LARGE VORTICITY STABLE SOLUTIONS TO THE GINZBURG-LANDAU EQUATIONS

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ABSTRACT. We construct local minimizers to the Ginzburg-Landau functional of superconductivity whose number of vortices N is prescribed and blows up as the parameter ε, inverse of the Ginzburg-Landau parameter κ, tends to 0. We treat the case of N as large as $|\log \varepsilon|$, and a wide range of intensity of external magnetic field. The vortices of our solutions arrange themselves with uniform density over a subregion of the domain bounded by a "free boundary" determined via an obstacle problem, and asymptotically tend to minimize the "Coulombian renormalized energy" W introduced in [14]. The method, inspired by [22], consists in minimizing the energy over a suitable subset of the functional space, and in showing that the minimum is achieved in the interior of the subset. It also relies heavily on refined asymptotic estimates for the Ginzburg-Landau energy obtained in [14].

1. INTRODUCTION

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded and smooth domain. We are interested in finding local minimizers of the 2D Ginzburg-Landau functional of superconductivity

$$(1.1) \qquad G_\varepsilon(u, A) := \frac{1}{2} \int_\Omega |(\nabla - iA)u|^2 + |\nabla \times A - h_{ex}|^2 + \frac{(1 - |u|^2)^2}{2\varepsilon^2}.$$

Here, the complex-valued function u is a pseudo wave function, the "order parameter" in physics, that indicates the local state of the superconductor. The material is in the superconducting state in the regions where $|u| \approx 1$, and in the normal state where $|u| \approx 0$. The zeros of the order parameter carrying a non-zero degree are called *vortices*. The parameter $h_{ex} > 0$ is the intensity of the applied magnetic field, A is the induced magnetic potential, $h := \nabla \times A$ is the induced magnetic field, and finally ε is the inverse of the "Ginzburg-Landau parameter." For more background, we refer to standard physics textbooks, such as [24, 23], or to [19] and references therein.

We are interested in the asymptotic regime $\varepsilon \to 0$. This corresponds to the characteristic lengthscale of the vortices going to 0.

The vortices of a critical point of (1.1) are conveniently studied via the vorticity measure

$$\mu(u, A) := \nabla \times j(u, A) + \nabla \times A,$$

where $j(u, A) := (iu, \nabla_A u)$ denotes the superconducting current associated to a configuration and $\nabla_A = \nabla - iA$. We also write $j_\varepsilon := j(u_\varepsilon, A_\varepsilon)$, for $(u_\varepsilon, A_\varepsilon)$ a pair depending on ε. In the limit $\varepsilon \to 0$, the vorticity measure is well approximated by a sum of Dirac masses at the vortex centers, for a precise statement see the Jacobian estimate (1.7) below.

In what follows we write, for quantities A_ε and B_ε depending on ε, $A_\varepsilon \lesssim B_\varepsilon$ when there is a constant C independent of ε such that $A_\varepsilon \le CB_\varepsilon$. We also write $A_\varepsilon \ll B_\varepsilon$ when $\lim_{\varepsilon \to 0} \frac{A_\varepsilon}{B_\varepsilon} = 0$.

The asymptotic study of critical points of (1.1), as $\varepsilon \to 0$ for external fields $h_{ex} \ll \frac{1}{\varepsilon^2}$, has been carried out in a series of papers by Sandier and Serfaty, starting from [20, 21]; continuing in [16, 17, 18]; the monograph [19], and more recently [14]. It was shown in [16] that for $h_{ex} = o(|\log \varepsilon|)$ there is a critical value λ_* of $\lambda = \lim_{\varepsilon \to 0} \frac{h_{ex}}{|\log \varepsilon|}$ below which minimizers have no vortices. It was further shown in [17] that if $h_{ex} - \lambda_*|\log \varepsilon| \lesssim \log|\log \varepsilon|$ then the number of vortices of a minimizer remains bounded independent of ε. For $\log|\log \varepsilon| \ll h_{ex} - \lambda_*|\log \varepsilon| \ll |\log \varepsilon|$, minimizers have N vortices with $1 \ll N \ll h_{ex}$ as shown in [19], Chap. 9, while for $\lambda > \lambda_*$ possibly $= +\infty$, minimizers exhibit a number N proportional to h_{ex} of vortices with uniform density in a subregion determined by an obstacle problem (see [18], [19] Chap. 7, and [14]).

The behavior of critical points of (1.1) that are not necessarily minimizers of the energy, was the object of study of [18]. In [18] the vorticity measures $\mu(u_\varepsilon, A_\varepsilon)$ associated to critical points $(u_\varepsilon, A_\varepsilon)$ of (1.1), normalized by the number of vortices, are shown to converge weakly to a measure μ. The measure μ has the form $\mu := -\Delta h + h$, where h is a "stationary function" for a free boundary problem (see (1.2) below), and satisfies in some weak sense the relation $\mu \nabla h = 0$. This expresses that the average force acting on the vortices (∇h) vanishes where the vortices are, i.e. on the support of μ, thus allowing them to be at equilibrium. This result was later proved under less restrictive assumptions in [19], Chapter 13, and included cases not previously covered.

It is then natural to ask the converse, that is, given a measure μ associated to a stationary function h as above, can one find $(u_\varepsilon, A_\varepsilon)$ critical points of (1.1) such that $\mu(u_\varepsilon, A_\varepsilon)$, suitably normalized, converges weakly to μ? We will answer this question partially, by building solutions to the Ginzburg-Landau equations which are local (but not global) minimizers of the energy, and have a prescribed and divergent (as $\varepsilon \to 0$) number of vortices N. The same thing has already been accomplished for a bounded number of vortices in [22], and also in [19], Chap. 11 for N slowly diverging as $\varepsilon \to 0$.

Note that the question of inverse problems, i.e. whether given a critical point of a limiting energy, one can find corresponding critical points of the original problem, has been addressed in a variety of contexts, and for Ginzburg-Landau in particular : see [8, 4] for the situation of a finite number of vortices in 2D, and [1, 10, 11] for lines of vortices in 3D. Our approach, which is not local inversion but rather local minimization, is more in the spirit of [10, 11] for vortex lines in 3D. All previous results however, concerned only finite numbers of vortices. Another related context where such a question is addressed is that of the Allen-Cahn equation, whose limiting energy is the perimeter functional, for results there see [6, 7, 13].

The stable solutions found in [22] and [19] Chap. 11, are shown to exist for a wide range of h_{ex} (they are in fact globally minimizing for only one value of h_{ex}): they exist for fields of intensity up to $\sim \frac{1}{\varepsilon^{\alpha_0}}$, where $\alpha_0 < \frac{1}{2}$, and also for fields almost as small as constant. They thus form branches of solutions, each corresponding to a number N of vortices, with N bounded with ε, or even unbounded, but subject to the restrictions $N^2 \leq \eta h_{ex}$, $h_{ex} \leq N \varepsilon^{\frac{-2\eta}{N^2+\eta}}$, where $\eta > 0$ is a small but fixed parameter. These restrictions imply in particular (cf. [19]) that in any case $N \ll (|\log \varepsilon|)^{\frac{1}{2}}$ and if $N \to \infty$, then $h_{ex} \ll \frac{1}{\varepsilon^\alpha}$ for any $\alpha > 0$, thus leaving open the question of existence of stable solutions with $(|\log \varepsilon|)^{\frac{1}{2}} \lesssim N \lesssim h_{ex}$ vortices. We address this question and prove for the first time that branches of local minimizers with N

vortices continue to exist up to $N = O(h_{ex})$, with restrictions on N and h_{ex} specified below, see (1.5)–(1.6).

The analysis in [22, 19] consisted, roughly, in minimizing the energy G_ε "among configurations with N vortices" and exploiting the quantization of N through a careful expansion of the minimal energy of a configuration in terms of its vortices. This expansion came with an error $o(N^2)$. However, the method followed there only tolerates an error of $o\left(|\log \varepsilon|\right)$ (hence at least the condition $N^2 \lesssim |\log \varepsilon|$) which thus limited the range of applicability of the approach. What we do here is to follow the same general idea of local minimization but this time over a class which is defined so as to exploit the results and methods of [14] to get a more precise energy expansion with an error of only $o(N)$. In this way we can cover the range of N even comparable to $|\log \varepsilon|$, and fields as large as a power of $\frac{1}{\varepsilon}$.

1.1. **Statement of main result.**

In order to state the main result of the article in more accurate terms, we first proceed to introducing some auxiliary function.

An Obstacle Problem. Let N be a positive number (typically an integer representing the number of vortices). Following [14], we denote by $h_{\varepsilon,N}$ the minimizer of

$$(1.2) \qquad \frac{1}{2} \int_\Omega |\nabla h|^2 + |h - h_{ex}|^2,$$

over the set of functions h that are equal to h_{ex} on the boundary and such that

$$\int_\Omega |-\Delta h + h| = 2\pi N.$$

As long as $2\pi N \le h_{ex}|\Omega|$ (an assumption we will need to make), this minimizer $h_{\varepsilon,N}$ exists and it is the solution to an obstacle problem (see Lemma 5.1 and Appendix A in [14]). There exists a coincidence set $\omega_{\varepsilon,N} \subset \Omega$ and a number $0 < m_{\varepsilon,N} \le 1$ (for us the normalized vortex density) such that

$$(1.3) \qquad \mu_{\varepsilon,N} := -\Delta h_{\varepsilon,N} + h_{\varepsilon,N} = h_{ex}\, m_{\varepsilon,N} \mathbf{1}_{\omega_{\varepsilon,N}} \text{ and } h_{ex}\, m_{\varepsilon,N} \, |\omega_{\varepsilon,N}| = 2\pi N,$$

where $\mathbf{1}$ denotes the characteristic function of a set and $|\cdot|$ its area. If Ω is convex, then $\omega_{\varepsilon,N}$ is also convex, by a result in [3]. To avoid technical difficulties, we will assume in the sequel that Ω is convex. It is also checked in [14] that $m_{\varepsilon,N}$ is bounded universally from below by a positive constant m_0 and that $m_{\varepsilon,N} \mapsto |\omega_{\varepsilon,N}|$ is increasing. Without loss of generality we can thus assume that

$$(1.4) \qquad \lim_{\varepsilon \to 0} m_{\varepsilon,N} =: m \ge m_0 > 0,$$

where N may also depend on ε.

As in [14] (cf. Lemma 5.2), the function $h_{\varepsilon,N}$ will serve to "split" the energy G_ε in a convenient way. The vorticity of the solutions we construct will ressemble $\mu_{\varepsilon,N}$, which is a constant over the subregion $\omega_{\varepsilon,N}$, hence we will have a uniform density of vortices there, and a vanishing one outside.

