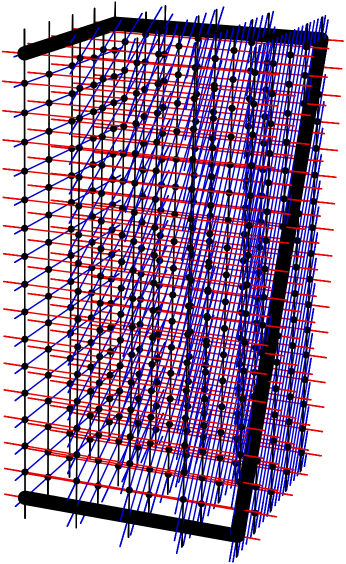

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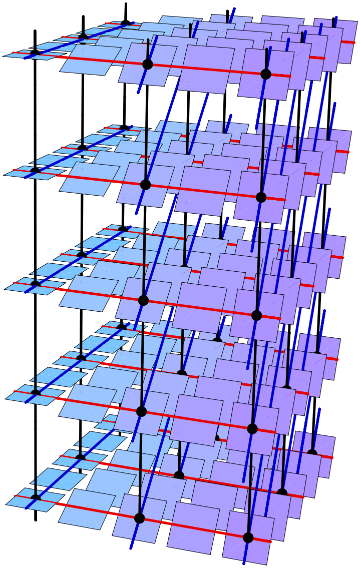
# From Cayley graph to sub-riemannian metric



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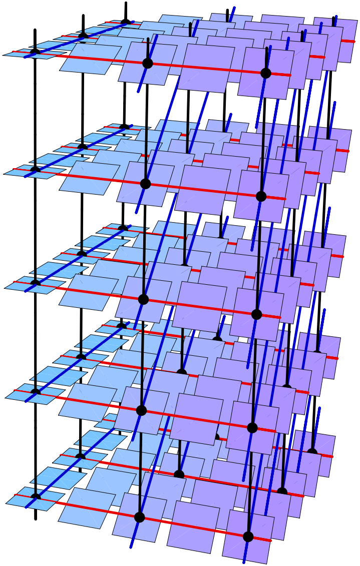

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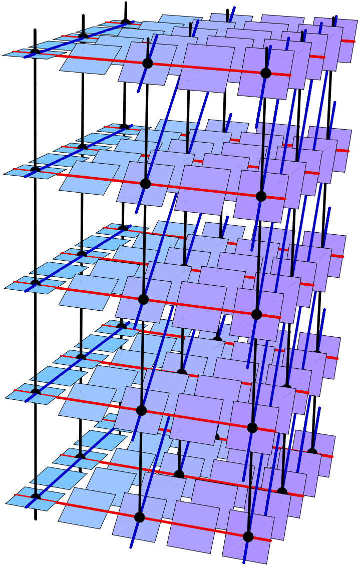
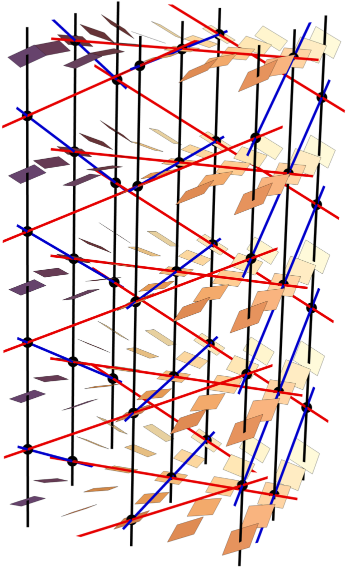


- $d(u, v) = \inf\{\ell(\gamma) \mid \gamma$ is a horizontal curve from $u$ to $v\}$
- The map $s_t(x, y, z) = (tx, ty, t^2 z)$ scales the metric
- So $H_n$ has topological dimension $2n + 1$ but Hausdorff dimension $2n + 2$

# Symmetries of $H_n$



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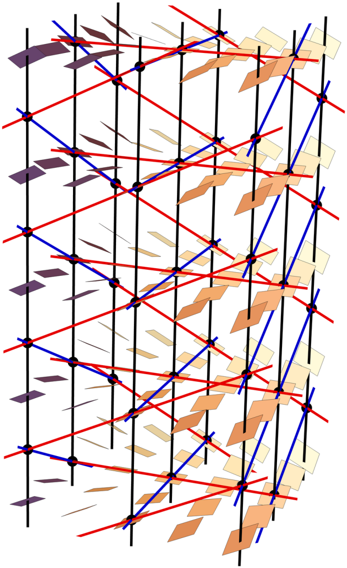

- The unitary group $U(n)$ acts on $H_n$ by isometries
- Any one-parameter horizontal subgroup is a line. We call these *horizontal lines*.