Range of fields and vorticity studied. We consider fields and vortex numbers satisfying the following conditions

$$(1.5) \qquad\qquad 1 \ll N \leq \min\left\{ c\frac{h_{ex}|\Omega|}{2\pi}, K\log\frac{1}{\varepsilon\sqrt{h_{ex}}} \right\},$$

for constants $0 < c < 1$ and $K > 0$ independent of ε, and

$$(1.6) \qquad\qquad h_{ex} \lesssim \varepsilon^{-s_0} \text{ for some } s_0 < \frac{1}{7}.$$

A few comments are in order. The first condition in (1.5) $N \leq c\frac{h_{ex}|\Omega|}{2\pi}$ with $c < 1$, seems like a natural one. It corresponds to the maximum number of vortices that a certain domain can fit according the the applied field, at least it is the maximum permitted at the level of the limiting obstacle problem. On the other hand, we do not believe the second condition $N \leq K\log\frac{1}{\varepsilon\sqrt{h_{ex}}}$ and (1.6) to be optimal, but rather to be technical assumptions. Optimal conditions should rather be of the type $N \leq \frac{h_{ex}|\Omega|}{2\pi}$ and h_{ex} less than some superheating field at which vortices lose their stability, probably of order $1/\varepsilon$ (but this is all a bit speculative). In any case, we expect our condition (1.5) on N to be optimal when $h_{ex} = |\log\varepsilon|$.

We also note that $N \leq K\log\frac{1}{\varepsilon\sqrt{h_{ex}}}$ could equivalently be written $N \leq K|\log\varepsilon|$ (changing K if necessary). Let us also point out that (1.5) implies $h_{ex} \gg 1$ and $0 < m_0 \leq m < 1$ (see (1.3)), and thus $1 - m_{\varepsilon,N}$ is always bounded away from 0 for ε small enough, and $h_{ex}(1 - m_{\varepsilon,N}) \gg 1$. We will make use of these facts repeatedly during the proofs.

Our main result is:

Theorem 1.1. *Let $\Omega \subseteq \mathbb{R}^2$ be a convex smooth bounded set. Assume that $N = N(\varepsilon)$ a family of positive integers and $h_{ex} = h_{ex}(\varepsilon)$ satisfy conditions (1.5)–(1.6). Then, for ε small enough (depending on the constants above), there exists a local minimizer $(u_\varepsilon, A_\varepsilon)$ of (1.1) such that the following holds:*

(i) There exists a measure ν_ε of the form $2\pi\sum_i d_i\delta_{a_i}$, with $d_i \in \mathbb{Z}$ and $a_i \in \Omega$, such that

$$(1.7) \qquad\qquad \|\mu(u_\varepsilon, A_\varepsilon) - \nu_\varepsilon\|_{(C_0^{0,1}(\Omega))^*} \leq \varepsilon^\alpha$$

   for some $\alpha > 0$, and we have

$$(1.8) \qquad\qquad \nu_\varepsilon(\Omega) = 2\pi N,$$

   i.e. the total vorticity of u_ε is N.

(ii) The asymptotics

$$(1.9) \qquad G_\varepsilon(u_\varepsilon, A_\varepsilon) = G_\varepsilon^N + \pi N\log\frac{1}{\varepsilon\sqrt{h_{ex}}} + NW_m - 2\pi Nh_{ex}(1 - m_{\varepsilon,N}) + o(N),$$

   hold as $\varepsilon \to 0$, where W_m and G_ε^N are constants explicited in (1.11) and (1.12) respectively, and m is as in (1.4).

(iii) For any $1 < p < 2$,

$$(1.10) \qquad\qquad \|\mu(u_\varepsilon, A_\varepsilon) - \mu_{\varepsilon,N}\|_{W^{-1,p}(\Omega)} \leq C_p\sqrt{N},$$

   where C_p depends only on p.

(iv) *For $1 < p < 2$, denoting by P_ε the probability measure on $L^p_{loc}(\mathbb{R}^2, \mathbb{R}^2)$ which is the push-forward of the normalized uniform measure on $\omega_{\varepsilon,N}$ by the map $x \mapsto \sqrt{h_{ex}} j_\varepsilon (\sqrt{h_{ex}}(x + \cdot))$,* [1] *we have that, up to subsequence, P_ε converges weakly to a translation invariant probability measure P on $L^p_{loc}(\mathbb{R}^2, \mathbb{R}^2)$ such that P-a.e. $j \in \mathcal{A}_m$ and P-a.e. j minimizes W over $\mathcal{A}_m$, where W and $\mathcal{A}_m$ are defined just below.*

As announced, this theorem proves the existence of locally minimizing solutions of Ginzburg-Landau with N vortices. Moreover, their vortices tend to follow the average distribution $\mu_{\varepsilon,N}$, and their superconducting currents, after blow-up at scale $\sqrt{h_{ex}}$, tend to minimize the "Coulombian renormalized energy" W, introduced in [14], whose definition we recall below.

1.2. **The renormalized energy W.** For each $m > 0$, consider the class of currents j such that div $j = 0$ and $\nabla \times j = \sum_{p \in \mathcal{P}} 2\pi \delta_p - m$ for a discrete set $\mathcal{P}$, and $\#(\mathcal{P} \cap B_R)/R^2$ is bounded for $R > 1$. We call this class $\mathcal{A}_m$.

For any family of sets $\{\mathbf{U}_R\}_{R>0}$ in $\mathbb{R}^2$, $\chi_{\mathbf{U}_R}$ will be a family of associated positive cut-off functions satisfying

$$|\nabla \chi_{\mathbf{U}_R}| \le C, \quad \mathrm{Supp}(\chi_{\mathbf{U}_R}) \subset \mathbf{U}_R, \quad \chi_{\mathbf{U}_R}(x) = 1 \text{ if } d(x, (\mathbf{U}_R)^c) \ge 1,$$

where the constant C is independent of R. We specialize this to the family of balls B_R of radius R and center the origin.

Definition 1.2. *W is defined, for $j \in \mathcal{A}_m$, by*

$$W(j) = \limsup_{R \to \infty} \frac{W(j, \chi_{B_R})}{|B_R|},$$

where for any positive function χ, we denote

$$W(j, \chi) = \lim_{\eta \to 0} \left(\frac{1}{2} \int_{\mathbb{R}^2 \setminus \cup_{p \in \mathcal{P}} B(p, \eta)} \chi |j|^2 + \pi \log \eta \sum_{p \in \mathcal{P}} \chi(p) \right).$$

For properties of W, we refer to [14]. It is proven there in particular that $\min_{j \in \mathcal{A}_m} W$ is achieved and finite. We also set γ

$$\gamma := \lim_{R \to \infty} \left[\frac{1}{2} \int r f'^2 + \frac{(1 - f^2)^2 r}{2} + \frac{f^2}{r} \, dr - \pi \ln R \right], \text{ constant introduced in [2]},$$

where f is such that $f(r)e^{i\theta}$ is the unique (cf. [9]), up to a phase shift, degree one radial solution of

$$-\Delta u + (1 - |u|^2)u = 0 \text{ in } \mathbb{R}^2.$$

We finally let

(1.11) $$W_m := \frac{2\pi}{m} \min_{j \in \mathcal{A}_m} W + \gamma.$$

In [14], it is conjectured that the minimum of W over $\mathcal{A}_m$ is achieved for currents corresponding to a configuration of points in a perfect hexagonal lattice (of suitable volume). A small hint to this is provided there by the fact that W is uniquely minimized, among perfect

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[1]Here j_ε is extended by 0 outside Ω.

lattices of fixed volume, by the hexagonal lattice. In superconductivity, the hexagonal lattice also corresponds to the famous "Abrikosov lattice", predicted from (1.1) and observed in experiments. It was proven in [14] that global minimizers of the Ginzburg-Landau functional (1.1) have currents which minimize W, after a suitable blow inside the free-boundary region that contains the vorticity. If the conjecture about the minimum of W is true, then one expects the vortices of global minimizers to arrange themselves in such an Abrikosov hexagonal lattice. From item (iv) of Theorem 1.1 the exact same thing is expected of the lo cally minimizing solutions found here. The only qualitative difference with the global minimizers is in the total number of vortices N, which manifests itself in a different $\mu_{\varepsilon,N}$ hence a different local density of vortices and a different location for the free-boundary that encloses them.

1.3. **The local minimization method.** The local minimization first relies on the same *energy splitting* introduced in [14], which we now recall. For any A, set

$$A_1 := A - \nabla^\perp h_{\varepsilon,N}$$

where $h_{\varepsilon,N}$ is the solution of the obstacle problem (1.2) above. Let us also define accordingly

$$J_{\varepsilon,N}(u, A) := \frac{1}{2} \int_\Omega |(\nabla - iA)u|^2 + |\nabla \times A - \mu_{\varepsilon,N}|^2 + \frac{(1 - |u|^2)^2}{2\varepsilon^2} \, dx,$$

with $\mu_{\varepsilon,N}$ as in (1.3), the configuration independent quantity

(1.12) $$G_\varepsilon^N := \|h_{\varepsilon,N} - h_{ex}\|_{H^1(\Omega)}^2 + 2\pi N(m_{\varepsilon,N} - 1)h_{ex},$$

and finally the function $\xi := h_{ex} - h_{\varepsilon,N}$ (implicitly depending on ε and N, note that it is nonnegative in Ω and vanishes on $\partial\Omega$). We denote

(1.13) $$\xi_{\max} = \max_\Omega \xi = h_{ex}(1 - m_{\varepsilon,N}).$$

Lemma 1.3. *For any (u, A), $G_\varepsilon(u, A)$ can be decomposed as:*

(1.14) $$G_\varepsilon(u, A) = G_\varepsilon^N + J_{\varepsilon,N}(u, A_1) - \int_\Omega \xi \, \mu(u, A_1) + \frac{1}{2} \int_\Omega (|u|^2 - 1) \, |\nabla h_{\varepsilon,N}|^2.$$

Proof. This splitting formula was written with slightly different notation in [14]. Decomposition (1.14) follows easily from the relations:

$$|(\nabla - iA)u|^2 = |(\nabla - iA_1)u|^2 + |u|^2 \left|\nabla^\perp h_{\varepsilon,N}\right|^2 - 2j(u, A_1) \cdot \nabla^\perp h_{\varepsilon,N},$$

and

$$\begin{aligned}
|\nabla \times A - h_{ex}|^2 &= |\nabla \times A_1 + \Delta h_{\varepsilon,N} - h_{\varepsilon,N} + h_{\varepsilon,N} - h_{ex}|^2 \\
&= |\nabla \times A_1 - \mu_{\varepsilon,N}|^2 + |h_{\varepsilon,N} - h_{ex}|^2 + 2(\nabla \times A_1 - \mu_{\varepsilon,N})(h_{\varepsilon,N} - h_{ex}).
\end{aligned}$$

$\square$

The last term in the right-hand side of (1.14) will be proven later to be $o(1)$ for the configurations of interest to us.

As explained above for the case of smaller values of N, we still seek our local minimizers as configurations that minimize G_ε among configurations with N vortices. The question is to

define (following [22]) a suitable set corresponding to this heuristic idea. This is achieved here by minimizing G_ε over

(1.15)
$$S^\beta_{\varepsilon,N} := \left\{ (u, A) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2) \text{ s.t. } J_{\varepsilon,N}(u, A_1) \leq \pi N \log \frac{1}{\varepsilon \sqrt{h_{ex}}} + N W_m + \beta N; \right.$$

$$\left. \int_\Omega \xi \mu \geq 2\pi N h_{ex}(1 - m_{\varepsilon,N}) - \beta N \right\},$$

where β is chosen to be a small enough positive constant. The idea of this class is that it allows for at most N vortices when β is small, it leaves enough room for a particular test function to be admissible (the precision with which we can compute its energy is $o(N)$ while the error allowed is $\gtrsim N$), and it can easily be related to the Ginzburg-Landau energy G_ε via (1.14). As in [22, 19], the task is then, exploiting the quantization of N, to show that $\min_{S^\beta_{\varepsilon,N}} G_\varepsilon$ is achieved in the interior of $S^\beta_{\varepsilon,N}$, thus providing a local minimizer of G_ε, and to show it has the other desired properties. The class $S^\beta_{\varepsilon,N}$ differs from those in [22, 19]. Indeed, following these works it would be tempting to try minimizing over the set of configurations (u, A) such that:
$$\left| J_{\varepsilon,N}(u, A_1) - \left(\pi N \log \frac{1}{\varepsilon \sqrt{h_{ex}}} + N W_m \right) \right| \leq \beta N.$$
However, if $J_{\varepsilon,N}(u, A_1) = \pi N \log \frac{1}{\varepsilon \sqrt{h_{ex}}} + N W_m - \beta N$ we cannot conclude that u has strictly less than N vortices, which is crucial to assure that such a configuration on the boundary of the set would have an energy that exceeds that of a local minimizer. If one replaces βN by $\beta \log \frac{1}{\varepsilon \sqrt{h_{ex}}}$ a similar problem occurs; in this scenario one is able to prove that if (u, A) is in the corresponding lower boundary then it has at most $N - 1$ vortices, but the $-\beta \log \frac{1}{\varepsilon \sqrt{h_{ex}}}$ in the value of $J(u, A_1)$ makes the energy of the hypothetical configuration possibly drop below the one in Proposition 3.1, rendering that approach doomed. To overcome this difficulty (and this is a novelty of the paper) we proceed by bounding the total vorticity from below indirectly by requiring the elements of $S^\beta_{\varepsilon,N}$ to satisfy

$$\int_\Omega \xi \mu(u, A_1) \geq 2\pi h_{ex}(1 - m_{\varepsilon,N})N - \beta N,$$

this implies there are at least N vortices as long as there are no vortices with negative degree outside the free boundary $\omega_{\varepsilon,N}$. This is proven to be the case in Proposition 4.1.

Finally, the question of treating the case of N larger than the order $|\log \varepsilon|$ remains open, we believe it would require much more precise estimates on the energy than those provided by [14] that we use here, and this seems out of reach for the moment.

The plan of the paper is as follows. In Section 2 we introduce notation and recall the Jacobian estimate. In Section 3 we construct an explicit configuration with N vortices uniformly distributed according to $\mu_{\varepsilon,N}$ and a superconducting current almost minimizing W, we check the test configuration belongs to the admissible class, and by computing its energy

G_ε, we obtain an almost optimal upper bound. In Section 4 we analyze general properties of the elements of $S_{\varepsilon,N}^\beta$, in particular characterize their total vorticity as well as give lower bounds for the different components of the splitting of G_ε. We conclude with the proof of Theorem 1.1 in Section 5, by showing minimizers do not lie on the boundary of the admissible class.

2. Preliminaries

We will mostly work in blown-up coordinates at the characteristic lengthscale $1/\sqrt{h_{ex}}$. Let

$$(2.1) \qquad\qquad \varepsilon' := \varepsilon\sqrt{h_{ex}} \text{ and } x' := \sqrt{h_{ex}}\,x.$$

With this notation (1.5) becomes $N \leq K\,|\log\varepsilon'|$ in the regime where $|\log\varepsilon'| \ll h_{ex}$. The domains Ω_ε and $(\omega_{\varepsilon,N})'$ correspond to Ω and $\omega_{\varepsilon,N}$ in the blown-up coordinates x'. We transform the rest of the variables in a natural fashion, that is

$$(2.2) \qquad\qquad u'(x') = u(x),\ \xi'(x') = \xi(x) \text{ and } A'(x') = \frac{1}{\sqrt{h_{ex}}}A(x),$$

for any magnetic potential A.

The blown up currents and measures are defined in terms of these quantities in a clear way, also denoted with a prime.

We define, for $U \subset \Omega_\varepsilon$, the sets:

$$(2.3) \qquad\qquad \widehat{U} := \{x \in \Omega_\varepsilon/d(x,U) \leq 1\}$$

$$(2.4) \qquad\qquad \check{U} := \{x \in U/d(x,\partial U) \geq 1\}$$

and

$$(2.5) \qquad\qquad \widetilde{U} := \{x \in U/d(x,\partial U) \geq 2\}.$$

We now recall the basic concentration estimate relating the Jacobian and the total vorticity, à la Jerrard-Soner [12].

Theorem 2.1. *(cf. [14], Theorem 5) Assume $G_\varepsilon(u_\varepsilon, A_\varepsilon) \lesssim (\varepsilon')^{-s}$, where $s < 1$. Then there exists a measure ν'_ε that depends only on u_ε (not on A_ε) of the form $\nu_\varepsilon = 2\pi\sum_i d_i\,\delta_{a_i}$, for $d_i \in \mathbb{Z}$ and $a_i \in \Omega_\varepsilon$, with*

$$(2.6) \qquad\qquad \|\mu' - \nu'_\varepsilon\|_{(C_0^{0,1}(\Omega_\varepsilon))^*} \lesssim (\varepsilon\sqrt{h_{ex}})^{\frac{1}{2}}G_\varepsilon(u_\varepsilon, A_\varepsilon),$$

and there exists a constant $C_1 > 0$ such that for any measurable set $E \subseteq \Omega_\varepsilon$

$$(2.7) \qquad\qquad |\nu'_\varepsilon|\,(E) \leq C_1\frac{\int_{\widehat{E}} e_\varepsilon}{|\log\varepsilon'|},$$

where

$$(2.8) \qquad\qquad e_\varepsilon(u, A) := |\nabla_A u|^2 + |\nabla \times A|^2 + \frac{(1 - |u|^2)^2}{2(\varepsilon')^2}.$$

3. Local minimization class and energy upper bound

As mentioned, we follow the idea of [22], and Chapter 11 of [19], and seek for local minimizers of G_ε among configurations with "N vortices" by minimizing G_ε over the class $S_{\varepsilon,N}^\beta$, cf. (1.15), with β to be determined later. From now on we write $(u_\varepsilon, A_\varepsilon)$ an element in the Argmin of G_ε in $S_{\varepsilon,N}^\beta$. The fact that such a $(u_\varepsilon, A_\varepsilon)$ actually exists follows from classical arguments (a proof can be adapted from [22] or Chapter 11 in [19]). The task is then to show a minimizer cannot lie on the boundary.

We first show that the classes $S_{\varepsilon,N}^\beta$ are non-empty and provide a priori estimates on the energy of a minimizer in those classes.

Proposition 3.1. *The classes $S_{\varepsilon,N}^\beta$ are non-empty. Furthermore, if $(u_\varepsilon, A_\varepsilon)$ is a minimizer of G_ε over $S_{\varepsilon,N}^\beta$, then, as $\varepsilon \to 0$,*

$$(3.1) \qquad G_\varepsilon(u_\varepsilon, A_\varepsilon) \leq G_\varepsilon^N + \pi N |\log \varepsilon'| + N W_m - 2\pi N h_{ex}(1 - m_{\varepsilon,N}) + o(N).$$

Proof. The proof relies on computing the energy of a carefully built test configuration $(\widehat{u_\varepsilon}, \widehat{A_\varepsilon})$. The construction is carried out in Theorem 6 in [14] where the argument makes use of the fact that in their case N corresponds to the number of vortices of a global minimizer. We show here that that the construction still works for N satisfying (1.5) and also make explicit why the test configuration belongs to $S_{\varepsilon,N}^\beta$, an issue not present in the global minimization setting of [14].

In Theorem 6 of [14], a test configuration $(\widehat{u_\varepsilon}, \widehat{A_\varepsilon})$ is built in the following way.

Step 1:

For $M > 0$, consider a discrete set of points $\mathcal{P}_\varepsilon$, such that $\cup_{p \in \mathcal{P}_\varepsilon} B(p, M\varepsilon) \subset \omega_{\varepsilon,N}$ and such that $|\mathcal{P}_\varepsilon| = N$. A current $\widehat{j_\varepsilon}$ is built that satisfies

$$(3.2) \quad \limsup_{\eta \to 0} \frac{1}{m_{\varepsilon,N} h_{ex} |\omega_{\varepsilon,N}|} \left(\frac{1}{2} \int_{\Omega \backslash \bigcup_{p \in \mathcal{P}_\varepsilon} B\left(p, \eta(m_{\varepsilon,N} h_{ex})^{-\frac{1}{2}}\right)} |\widehat{j_\varepsilon}|^2 + \pi N \log \eta \right) \leq \min_{\mathcal{A}_1} W + o_\varepsilon(1).$$

The current $\widehat{j_\varepsilon}$ is constructed under the assumption that Ω is convex (Proposition 4.2 and Corollary 7.4 in [14]). In addition it is necessary that the coincidence set $\omega_{\varepsilon,N}$ be well contained inside Ω. More precisely one must guarantee

$$(3.3) \qquad \qquad \text{dist}\,(\omega_{\varepsilon,N}, \Omega^c) \gtrsim (m_{\varepsilon,N} h_{ex})^{-\frac{1}{2}}$$

In [14] an explicit lower bound for dist $(\omega_{\varepsilon,N}, \Omega^c)$ was proven in the case where N is the number of vortices corresponding to global minimizers of G_ε. In our case we recall that (1.5) implies $m < 1$ and therefore $\sqrt{1 - m_{\varepsilon,N}}$ is bounded from below by a positive number. From Appendix A in [14], we know that

$$(3.4) \qquad \qquad \|\nabla h_{\varepsilon,N}\|_\infty \lesssim h_{ex}\sqrt{1 - m_{\varepsilon,N}}.$$

Then since $h_{\varepsilon,N}$ is equal to $m_{\varepsilon,N} h_{ex}$ on $\omega_{\varepsilon,N}$ and to h_{ex} on the boundary, $\xi = h_{ex} - h_{\varepsilon,N}$ and (1.13), one has:

$$\text{dist}\,(\omega_{\varepsilon,N}, \Omega^c) \gtrsim \frac{\xi_{\max}}{\|\nabla h_{\varepsilon,N}\|_{L^\infty}} \gtrsim \sqrt{1 - m_{\varepsilon,N}}.$$

On the other hand, $m_{\varepsilon,N} \geq m_0 > 0$, $1 - m_{\varepsilon,N}$ is bounded below, and $h_{ex} \gg 1$, therefore

$$\sqrt{1 - m_{\varepsilon,N}} \gtrsim \frac{1}{\sqrt{m_{\varepsilon,N} h_{ex}}},$$

which proves (3.3) in our case.