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- Surfaces in $H_n$ behave differently when $n = 1$ and $n \geq 2$
- This stems from the geometry of vertical planes in $H_n$
- Because of the different geometry, we can use different techniques to study surfaces in $H_n$ and $H_1$.

# Vertical planes

A *vertical plane* is a codimension–1 plane parallel to the $Z$–axis.

- When $n = 1$, up to isometry, this is $\langle Y, Z\rangle \cong \mathbb{R} \times \mathbb{R}$ with the parabolic metric

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  with the product metric.
- When $n > 1$, this is horizontally connected, when $n = 1$, this is horizontally **disconnected**.

# Smooth surfaces in $H_n$

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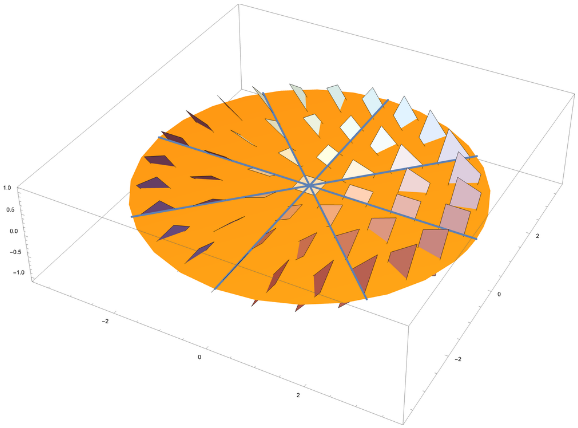
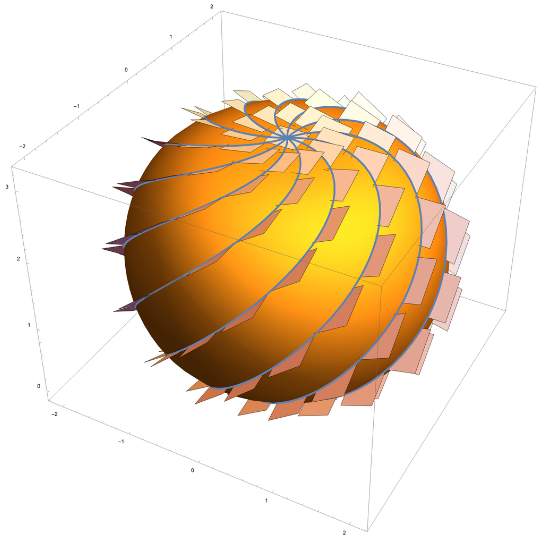
- Otherwise, $s_t(p^{-1}P_p^{\mathbb{R}})$ converges to a vertical plane as $t \to \infty$, which we call the *(intrinsic) tangent plane* $P_p$.

# Tangent planes in $H_1$




A horizontal plane through the origin

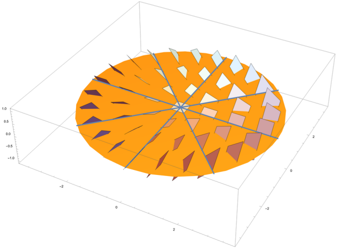
# Tangent planes in $H_1$




A Pansu bubble set

# Horizontal mean curvature

If $\Sigma \subset H_1$ is smooth and has no characteristic points, then the first variation of area is determined by *horizontal mean curvature*, the curvature of the projection of its horizontal curves to the $xy$–plane:



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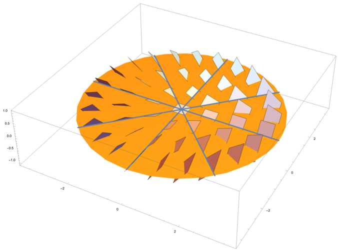