$\underline{\text{Step 2:}}$

The functions $\widehat{u_\varepsilon}$ and $\widehat{A_\varepsilon}$ are then built satisfying

$$(3.5) \qquad\qquad\qquad |\widehat{u_\varepsilon}| = 1 \text{ on } \Omega \setminus \cup_{p \in \mathcal{P}_\varepsilon} B(p, M\varepsilon).$$

$$(3.6) \qquad\qquad\qquad \widehat{A_\varepsilon} := \widehat{A_{1,\varepsilon}} + \nabla^\perp h_{\varepsilon,N},$$

where $\widehat{A_{1,\varepsilon}}$ is chosen to satisfy $\nabla \times \widehat{A_{1,\varepsilon}} = \mu_{\varepsilon,N}$ and

$$(3.7) \qquad \frac{1}{2} \int_{B(p,M\varepsilon)} \left| \nabla_{\widehat{A_{1,\varepsilon}}} \widehat{u_\varepsilon} \right|^2 + \frac{(1 - |\widehat{u_\varepsilon}|^2)^2}{2\varepsilon^2} = \pi \log M + \gamma + o_M(1) + o_\varepsilon(1),$$

for all $p \in \mathcal{P}_\varepsilon$, where $o_M(1)$ is a quantity that goes to zero as $M \to \infty$, uniformly with respect to ε.

$$(3.8) \quad \frac{1}{2} \int_{B\left(p, \eta(m_{\varepsilon,N} h_{ex})^{-\frac{1}{2}}\right) \setminus B(p,M\varepsilon)} \left| \nabla_{\widehat{A_{1,\varepsilon}}} \widehat{u_\varepsilon} \right|^2 + \frac{(1 - |\widehat{u_\varepsilon}|^2)^2}{2\varepsilon^2} \leq \pi \log \left(\frac{\eta}{\sqrt{m_{\varepsilon,N} h_{ex}} M\varepsilon} \right) + \mathcal{O}(\eta).$$

$$(3.9) \qquad \int_\Omega \mu(\widehat{u_\varepsilon}, \widehat{A_{1,\varepsilon}}) = \int_\Omega \mu_{\varepsilon,N} = 2\pi N, \text{ and } \mu(\widehat{u_\varepsilon}, \widehat{A_{1,\varepsilon}}) \equiv 0 \text{ on } (\omega_{\varepsilon,N})^c.$$

From this it easily follows that $(\widehat{u_\varepsilon}, \widehat{A_\varepsilon}) \in S_{\varepsilon,N}^\beta$. Indeed

$$\begin{aligned} J_{\varepsilon,N}(\widehat{u_\varepsilon}, \widehat{A_{1,\varepsilon}}) &= \frac{1}{2} \int_\Omega \left| \nabla_{\widehat{A_{1,\varepsilon}}} \widehat{u_\varepsilon} \right|^2 + \left| \nabla \times \widehat{A_{1,\varepsilon}} - \mu_{\varepsilon,N} \right|^2 + \frac{(1 - |\widehat{u_\varepsilon}|^2)^2}{2\varepsilon^2} \\ &= \frac{1}{2} \int_\Omega \left| \nabla_{\widehat{A_{1,\varepsilon}}} \widehat{u_\varepsilon} \right|^2 + \frac{(1 - |\widehat{u_\varepsilon}|^2)^2}{2\varepsilon^2}. \end{aligned}$$

Above, the second term disappears because of (3.6).

The identity $|\nabla_A u|^2 = |\nabla |u||^2 + |u|^2 |j|^2$ implies, together with (3.5) that

$$\left| \nabla_{\widehat{A_{1,\varepsilon}}} \widehat{u_\varepsilon} \right|^2 = |\widehat{j_\varepsilon}|^2 \text{ on } \Omega \setminus \cup_{p \in \mathcal{P}_\varepsilon} B(p, M\varepsilon).$$

Therefore, combining (3.2), (3.7) and (3.8), one gets

$$J_{\varepsilon,N}(\widehat{u_\varepsilon}, \widehat{A_{1,\varepsilon}}) \leq \pi N \left|\log \varepsilon\right| + 2\pi N \min_{\mathcal{A}_1} W + \pi N \log \frac{1}{\sqrt{m_{\varepsilon,N} \, h_{ex}}} + N \left(\gamma + o_\varepsilon(1) + o_M(1) + \mathcal{O}(\eta)\right).$$

Next, passing to a subsequence, one can let $M \to \infty$ as $\varepsilon \to 0$, so that the above becomes, after taking $\eta \to 0$:

$$(3.10) \qquad J_{\varepsilon,N}(\widehat{u_\varepsilon}, \widehat{A_{1,\varepsilon}}) \leq \pi N \left|\log \varepsilon\right| + 2\pi N \min_{\mathcal{A}_1} W + \pi N \log \frac{1}{\sqrt{m_{\varepsilon,N} \, h_{ex}}} + N\gamma + o(N).$$

By assumption $m_{\varepsilon,N} \to m$, therefore $N \left|\log m_{\varepsilon,N}\right| = N \left|\log m\right| + o(N)$. According to (1.17) and (1.18) of [14], by scaling, one has

$$\min_{\mathcal{A}_1} W = \frac{1}{m} \min_{\mathcal{A}_m} W + \frac{1}{4} \log m,$$

Thus, by definition of W_m and ε', (3.10) is equal to

$$(3.11) \qquad\qquad J_{\varepsilon,N}(\widehat{u_\varepsilon}, \widehat{A_{1,\varepsilon}}) \leq \pi N \left|\log \varepsilon'\right| + N W_m + o(N).$$

This shows that $(\widehat{u_\varepsilon}, \widehat{A_{1,\varepsilon}})$ satisfies the upper bound in the definition of $S_{\varepsilon,N}^\beta$. To see it also satisfies the lower bound, we see by (3.9) that

$$\int_\Omega \xi \mu(\widehat{u_\varepsilon}, \widehat{A_{1,\varepsilon}}) = \int_{\omega_{\varepsilon,N}} \xi \mu(\widehat{u_\varepsilon}, \widehat{A_{1,\varepsilon}}) = \xi_{\max} \int_{\omega_{\varepsilon,N}} \mu(\widehat{u_\varepsilon}, \widehat{A_{1,\varepsilon}})$$

$$(3.12) \qquad = \xi_{\max} \int_\Omega \mu(\widehat{u_\varepsilon}, \widehat{A_{1,\varepsilon}}) = \xi_{\max} \int_\Omega \mu_{\varepsilon,N} = 2\pi N h_{ex}(1 - m_{\varepsilon,N}).$$

By (3.11) and (3.12) we obtain that $(\widehat{u_\varepsilon}, \widehat{A_\varepsilon}) \in S_{\varepsilon,N}^\beta$.

To conclude, from the splitting formula (1.14), we see that

$$G_\varepsilon(\widehat{u_\varepsilon}, \widehat{A_\varepsilon}) = G_\varepsilon^N + J_{\varepsilon,N}(\widehat{u_\varepsilon}, \widehat{A_{1,\varepsilon}}) - \int_\Omega \xi \mu(\widehat{u_\varepsilon}, \widehat{A_{1,\varepsilon}}) + \frac{1}{2} \int_\Omega (|\widehat{u_\varepsilon}|^2 - 1) \, |\nabla h_{\varepsilon,N}|^2.$$

The last term is identically zero because $|\nabla h_{\varepsilon,N}| = 0$ on $\omega_{\varepsilon,N}$, while $|\widehat{u_\varepsilon}| = 1$ on $(\omega_{\varepsilon,N})^c$ by (3.5). Hence, by (3.11) and (3.12) again, we see that

$$G_\varepsilon(\hat{u_\varepsilon}, \hat{A_\varepsilon}) \leq G_\varepsilon^N + \pi N |\log \varepsilon'| + N W_m - 2\pi N h_{ex}(1 - m_{\varepsilon,N}) + o(N),$$

and the proposition is proved. $\square$

In the next section, the lower bound component of Theorem 1.1 requires a very precise approximation of the vorticity measure by ν_ε. We also need a better control on the rightmost term in the splitting formula (1.14).

Lemma 3.2. *Let $(u_\varepsilon, A_\varepsilon)$ be a minimizing pair on $S_{\varepsilon,N}^\beta$. Assume N and h_{ex} satisfy (1.5) and (1.6). Then there exists an $\alpha > 0$ such that*

$$(3.13) \qquad\qquad \|\mu' - \nu_\varepsilon'\|_{(C_0^{0,1}(\Omega_\varepsilon))^*} = o(\varepsilon^\alpha)$$

and

$$(3.14) \qquad \int_\Omega (1 - |u_\varepsilon|^2) |\nabla h_{\varepsilon,N}|^2 = o(1).$$

Proof. Let us assume first $h_{ex} \gg |\log \varepsilon|$. It is easy to see that in this case $G_\varepsilon^N \lesssim h_{ex}^2$, while the rest of the positive terms in (3.1) are $o(h_{ex}^2)$. Therefore

$$G_\varepsilon(u_\varepsilon, A_\varepsilon) \lesssim h_{ex}^2.$$

From (2.6) and (1.6) we can deduce

$$(3.15) \qquad \|\mu' - \nu'_\varepsilon\|_{(C_0^{0,1}(\Omega_\varepsilon))^*} \lesssim \varepsilon^{\frac{1}{2}} h_{ex}^{\frac{1}{4}} \cdot h_{ex}^2 \lesssim \varepsilon^{\frac{1}{2} - \frac{9}{4} s_0},$$

The fact that this is $o(\varepsilon^\alpha)$ for some $\alpha > 0$ follows from (1.6). Thus, we have proved (3.13), and (1.7) also follows. For the second assertion, Cauchy-Schwarz yields

$$\left| \int_\Omega (|u_\varepsilon|^2 - 1) |\nabla h_{\varepsilon,N}|^2 \right| \leq \left(\int_\Omega (1 - |u_\varepsilon|^2)^2 \right)^{\frac{1}{2}} \|\nabla h_{\varepsilon,N}\|_\infty^2 |\Omega|^{\frac{1}{2}} \lesssim \sqrt{\varepsilon^2 G_\varepsilon(u_\varepsilon, A_\varepsilon)} \|\nabla h_{\varepsilon,N}\|_\infty^2$$

Using (3.4) and the bound $h_{ex}^2 \gtrsim G_\varepsilon(u_\varepsilon, A_\varepsilon)$, the above leads to

$$\int (1 - |u_\varepsilon|^2) |\nabla h_{\varepsilon,N}|^2 \lesssim \varepsilon h_{ex}^3 \leq \varepsilon^{(1-3s_0)} = o(1),$$

by (1.6).

The case $h_{ex} \lesssim |\log \varepsilon|$ can be dealt with similarly. $\qquad \square$

4. FURTHER PROPERTIES OF MINIMIZERS AND LOWER BOUND FOR $G_\varepsilon(u_\varepsilon, A_\varepsilon)$

The next proposition, which is the crucial part of the proof, shows that the elements of $S_{\varepsilon,N}^\beta$ have N vortices and also that the upper bound in (3.1) is sharp.

We recall that Ω_ε and $\omega'_{\varepsilon,N}$ correspond to Ω and $\omega_{\varepsilon,N}$ in the blown up coordinates. At this point, it is convenient to introduce the notion of "vorticity adapted" family of functions. To that end, for a function ψ_ε, let $\omega^{\psi_\varepsilon} := \left\{ x \in \Omega_\varepsilon \, s.t. \, \psi_\varepsilon(x) = \frac{1}{2}|\log \varepsilon'| \right\}$. We will restrict to considering families $\{\psi_\varepsilon\}_{\varepsilon > 0}$ such that

$$(4.1) \qquad \|\nabla \psi_\varepsilon\|_\infty = o(|\log \varepsilon'|).$$

Now, define the class:

$$\mathcal{VA} := \left\{ \{\psi_\varepsilon\}_{\varepsilon > 0} \text{ such that } \psi_\varepsilon : \Omega_\varepsilon \to \mathbb{R}, \psi_\varepsilon \text{ satisfies } (4.1), \omega^{\psi_\varepsilon} \supset \omega'_{\varepsilon,N}; \right.$$

$$\left. 0 \leq \psi_\varepsilon(x) < \frac{1}{2}|\log \varepsilon'| \text{ if } x \in \Omega_\varepsilon \setminus \omega^{\psi_\varepsilon}; \psi_\varepsilon = 0 \text{ on } \partial\Omega_\varepsilon \right\}$$

Abusing notation, we sometimes call ψ_ε a vorticity adapted function when $\{\psi_\varepsilon\}_{\varepsilon > 0}$ is a vorticity adapted family. This notion will help us in making use of a family of estimates available in [14] in a transparent way. The proof in [14] of such estimates, is presented for a particular vorticity adapted function, while here we use the estimates in their full generality. Ultimately,

these estimates, together with the quantization of the vorticity measure, will prove that $S_{\varepsilon,N}^{\beta}$ consists only of functions with N essential vortices.

In what follows, a $+$ or $-$ superscript denotes the positive resp. negative part of a function or measure.

Proposition 4.1. *Let N and h_{ex} be sequences satisfying (1.5) and (1.6). Assume $(u^{\varepsilon}, A^{\varepsilon}) \in S_{\varepsilon,N}^{\beta}$ where $\beta > 0$ is small enough but fixed independent of ε. Then, for ε small, we have:*

(i) *There are no vortices with negative associated degree outside the free boundary, i.e.:*
$$\nu_{\varepsilon} = \nu_{\varepsilon}^{+} \ \text{ on } (\omega_{\varepsilon,N})^{c}.$$

(ii) *The total vorticity is N, in other words:*
$$\nu_{\varepsilon}(\Omega) = 2\pi N.$$

(iii)
$$\int_{\Omega} \xi \mu(u^{\varepsilon}, A_1^{\varepsilon}) \leq 2\pi N h_{ex}(1 - m_{\varepsilon,N}) + o(1).$$

(iv)
$$J_{\varepsilon,N}(u^{\varepsilon}, A_1^{\varepsilon}) \geq \pi N \left|\log \varepsilon'\right| + N W_m + o(N).$$

Proof. Recall that $(u^{\varepsilon}, A^{\varepsilon}) \in S_{\varepsilon,N}^{\beta}$ means that

(4.2)
$$J_{\varepsilon,N}(u^{\varepsilon}, A_1^{\varepsilon}) \leq \pi N \left|\log \varepsilon'\right| + N W_m + \beta N,$$

and

(4.3)
$$\int_{\Omega} \xi \mu(u^{\varepsilon}, A_1^{\varepsilon}) \geq 2\pi N h_{ex}(1 - m_{\varepsilon,N}) - \beta N.$$

The proof relies on lower bounds on quantities of the form

(4.4)
$$J_{\varepsilon,N}(u^{\varepsilon}, A_1^{\varepsilon}) - \int_{\Omega} \psi_{\varepsilon}(\sqrt{h_{ex}} \cdot)\mu(u^{\varepsilon}, A_1^{\varepsilon}),$$

where $\{\psi_{\varepsilon}\}_{\varepsilon>0} \in \mathcal{VA}$. More precisely, in Steps 3-6 below we derive assertions (i)-(iv) with the aid of sufficiently good lower bounds on (4.4) that are available for pairs $(u^{\varepsilon}, A^{\varepsilon})$ such that their superconducting current j is divergence free. We first show that we can reduce to this case by proving (4.6) below for a A_0^{ε} such that $j(u^{\varepsilon}, A_0^{\varepsilon})$ is divergence free.

<u>Step 1:</u> First we claim we may assume that $(u^{\varepsilon}, A^{\varepsilon})$ satisfies

(4.5)
$$\text{div } j(u^{\varepsilon}, A^{\varepsilon}) = 0.$$

This assumption is justified by the following: we claim there is a A_0^{ε} such that div $j(u^{\varepsilon}, A_0^{\varepsilon}) = 0$ and

$$J_{\varepsilon,N}(u^{\varepsilon}, A_1^{\varepsilon}) - \int_{\Omega} \psi_{\varepsilon}(\sqrt{h_{ex}} \cdot)\, \mu(u^{\varepsilon}, A_1^{\varepsilon}) \ \geq \ J_{\varepsilon,N}(u^{\varepsilon}, A_{0,1}^{\varepsilon}) - \int_{\Omega} \psi_{\varepsilon}(\sqrt{h_{ex}} \cdot)\, \mu(u^{\varepsilon}, A_{0,1}^{\varepsilon}) + o(1),$$

(4.6)

holds for any $\{\psi_{\varepsilon}\}_{\varepsilon>0} \in \mathcal{VA}$. Indeed, letting A_0^{ε} be such that $G_{\varepsilon}(u^{\varepsilon}, A_0^{\varepsilon}) = \min_{A \in H^1(\Omega, \mathbb{R}^2)} G_{\varepsilon}(u^{\varepsilon}, A)$ (such a minimizer is easily seen to exist), by minimality $(u^{\varepsilon}, A_0^{\varepsilon})$ satisfies the second Ginzburg-Landau equations, that is $-\nabla^{\perp}(\nabla \times A_0^{\varepsilon}) = (iu^{\varepsilon}, \nabla_{A_0^{\varepsilon}} u^{\varepsilon})$, whence $j(u^{\varepsilon}, A_0^{\varepsilon})$ is divergence free.

By definition of A_0^ε, we have

$$(4.7) \qquad G_\varepsilon(u^\varepsilon, A_0^\varepsilon) \leq G_\varepsilon(u^\varepsilon, A^\varepsilon) \lesssim \max\{h_{ex}^2, h_{ex}\, |\log \varepsilon'|\},$$

which follows from the decomposition (1.14) together with $G_\varepsilon^N \lesssim \max\{h_{ex}^2, h_{ex}\, |\log \varepsilon'|\}$ and the definition of $S_{\varepsilon,N}^\beta$. As in the proof of Lemma 3.2 this upper bound implies

$$\int_\Omega (1 - |u^\varepsilon|^2)\, |\nabla h_{\varepsilon,N}|^2 = o(1).$$

We can use the splitting formula (1.14) to conclude:

$$G_\varepsilon^N + J_{\varepsilon,N}(u^\varepsilon, A_1^\varepsilon) - \int_\Omega \xi \mu(u^\varepsilon, A_1^\varepsilon) \;\geq\; G_\varepsilon^N + J_{\varepsilon,N}(u^\varepsilon, A_{0,1}^\varepsilon) - \int_\Omega \xi \mu(u^\varepsilon, A_{0,1}^\varepsilon) + o(1).$$

From this, we can assert that

$$J_{\varepsilon,N}(u^\varepsilon, A_1^\varepsilon) - \int_\Omega \psi(\sqrt{h_{ex}}\,\cdot)\mu(u^\varepsilon, A_1^\varepsilon) \geq J_{\varepsilon,N}(u^\varepsilon, A_{0,1}^\varepsilon) - \int_\Omega \psi_\varepsilon(\sqrt{h_{ex}}\,\cdot)\mu(u^\varepsilon, A_{0,1}^\varepsilon)$$

$$- \int_\Omega \left(\psi_\varepsilon(\sqrt{h_{ex}}\,\cdot) - \xi \right) \left(\mu(u^\varepsilon, A_1^\varepsilon) - \mu(u^\varepsilon, A_{0,1}^\varepsilon) \right) + o(1).$$

If the field h_{ex} is not too large, the Jacobian estimate (2.6) implies that the last term is $o(1)$. To see this, recall that the measure ν_ε in (2.6) does not depend on A_ε and therefore:

$$\left| \int_\Omega \left(\psi(\sqrt{h_{ex}}\,\cdot) - \xi \right) \left(\mu(u^\varepsilon, A_{0,1}^\varepsilon) - \mu(u^\varepsilon, A_1^\varepsilon) \right) \right|$$

$$\leq \left| \int_\Omega \left(\psi(\sqrt{h_{ex}}\,\cdot) - \xi \right) \left(\mu(u^\varepsilon, A_{0,1}^\varepsilon) - d\nu_\varepsilon \right) \right| + \left| \int_\Omega \left(\psi(\sqrt{h_{ex}}\,\cdot) - \xi \right) \left(d\nu_\varepsilon - \mu(u^\varepsilon, A_1^\varepsilon) \right) \right|$$

$$\lesssim \left\| \psi(\sqrt{h_{ex}}\,\cdot) - \xi \right\|_\infty \cdot \max\left\{ \left\| \nu_\varepsilon - \mu(u^\varepsilon, A_{0,1}^\varepsilon) \right\|_{(C_0^{0,1}(\Omega))^*}, \left\| \nu_\varepsilon - \mu(u^\varepsilon, A_1^\varepsilon) \right\|_{(C_0^{0,1}(\Omega))^*} \right\}$$

$$\lesssim (\|\xi\|_\infty + \|\nabla \psi\|_\infty) \cdot \left(\sqrt{\varepsilon'} G_\varepsilon(u^\varepsilon, A^\varepsilon) \right) \lesssim (h_{ex} + h_{ex}^{\frac{1}{2}}|\log \varepsilon|)\, \varepsilon^{\frac{1}{2}}\, h_{ex}^{\frac{1}{4}}\, h_{ex}^2 = o(1),$$

thanks to (1.6) and (4.7). This proves the claim.