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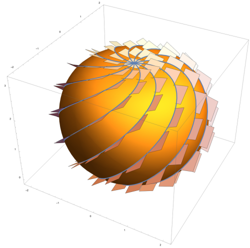
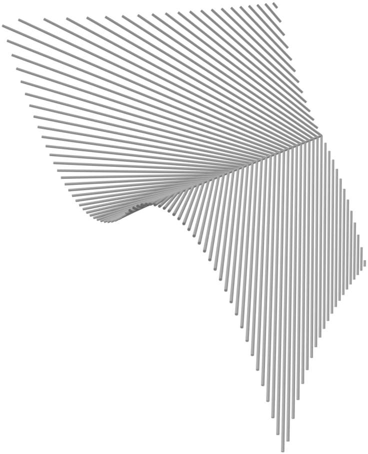
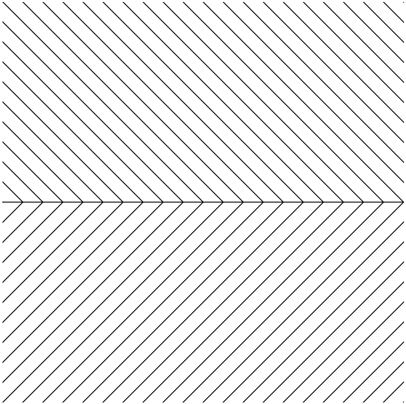


Horizontal curves project to circles, $H_{\text{horiz}}$ is constant.

# Some minimal surfaces in $H_1$



Herringbone surface (Pauls)



From above

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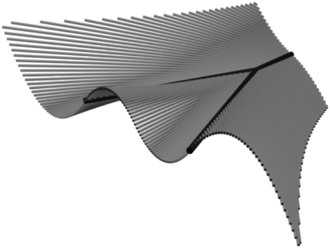
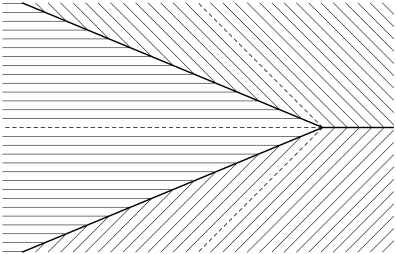


Branched singularity (Ritoré)



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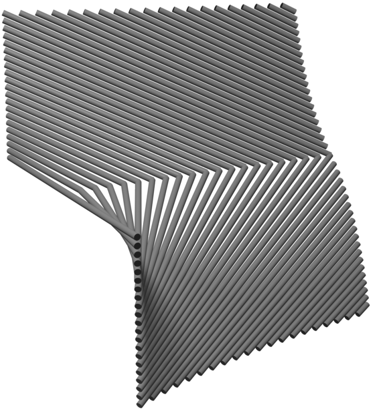
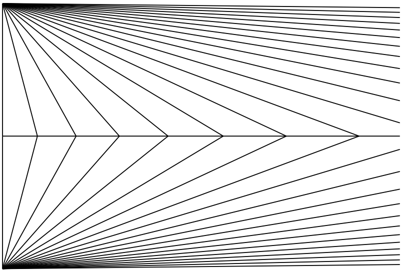


Minimal surface with boundary



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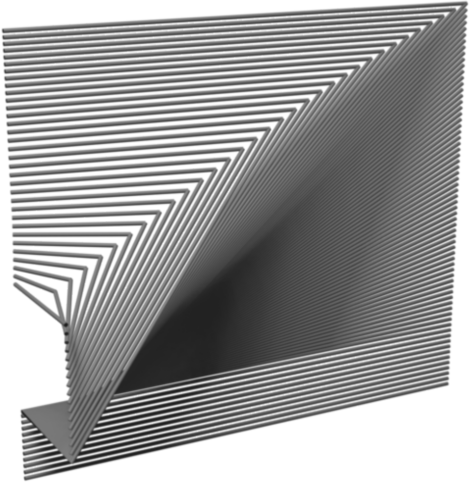

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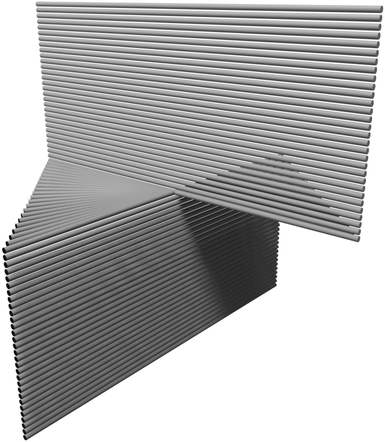