$\underline{\text{Step 2}}$: **Lower bound with vorticity adapted functions.** We switch to blown-up coordinates in order to appeal to the proof of Proposition 6.1 in [14]. Recalling that the Jacobian estimate allows us to approximate μ' by ν_ε', we can replace bounding below (4.4) by bounding below

$$J_{\varepsilon,N}'((u^\varepsilon)', (A_1^\varepsilon)') - \int_{\Omega_\varepsilon} \psi_\varepsilon\, d\nu_\varepsilon'.$$

But it was proved in [14] that there exists a measure g_ε' depending only on $((u^\varepsilon)', (A^\varepsilon)')$ such that for any vorticity adapted function ψ_ε, and for some constant $C_1 > 0$,

$$J'_{\varepsilon,N}((u^\varepsilon)', (A_1^\varepsilon)') - \int_{\Omega_\varepsilon} \psi_\varepsilon \, d\nu'_\varepsilon \geq \frac{1}{6} \int_{\Omega_\varepsilon} \left(1 - \frac{2\psi_\varepsilon}{|\log \varepsilon'|}\right) e_\varepsilon + \frac{1}{C_1}(g'_\varepsilon)^+ \left((\widetilde{\omega'_{\varepsilon,N}})^c\right)$$

$$(4.8) \qquad\qquad\qquad\qquad\qquad +g'_\varepsilon\left((\widetilde{\omega'_{\varepsilon,N}})^c\right) + o(|\omega'_{\varepsilon,N}|),$$

where e_ε is as in (2.8).

The argument can be found in (6.28)–(6.35) of [14] where ζ'_ε can be replaced by ψ_ε; indeed it can be readily checked that the only fact about ζ'_ε that was used in the proof was that it was vorticity adapted. Note that here we also use the fact that div $j(u^\varepsilon, A^\varepsilon) = 0$, obtained in Step 1.

We also record the followings facts from [14, 15]

1. There exists a family of disjoint balls $\{B_i\}_{i\in I}$ of bounded total radius such that the support of ν'_ε is contained in $\cup_{i\in I} B_i$, a universal constant C_2 and functions $\{g_\varepsilon^{B_i}\}$ such that if $\nu'_\varepsilon(B_i) < 0$

$$(4.9) \qquad \int_{B_i} g_\varepsilon^{B_i} \geq \frac{1}{8}\left(|\log \varepsilon'| - C_2\right)|\nu_\varepsilon(B_i)| \geq \frac{1}{16}|\log \varepsilon'| \, |\nu_\varepsilon(B_i)|$$

(The family $\{B_i\}_{i\in I}$ is built in [15], where (4.9) is proved in Proposition 3.2)

2. There exists a universal constant C_3 such that

$$(4.10) \qquad (g'_\varepsilon)^+ \geq \frac{3}{4}\left(\sum_{\{i,\, \nu'_\varepsilon(B_i)<0\}} \left(g_\varepsilon^{B_i} - C_3\right)\mathbf{1}_{B_i}\right)$$

where both sides are understood as measures. (This is a consequence of the definition of g'_ε in [14], combined with (3.22) in [15] and (4.9)).

3. Let P_ε be the probability measure on $L^p_{loc}(\mathbb{R}^2, \mathbb{R}^2)$ which is the push-forward of the normalized uniform measure on $\omega_{\varepsilon,N}$ by the map $x \mapsto \sqrt{h_{ex}} j_\varepsilon(\sqrt{h_{ex}}(x + \cdot))$. Let $x'_\varepsilon \in \omega'_{\varepsilon,N}$ where $d(x'_\varepsilon, (\omega_{\varepsilon,N})^c) \to +\infty$. By the definition of g'_ε, one can see as in the proof of Proposition 6.1 in [14] that if

$$\forall R > 0, \quad \limsup_{\varepsilon \to 0} \fint_{B(x'_\varepsilon, R)} dg'_\varepsilon(x'_\varepsilon + \cdot) < +\infty$$

then assumption (4.5) together with $m_{\varepsilon,N} \to m$ implies that $j((u^\varepsilon)', (A_1^\varepsilon)')(x'_\varepsilon + \cdot)$ converges in $L^p_{loc}(\mathbb{R}^2, \mathbb{R}^2)$ for any $p < 2$ to some $j \in \mathcal{A}_m$. Thus, since the field h_{ex} is not too large (cf. (1.6)), Proposition 6.3 in [14] tells us that P_ε converges weakly to a translation invariant probability measure P on $L^p_{loc}(\mathbb{R}^2, \mathbb{R}^2)$ such that P-a.e. $j \in \mathcal{A}_m$ and furthermore (recalling the definitions (1.11) and (2.5))

$$(4.11) \qquad g'_\varepsilon(\widetilde{\omega'_{\varepsilon,N}}) \geq N\left(\frac{2\pi}{m}\int W(j)\, dP(j) + \gamma + o(1)\right) + o(|\omega'_{\varepsilon,N}|) \geq NW_m + o(N),$$

since, in view of (1.3)

$$\left|\omega'_{\varepsilon,N}\right| = |\omega_{\varepsilon,N}| \, h_{ex} = \frac{2\pi N}{m_{\varepsilon,N}} \leq \frac{2\pi N}{m_0} = \mathcal{O}(N). \tag{4.12}$$

Using (4.8) together with (4.11)–(4.12), we have thus found that for any $\psi_\varepsilon \in \mathcal{VA}$,

$$J'_{\varepsilon,N}((u^\varepsilon)', (A_1^\varepsilon)') - \int_{\Omega_\varepsilon} \psi_\varepsilon \, d\nu'_\varepsilon \geq \frac{1}{6} \int_{\Omega_\varepsilon} \left(1 - \frac{2\psi_\varepsilon}{|\log \varepsilon'|}\right) e_\varepsilon + N W_m + o(N). \tag{4.13}$$

Step 3: **All vortices of u^ε are contained in a neighborhood of $\omega'_{\varepsilon,N}$.**

We will quite often need to use the fact that for any measurable set A,

$$\nu'_\varepsilon(A) \in 2\pi\mathbb{Z}, \quad \text{while} \quad \left|(\nu'_\varepsilon)^+(A)\right|, \, \left|(\nu'_\varepsilon)^-(A)\right|, \, |\nu'_\varepsilon|\,(A) \in 2\pi\mathbb{N}. \tag{4.14}$$

In what remains, α is set to be the constant

$$\alpha := \frac{|\log \varepsilon'|}{2h_{ex}(1 - m_{\varepsilon,N})}.$$

Note that α depends on ε and N but we omit to indicate this dependence. Also, from here on, all the primed quantities denote the blown up versions of the unprimed ones, according to the transformations (2.1) – (2.2). We begin by noticing that $\alpha\xi' \in \mathcal{VA}$. Indeed, since $h_{ex} \gg 1$, and $m = \lim_{\varepsilon \to 0} m_{\varepsilon,N} < 1$ one can readily see, using (3.4) and the fact that $h_{ex}(1 - m_{\varepsilon,N}) \gg 1$, that

(4.15)

$$\|\nabla(\alpha\xi')\|_\infty = \mathcal{O}\left(\frac{|\log \varepsilon'|}{h_{ex}(1 - m_{\varepsilon,N})}\right) \cdot \|\nabla\xi\|_\infty \cdot \frac{1}{\sqrt{h_{ex}}} = \mathcal{O}\left(\frac{|\log \varepsilon'|}{(h_{ex})^{\frac{3}{2}}(1 - m_{\varepsilon,N})}\right) \cdot \|\nabla h_{\varepsilon,N}\|_\infty$$

$$\lesssim \mathcal{O}\left(\frac{|\log \varepsilon'|}{(h_{ex})^{\frac{3}{2}}(1 - m_{\varepsilon,N})}\right) h_{ex}\sqrt{1 - m_{\varepsilon,N}} \lesssim \mathcal{O}\left(\frac{|\log \varepsilon'|}{\sqrt{h_{ex}(1 - m_{\varepsilon,N})}}\right) = o(|\log \varepsilon'|).$$

By (4.2), (4.3) and definition of α, we find that

$$J'_{\varepsilon,N}((u^\varepsilon)', (A_1^\varepsilon)') - \alpha \int_{\Omega_\varepsilon} \xi' \mu'((u^\varepsilon)', (A_1^\varepsilon)') \tag{4.16}$$

$$\leq \pi N|\log \varepsilon'| + N W_m + \beta N - \alpha(2\pi N h_{ex}(1 - m_{\varepsilon,N}) - \beta N)$$

$$= (1 + \alpha)\beta N + N W_m.$$

Conversely, the bound (4.13) applied to $\psi_\varepsilon = \alpha\xi'$ together with (4.16) will allow us to rule out the presence of vortices with negative degrees where ξ' is small. In turn, we shall see that this implies an upper bound on the total vorticity $\nu_\varepsilon(\Omega_\varepsilon)$.