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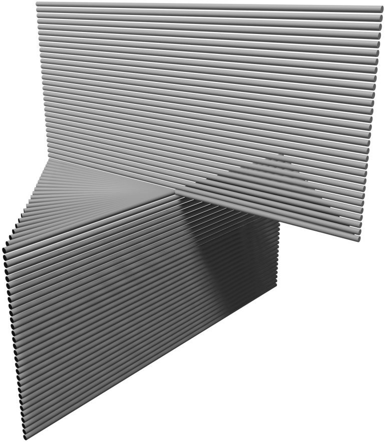

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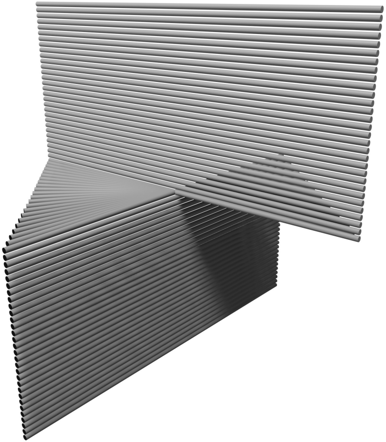

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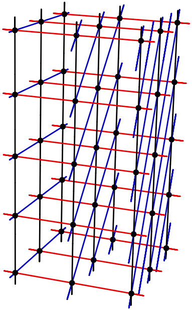
Open question: are all area-minimizing sets in $H_1$ like this?

# Intrinsic graphs



Let $X_n^t$ be the 1–parameter subgroup generated by $X_n$. Let $x_n : H_n \to \mathbb{R}$ be the $x_n$–coordinate function.

For $f : V_0 = \{x_n = 0\} \to \mathbb{R}$, we define the *intrinsic graph of f* as

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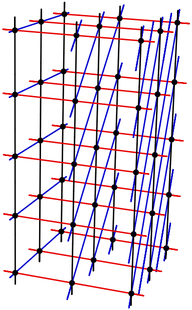

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$$\nabla_f g = (X_1 g, \dots, X_{n-1} g, Y_1 g, \dots, Y_{n-1} g, (Y_n + fZ)g),$$

where $X_i, Y_i, Z$ are the left-invariant fields generating $\mathfrak{h}_n$.

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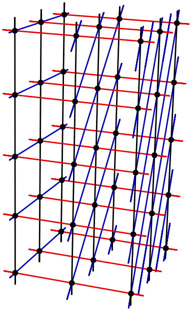

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where $X_i, Y_i, Z$ are the left-invariant fields generating $\mathfrak{h}_n$.
If $f$ is smooth, then $\nabla_f f$ gives the slope of the tangent plane to $\Gamma_f$.

# Intrinsic Lipschitz graphs

An intrinsic graph $\Gamma_f$ is an *intrinsic Lipschitz graph* if there is an $0 < L < 1$ such that for all $p, q \in \Gamma_f$,

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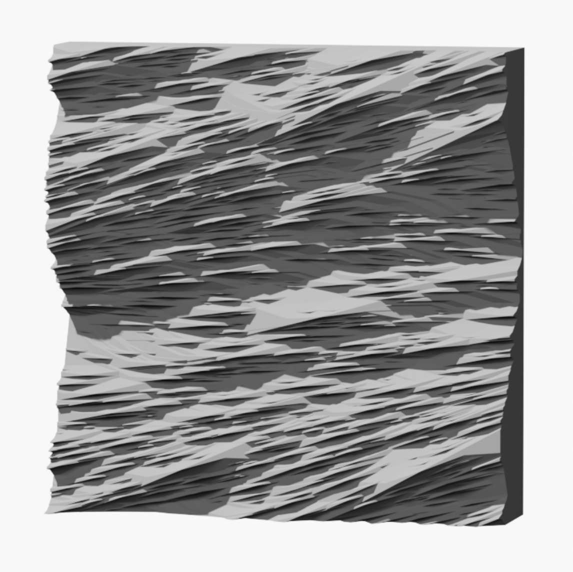
## Theorem (Bigolin–Caravenna–Serra Cassano)

*$\Gamma_f$ is an intrinsic Lipschitz graph if and only if $\nabla_f f$ (defined distributionally) is $L_\infty$.*

# An intrinsic Lipschitz graph



# Differentiability of intrinsic Lipschitz graphs

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$$\lim_{r \to 0} r^{-1} d_{Haus}(P \cap B(x,r), \Gamma \cap B(x,r)) = 0.$$

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Today: Can we quantify this? How fast does this limit converge?