As announced, we apply (4.13) to $\psi_\varepsilon = \alpha\xi'$ which is indeed vorticity adapted, combining with the Jacobian estimate (2.6), (4.16) and (4.13) we obtain that

$$
\begin{aligned}
(1+\alpha)\beta N + NW_m &\geq J'_{\varepsilon,N}((u^\varepsilon)', (A_1^\varepsilon)') - \alpha \int_{\Omega_\varepsilon} \xi' \mu'((u^\varepsilon)', (A_1^\varepsilon)') \\
&\geq J'_{\varepsilon,N}((u^\varepsilon)', (A_1^\varepsilon)') - \alpha \int_{\Omega_\varepsilon} \xi' \, d\nu'_\varepsilon + \mathcal{O}\left(\|\nabla(\alpha\xi')\|_\infty \|\mu' - \nu'_\varepsilon\|_{(C_0^{0,1}(\Omega_\varepsilon))^*} \right) \\
&\geq \frac{1}{6} \int_{\Omega_\varepsilon} \left(1 - \frac{2\alpha\xi'}{|\log \varepsilon'|}\right) e_\varepsilon + NW_m + o(N) + \mathcal{O}\left(|\log \varepsilon'| \, \varepsilon^{\frac{1}{2}} \, h_{ex}^{\frac{1}{4}} \, G_\varepsilon(u^\varepsilon, A^\varepsilon)\right).
\end{aligned}
$$
(4.17)

The last term is $o(1)$ because $G_\varepsilon(u^\varepsilon, A^\varepsilon) \lesssim \max\{h_{ex}^2, h_{ex}|\log \varepsilon'|\}$ and h_{ex} satisfies (1.6). Thus, we obtain

$$
\frac{1}{6} \int_{\Omega_\varepsilon} \left(1 - \frac{2\alpha\xi'}{|\log \varepsilon'|}\right) e_\varepsilon \leq \beta(1+\alpha)N + o(N).
$$
(4.18)

For any $Z > 0$ small, we introduce the following neighborhood of $\partial\Omega$:

$$
U^Z := \left\{ x'/\xi'(x') \leq \left(1 - \frac{6C_1\left(K + \frac{c|\Omega|(1-m_{\varepsilon,N})}{2\pi}\right)Z}{\pi}\right) h_{ex}(1 - m_{\varepsilon,N}) \right\},
$$

where C_1 and K are the constants in (2.7) and (1.5) respectively. From the definition of α, it follows that

$$
1 - \frac{2\alpha\xi'}{|\log \varepsilon'|} \geq \frac{6C_1\left(K + \frac{c|\Omega|(1-m_{\varepsilon,N})}{2\pi}\right)\beta}{\pi} \mathbf{1}_{U^\beta}.
$$

Using this together with (2.7), we obtain (cf. (2.4))

$$
\begin{aligned}
\int_{\Omega_\varepsilon} \left(1 - \frac{2\alpha\xi'}{|\log \varepsilon'|}\right) e_\varepsilon &\geq \frac{6C_1\left(K + \frac{c|\Omega|(1-m_{\varepsilon,N})}{2\pi}\right)\beta}{\pi} \int_{U^\beta} e_\varepsilon \\
&\geq \frac{6\left(K + \frac{c|\Omega|(1-m_{\varepsilon,N})}{2\pi}\right)\beta}{\pi} |\nu'_\varepsilon|(\check{U}^\beta) |\log \varepsilon'|.
\end{aligned}
$$
(4.19)

On the other hand, by (1.5), $N \leq K|\log \varepsilon'|$ and $\frac{\alpha N}{|\log \varepsilon'|} \leq \frac{c|\Omega|}{2\pi(1-m_{\varepsilon,N})}$, so combining (4.18) and (4.19), we get $\frac{|\nu'_\varepsilon|(\check{U}^\beta)}{\pi} \leq 1 + o\left(\frac{N}{|\log \varepsilon'|}\right) < 2$ (for ε small enough), which implies

$$
|\nu'_\varepsilon|(\check{U}^\beta) = 0,
$$
(4.20)

in light of (4.14).

Step 4: **There are no vortices with negative degree outside $\omega'_{\varepsilon,N}$**

We turn to the proof of item (i). We invoke (4.8) applied to $\alpha\xi'$ and (4.11)–(4.12) again to get

$$J'_{\varepsilon,N}((u^\varepsilon)',(A_1^\varepsilon)') - \int_{\Omega_\varepsilon} \alpha\xi' \, d\nu'_\varepsilon \geq \frac{1}{C_1}(g'_\varepsilon)^+((\widetilde{\omega'_{\varepsilon,N}})^c) + NW_m + o(N),$$

which, arguing as in (4.17) yields

$$(4.21) \qquad\qquad (1+\alpha)\beta N + o(N) \geq \frac{1}{C_1}(g'_\varepsilon)^+((\widetilde{\omega'_{\varepsilon,N}})^c).$$

Recall that from (1.5) we have $\alpha\beta N \lesssim \beta \left|\log \varepsilon'\right| \left(\frac{N}{h_{ex}}\right) \lesssim \beta \left|\log \varepsilon'\right|$ hence $(1+\alpha)\beta N + o(N) \lesssim \beta|\log \varepsilon'| + o(|\log \varepsilon'|)$.

On the other hand, from (4.9) and (4.10) one can bound $(g'_\varepsilon)^+((\widetilde{\omega'_{\varepsilon,N}})^c)$ from below by (letting C denote a generic positive constant)

$$(g'_\varepsilon)^+((\widetilde{\omega'_{\varepsilon,N}})^c) \geq C|\log \varepsilon'|\left|\nu_\varepsilon^-((\widetilde{\omega'_{\varepsilon,N}})^c)\right| - \frac{C_3}{2\pi}\sum_{i\in I}|B_i| \gtrsim |\log \varepsilon'|\left|\nu_\varepsilon^-((\widetilde{\omega'_{\varepsilon,N}})^c)\right|.$$

Going back to (4.21), this implies $\beta + o(1) \geq C\left|\nu_\varepsilon^-\left((\widetilde{\omega'_{\varepsilon,N}})^c\right)\right|$, where C is a positive constant independent of β and ε. Since $|\nu_\varepsilon^-| \in 2\pi\mathbb{N}$, we may choose β small so that for any ε small enough, it imposes

$$(4.22) \qquad\qquad \left|(\nu'_\varepsilon)^-\left((\widetilde{\omega'_{\varepsilon,N}})^c\right)\right| = 0.$$

Since $(\widetilde{\omega'_{\varepsilon,N}})^c$ contains $(\omega'_{\varepsilon,N})^c$, we have thus proved item (i).

Step 5: **The total vorticity of u^ε is N**

We proceed, once more, applying (4.13) only this time to a vorticity adapted function ψ_ε such that $\omega^{\psi_\varepsilon} \supseteq \Omega_\varepsilon \setminus U^\beta$. Such a function exists. Indeed, by (4.15), for $x \in \partial(\Omega_\varepsilon \setminus U^\beta)$ we have $|\alpha\xi'(x)| \leq o(|\log \varepsilon'|)d(x,\partial\Omega_\varepsilon)$ while $\alpha\xi'(x) = \frac{1}{2}(1-\mathcal{O}(1)\beta)|\log \varepsilon'|$. It follows that $d(\partial(\Omega_\varepsilon \setminus U^\beta), \partial\Omega_\varepsilon) \gg 1$ as $\varepsilon \to 0$ (if β is a small enough constant), and thus $d(\partial(\widehat{\Omega_\varepsilon \setminus U^\beta}), \partial\Omega_\varepsilon) \gg 1$. One can therefore build a family of functions ψ_ε which are equal to $\frac{1}{2}|\log \varepsilon'|$ in $\widehat{\Omega_\varepsilon \setminus U^\beta} = \Omega_\varepsilon \setminus \check{U}^\beta$, which vanish on $\partial\Omega_\varepsilon$ and satisfy $\|\nabla\psi_\varepsilon\|_\infty = o(|\log \varepsilon'|)$, i.e. belongs to $\mathcal{VA}$.

Applying (4.13) to that family yields

$$J'_{\varepsilon,N}((u^\varepsilon)',(A_1^\varepsilon)') - \int_{\Omega_\varepsilon} \psi_\varepsilon \, d\nu'_\varepsilon \geq NW_m + o(N).$$

We can then appeal to (3.13) again to obtain

$$J'_{\varepsilon,N}((u^\varepsilon)',(A_1^\varepsilon)') - \int_{\Omega_\varepsilon} \psi_\varepsilon\mu'((u^\varepsilon)',(A_1^\varepsilon)')$$

$$\geq J'_{\varepsilon,N}((u^\varepsilon)',(A_1^\varepsilon)') - \int_{\Omega_\varepsilon} \psi_\varepsilon^\beta \, d\nu'_\varepsilon + \mathcal{O}(\|\nabla\psi_\varepsilon\|_\infty \, \varepsilon^\alpha) \geq NW_m + o(N).$$

Since $(u^\varepsilon, A^\varepsilon) \in S^\beta_{\varepsilon,N}$, we must have

$$(4.23) \qquad \pi N \left| \log \varepsilon' \right| + \beta N \geq J'_{\varepsilon,N}((u^\varepsilon)', (A^\varepsilon_1)') - NW_m \geq \int_{\Omega_\varepsilon} \psi_\varepsilon \, d\nu'_\varepsilon + o(N).$$

Now, by (4.20) and definition of ψ_ε

$$(4.24) \qquad \int_{\Omega_\varepsilon} \psi_\varepsilon \, d\nu'_\varepsilon = \int_{\Omega_\varepsilon \setminus \check{U}^\beta} \psi_\varepsilon \, d\nu'_\varepsilon = \int_{\Omega_\varepsilon \setminus \check{U}^\beta} \frac{1}{2} \left| \log \varepsilon' \right| d\nu'_\varepsilon = \frac{1}{2} \left| \log \varepsilon' \right| \nu'_\varepsilon(\Omega_\varepsilon \setminus \check{U}^\beta).$$

Plugging this into (4.23) leads us to

$$N + \frac{\beta}{\pi} \frac{N}{\left| \log \varepsilon' \right|} \geq \frac{1}{2\pi} \nu'_\varepsilon(\Omega_\varepsilon \setminus \check{U}^\beta) + o\left(\frac{N}{\left| \log \varepsilon' \right|} \right).$$

Since N is taken satisfying (1.5), if β is chosen small enough, the above implies that for ε small enough,

$$(4.25) \qquad N \geq \frac{\nu'_\varepsilon(\Omega_\varepsilon \setminus \check{U}^\beta)}{2\pi} = \frac{\nu'_\varepsilon(\Omega_\varepsilon)}{2\pi},$$

because of (4.14) and (4.20). We now show that there is actually equality in (4.25). To that end we first prove

$$\int_{\Omega_\varepsilon} \xi' \mu((u^\varepsilon)', (A^\varepsilon_1)') \leq \nu_\varepsilon(\Omega_\varepsilon) \xi'_{\max} + o(1).$$

This is not immediately clear since ξ is not constant and ν_ε is not necessarily equal to ν^+_ε. However, from (1.6) and (2.6), we have

$$\int \xi' \mu((u^\varepsilon)', (A^\varepsilon_1)') = \int_{\Omega_\varepsilon} \xi' \, d\nu'_\varepsilon + \mathcal{O}(\varepsilon^{\frac{1}{2}} h^{\frac{1}{4}+2}_{ex} |\nabla \xi'|_\infty) = \int_{\widetilde{\omega'_{\varepsilon,N}}} \xi' \, d\nu'_\varepsilon + \int_{(\widetilde{\omega'_{\varepsilon,N}})^c} \xi' \, d\nu'_\varepsilon + o(\varepsilon^\alpha).$$

On the other hand, from (4.22),

$$\int_{(\widetilde{\omega'_{\varepsilon,N}})^c} \xi' \, d\nu'_\varepsilon = \int_{(\widetilde{\omega'_{\varepsilon,N}})^c} \xi' \, d(\nu'_\varepsilon)^+ \leq \xi_{\max}(\nu'_\varepsilon)^+((\widetilde{\omega'_{\varepsilon,N}})^c).$$

Also,

$$\int_{(\widetilde{\omega'_{\varepsilon,N}})} \xi' \, d\nu'_\varepsilon = \xi_{\max} \nu'_\varepsilon(\widetilde{\omega'_{\varepsilon,N}})$$

because $\xi' = \xi_{\max}$ on $(\widetilde{\omega'_{\varepsilon,N}})$. Collecting these estimates we are led to

$$(4.26) \quad \int_{\Omega_\varepsilon} \xi' \mu'((u^\varepsilon)', (A^\varepsilon_1)') \leq \xi_{\max} \left(\nu'_\varepsilon(\widetilde{\omega'_{\varepsilon,N}}) + (\nu'_\varepsilon)^+((\widetilde{\omega'_{\varepsilon,N}})^c) \right) + o(1) = \xi_{\max} \nu'_\varepsilon(\Omega_\varepsilon) + o(1)$$

where the last equality follows from (4.22). Consequently $\nu'_\varepsilon(\Omega_\varepsilon) > 2\pi(N-1)$, for otherwise we would have (cf. (1.13))

$$\int_{\Omega_\varepsilon} \xi' \mu'((u^\varepsilon)', (A^\varepsilon_1)') \leq 2\pi(N-1)\xi_{\max} + o(1) = 2\pi(N-1)h_{ex}(1 - m_{\varepsilon,N}) + o(1)$$

and the right-hand side is strictly smaller than $2\pi N h_{ex}(1 - m_{\varepsilon,N}) - \beta N$, if β is small enough, by virtue of (1.5). This contradicts $(u^\varepsilon, A_1^\varepsilon) \in S_{\varepsilon,N}^\beta$. Therefore, $\nu_\varepsilon'(\Omega_\varepsilon) \geq 2\pi N$ (recall $\nu_\varepsilon'(\Omega_\varepsilon) \in 2\pi\mathbb{Z}$.) Thus, with (4.25), we have proved item (ii).

Step 6: **Proof of (iii) and (iv)**

Item (iii) follows readily from (4.26) since we already know that $\nu_\varepsilon'(\Omega_\varepsilon) = 2\pi N$. For item (iv), note that from (4.20), (4.23) and (4.24) we deduce

$$J_{\varepsilon,N}'((u^\varepsilon)', (A_1^\varepsilon)') \geq \int_{\Omega_\varepsilon} \psi_\varepsilon \, d\nu_\varepsilon' + NW_m + o(N)$$

$$= \frac{1}{2} |\log \varepsilon'| \, \nu_\varepsilon'(\Omega_\varepsilon \setminus \check{U}^\beta) + NW_m + o(N) = \pi N \, |\log \varepsilon'| + NW_m + o(N).$$

The proposition is proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

5. Conclusion

In this section we finish the proof of our main result.

Proof of Theorem 1.1

As mentioned earlier, a minimizer in $S_{\varepsilon,N}^\beta$ exists, so to show it actually corresponds to a local minimizer, it only remains to prove $(u_\varepsilon, A_\varepsilon) \notin \partial S_{\varepsilon,N}^\beta$.

We decompose $\partial S_{\varepsilon,N}^\beta = (\partial S_{\varepsilon,N}^\beta)^+ \bigcup (\partial S_{\varepsilon,N}^\beta)^-$ where

$$(\partial S_{\varepsilon,N}^\beta)^+ := \left\{ (u, A) \in H^1 \times H^1 \text{ such that: } J_{\varepsilon,N}(u, A_1) = \pi N \, |\log \varepsilon'| + NW_m + \beta N \right\}$$

and

$$(\partial S_{\varepsilon,N}^\beta)^- := \left\{ (u, A) \in H^1 \times H^1 \text{ such that: } \int_\Omega \xi \mu(u, A_1) = 2\pi N h_{ex}(1 - m_{\varepsilon,N}) - \beta N \right\}.$$

Since $(u_\varepsilon, A_\varepsilon)$ is a minimizer in $S_{\varepsilon,N}^\beta$, we may apply Lemma 3.2 and Proposition 4.1. By (1.14) and (3.14), we have

(5.1) $$G_\varepsilon(u_\varepsilon, A_\varepsilon) = G_\varepsilon^N + J_{\varepsilon,N}(u_\varepsilon, A_{1,\varepsilon}) - \int_\Omega \xi\mu(u_\varepsilon, A_{1,\varepsilon}) + o(1).$$

If $(u_\varepsilon, A_\varepsilon) \in (\partial S_{\varepsilon,N}^\beta)^+$ then by (iii) in Proposition 4.1, we have

$$G_\varepsilon(u_\varepsilon, A_\varepsilon) \geq G_\varepsilon^N + \pi N \, |\log \varepsilon'| + NW_m + \beta N - 2\pi N h_{ex}(1 - m_{\varepsilon,N}) + o(N).$$

If instead, $(u_\varepsilon, A_\varepsilon) \in (\partial S_{\varepsilon,N}^\beta)^-$, invoking (iv) in Proposition 4.1 yields

$$G_\varepsilon(u_\varepsilon, A_\varepsilon) \geq G_\varepsilon^N + \pi N \, |\log \varepsilon'| + NW_m - (2\pi N h_{ex}(1 - m_{\varepsilon,N}) - \beta N) + o(N).$$

In either case there is a contradiction with the upper bound (3.1), so necessarily $(u_\varepsilon, A_\varepsilon) \notin \partial S_{\varepsilon,N}^\beta$, whence $(u_\varepsilon, A_\varepsilon)$ is a local minimizer.

From (5.1), (iii) and (iv) in Proposition 4.1, we must have

$$(5.2) \qquad G_\varepsilon(u_\varepsilon, A_\varepsilon) \geq G_\varepsilon^N + \pi N \left| \log \varepsilon' \right| + N W_m - 2\pi N h_{ex}(1 - m_{\varepsilon,N}) + o(N).$$

Thus, the asymptotic expansion (1.9) follows from (5.2) and (3.1). (1.7), (1.8) were proved in Lemma 3.2 and Proposition 4.1 respectively.

We prove item (iii). It essentially follows the proof of (1.34) in [14]. First, arguing as in item 2 of Proposition 6.1 in [14], replacing F_ε' with $J_{\varepsilon,N}' - \alpha \int_{\Omega_\varepsilon} \xi' \mu'$, we can sandwich the term $\int_\Omega |\nabla \times A_{1,\varepsilon} - \mu_{\varepsilon,N}|^2$ in $J_\varepsilon(u_\varepsilon, A_{1,\varepsilon})$ between the upper and lower bound of (4.17) and hence control it by $\mathcal{O}(N)$, i.e.

$$(5.3) \qquad \|\nabla \times A_{1,\varepsilon} - \mu_{\varepsilon,N}\|_{L^2} \lesssim \sqrt{N}$$

Using the same item of Proposition 6.1 in [14] also allows to control, for $1 < p < 2$,

$$\int_{\Omega_\varepsilon} |j(u_\varepsilon', A_{1,\varepsilon}')|^p \leq C_p \left(J_{\varepsilon,N}'(u_\varepsilon', A_{1,\varepsilon}') - \alpha \int_{\Omega_\varepsilon} \xi' \mu' + |\omega_{\varepsilon,N}'| \right) \leq C(N + |\omega_{\varepsilon,N}'|).$$

Rescaling we find that for any $1 < p < 2$,

$$(5.4) \qquad \|j(u_\varepsilon, A_{1,\varepsilon})\|_{L^p(\Omega)} \lesssim h_{ex}^{\frac{1}{2} - \frac{1}{p}} \left(N^{1/p} + |\omega_{\varepsilon,N}'|^{1/p} \right) \lesssim h_{ex}^{\frac{1}{2} - \frac{1}{p}} N^{1/p} \lesssim N^{\frac{1}{2}}$$

thanks to (4.12), then (1.5) giving $N \lesssim h_{ex}$. On the other hand, by direct calculation we have

$$\mu(u_\varepsilon, A_\varepsilon) - \mu(u_\varepsilon, A_{1,\varepsilon}) = \nabla \times ((1 - |u_\varepsilon|^2)\nabla^\perp h_{\varepsilon,N})$$

from (1.6)and (3.4)

$$\left\| (1 - |u_\varepsilon|^2) \left| \nabla h_{\varepsilon,N} \right| \right\|_{L^2} \lesssim \sqrt{\varepsilon^2 G_\varepsilon} \cdot |\nabla h_{\varepsilon,N}|_\infty \lesssim \varepsilon \max\{h_{ex}^2, h_{ex} |\log \varepsilon'|\} = o(1),$$

from which one obtains

$$(5.5) \qquad \|\mu(u_\varepsilon, A_{1,\varepsilon}) - \mu(u_\varepsilon, A_\varepsilon)\|_{W^{-1,p}} = o(1).$$

We may now write, using the triangle inequality

$$\|\mu(u_\varepsilon, A_\varepsilon) - \mu_{\varepsilon,N}\|_{W^{-1,p}} \leq \|\mu(u, A_\varepsilon) - \mu(u_\varepsilon, A_{1,\varepsilon})\|_{W^{-1,p}} + \|\mu_{\varepsilon,N} - \nabla \times A_{1,\varepsilon}\|_{W^{-1,p}}$$
$$+ \|\mu(u_\varepsilon, A_{1,\varepsilon}) - \nabla \times A_{1,\varepsilon}\|_{W^{-1,p}} \lesssim N$$

where we have used (5.3) (together with an embedding inequality), (5.5) and the fact that $\mu(u_\varepsilon, A_{1,\varepsilon}) - \nabla \times A_{1,\varepsilon} = \nabla \times j(u_\varepsilon, A_{1,\varepsilon})$ combined with (5.4). Consequently, item (iii) is proved.

We turn to the proof of (iv). Since $\alpha \xi' \in \mathcal{VA}$, (4.8), (4.11) and (1.9) allow us to assert

$$N W_m + o(N) \geq N \left(\frac{2\pi}{m} \int W(j)\, dP(j) + \gamma + o(1) \right)$$

which in turns translates into

$$W_m \geq \frac{2\pi}{m} \int W(j)\, dP(j) + \gamma.$$

To conclude, since P-a.e. $j \in \mathcal{A}_m$ (as given by [14]) this implies P-a.e. j minimizes W over $\mathcal{A}_m$. The result follows.

$\square$

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