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Theorem (Chousionis–Li–Y.)

*Let $\Gamma \subset H_n$ be $L$–intrinsic Lipschitz. For $x_0 \in \Gamma$,*

$$\int_0^1 \int_{B(x_0,1)} \beta_\Gamma(x,r)^p \, dx \, \frac{dr}{r} \lesssim_L 1,$$

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We say that $\Gamma$ is close to a plane at most points and most scales.

We can compare the theorem to a theorem of Dorronsoro:

### Theorem (Dorronsoro)

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So intrinsic Lipschitz graphs in $H_n$ are about as rough as Lipschitz graphs in $\mathbb{R}^n$, intrinsic Lipschitz graphs in $H_1$ are rougher.

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- When $n \geq 2$, we slice $\Gamma$ along vertical planes and apply a version of Dorronsoro to each slice.
- When $n = 1$, we study graphs in $H_1$ by studying the horizontal foliation.

# $H_n$: Slicing

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## Theorem (Fässler–Orponen)

*For any $n \geq 1$, let $\Gamma \subset H_n \times \mathbb{R}$ be the graph of a Lipschitz function $g : H_n \to \mathbb{R}$. For $x_0 \in \Gamma$,*

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We repeat this with different planes to get the full inequality.

# $H_1$: Horizontal curves

**Lemma**

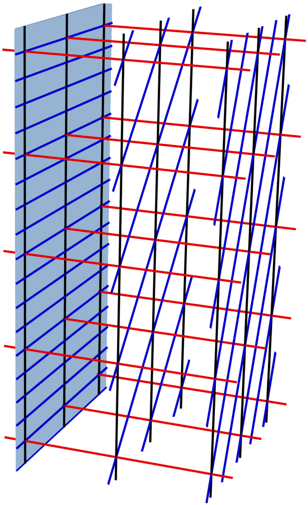
*Let $f$ be intrinsic Lipschitz. If $\gamma$ is an integral curve of the vector field $\nabla_f = Y + fZ$, then the graph*

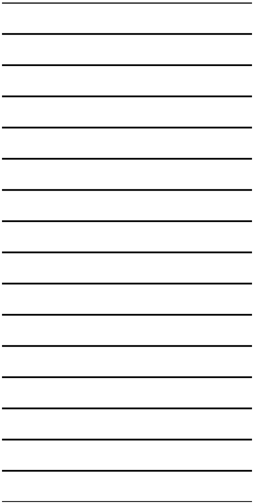
$$\tilde{\gamma}(t) = \gamma(t)X^{f(\gamma(t))}$$

*is a horizontal curve in $\Gamma_f$. We call $\gamma$ a characteristic curve.*

# $H_1$: Characteristic curves

Any smooth intrinsic graph induces a foliation of $V_0$:





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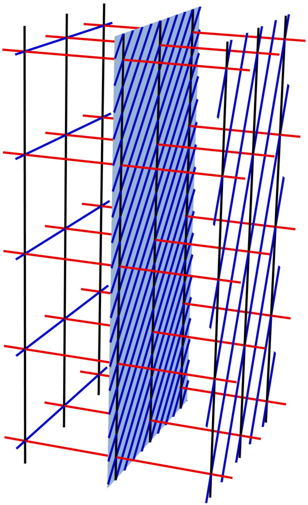

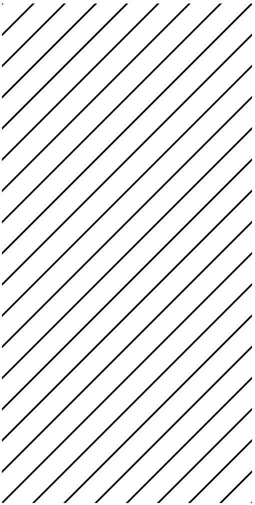

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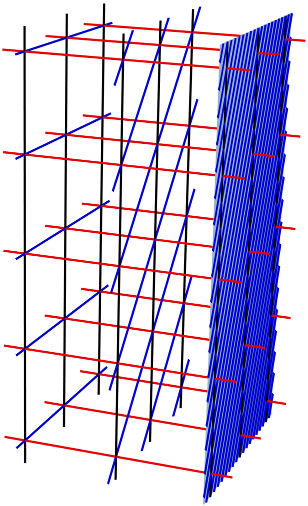

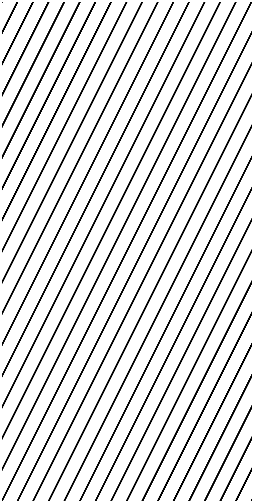

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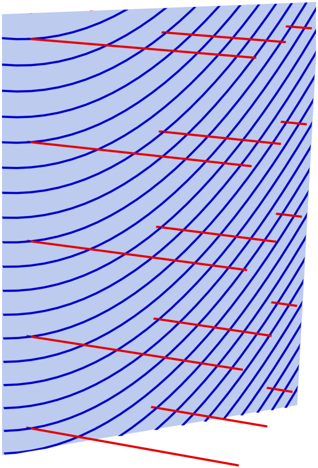

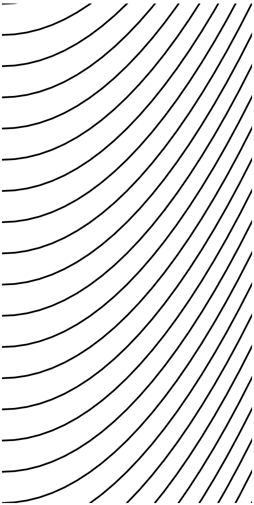

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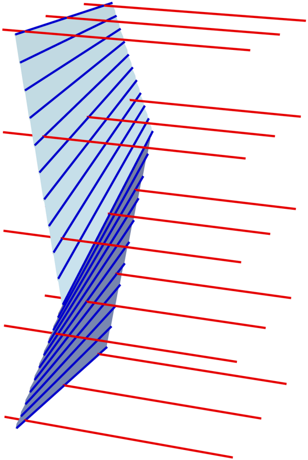
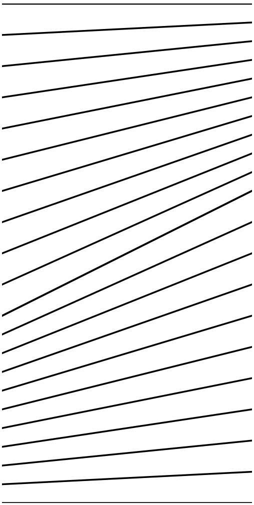
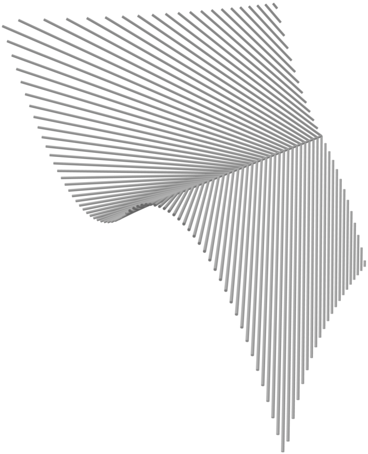


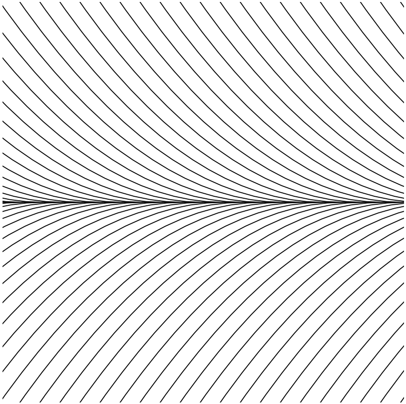


# $H_1$: Characteristic curves

Intrinsic Lipschitz graphs might not have unique integral curves:



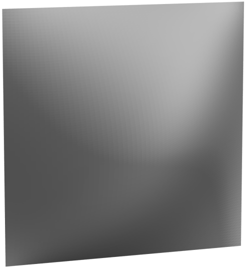
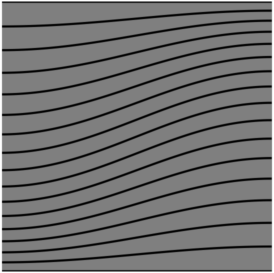
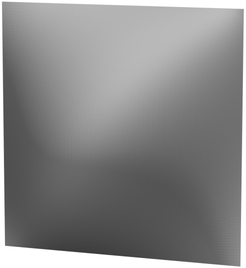
Herringbone surface



Characteristic curves

# $H_1$: Lower bound - constructing a bumpy surface

Any foliation with bounded second derivative corresponds to an intrinsic Lipschitz graph:

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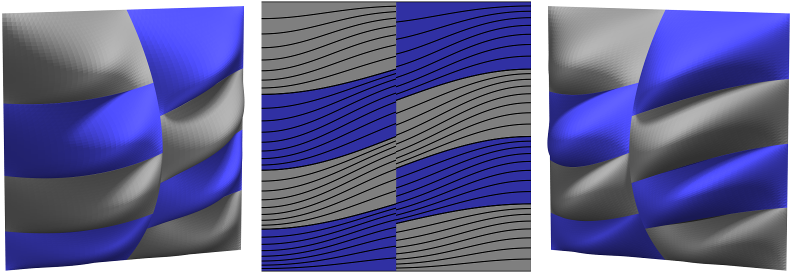

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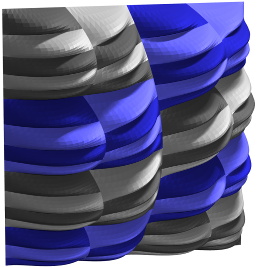
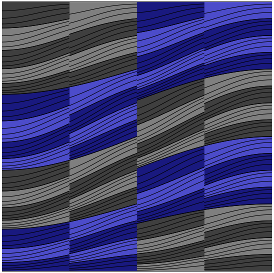
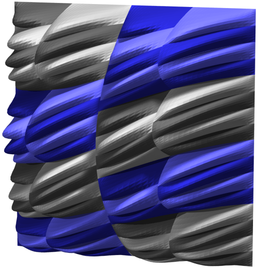

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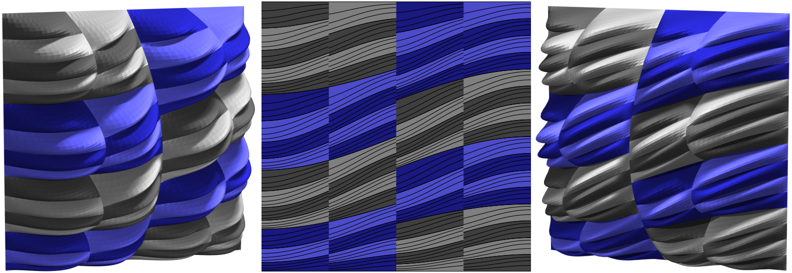
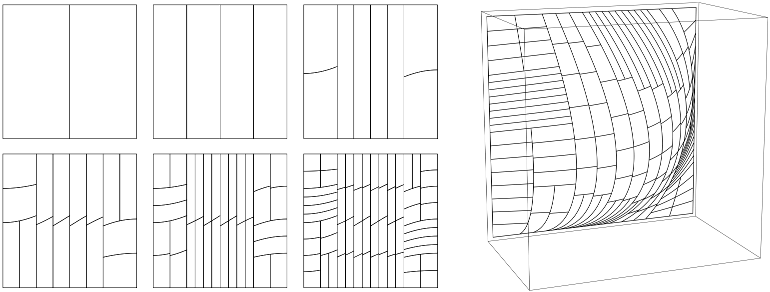


If we make the $\frac{\text{width}}{\text{height}}$ ratio large enough, we can perturb the plane by $\epsilon$ but only add $\epsilon^4$ area. If we do this $\epsilon^{-4}$ times, we get a surface that makes the inequality sharp.

# $H_1$: Upper bound - foliated corona decompositions

### Theorem (Naor–Y.)

*Any intrinsic Lipschitz graph has a foliated corona decomposition: we can cut $V_0$ along vertical lines and characteristic curves to get quadrilaterals satisfying certain bounds.*



We analyze this decomposition to prove the inequality.

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**Further questions:**

- What about surfaces of higher codimension?
- Can we classify the intrinsic Lipschitz graphs that are area-minimizing?
- We can construct minimal surfaces with a wide variety of singularities in $H_1$, but $H_n$ seems much more limited. Are minimal surfaces different in $H_1$ and $H_n$?