Mean-field dynamics for Ginzburg-Landau vortices with pinning and applied force

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Abstract

We consider the time-dependent Ginzburg-Landau equation in the whole plane with terms modeling pinning and applied forces. The Ginzburg-Landau vortices are then subjected to three forces: their mutual repulsive interaction, a constant applied force pushing them in a fixed direction, and the pinning force attracting them towards the local minima of the pinning potential. The competition between the three is expected to lead to possible glassy effects.

We first rigorously study the limit in which the number of vortices N_ε blows up as the inverse Ginzburg-Landau parameter ε goes to 0, and we derive via a modulated energy method the limiting fluid-like mean-field evolution equations. These results hold in the case of parabolic, conservative, and mixed-flow dynamics in appropriate regimes of $N_\varepsilon \to \infty$. We next consider the problem of homogenization of the limiting mean-field equations when the pinning potential oscillates rapidly: we formulate a number of questions and heuristics on the appropriate limiting stick-slip equations, as well as some rigorous results on the simpler regimes.

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1 Introduction

1.1 General overview

We are interested in the collective dynamics of many vortices in a superconductor with impurities, within the framework of the 2D Ginzburg-Landau model. This is important for practical applications, and is a main concern of current research in the physics community (see e.g. [10, 44, 70]). Indeed, superconductors are used in order to carry electric currents without energy dissipation. In most of the interesting superconducting materials (those with a high critical temperature), vortices occur for a very wide range of values of the applied magnetic fields, in the so-called "mixed state". When flowing an electric current through a superconducting wire, the vortices are set in motion by the Lorentz force exerted by the current, leading to energy dissipation. This problem is fixed in practice by introducing normal impurities in the material which "pin" the vortices to their locations if the applied current is not too strong, thus avoiding the energy dissipation.

Physicists are therefore very interested in understanding the effect of such impurities (which are typically randomly scattered around the sample) on the statics and dynamics of vortices. In particular, they want to understand the critical applied current needed to depin the vortices from their pinning sites, and the slow motion of vortices in the disordered sample — named *creep* — when the applied current has a small intensity and thermal or quantum effects are taken into consideration (see e.g. [10, 44, 70]).

We will thus study the motion of vortices for the Ginzburg-Landau equations including both a pinning potential and an applied electric current. The dynamics will be either parabolic (parabolic Ginzburg-Landau equation), conservative (Gross-Pitaevskii equation), or even mixed. These equations have been studied in [89, 84] in the mixed case and in [57] in the conservative case, for a fixed number N of vortices in the asymptotic limit when ε (the inverse Ginzburg-Landau parameter, which is also the characteristic lengthscale of the vortex cores) tends to 0. As seen there, vortices are subjected to three forces: their mutual interaction, which is a logarithmic repulsion, the Lorentz force F due to the applied current of intensity J_{ex}, and the pinning force equal to $-\nabla \log a$ in terms of the pinning weight a. The effective vortex dynamics then corresponds to a system of ODEs of the form

$$(\alpha + \mathbb{J}\beta)\dot{x}_i = -N^{-1}\nabla_{x_i} W_N(x_1, \ldots, x_N) - \nabla h(x_i) + F(x_i), \qquad 1 \le i \le N, \tag{1.1}$$
$$h := \log a, \qquad W_N(x_1, \ldots, x_N) := -\sum_{i\neq j}^{N} \log|x_i - x_j|,$$

where the x_i's are the effective vortex trajectories, where $\mathbb{J}$ denotes the rotation of vectors by angle $\pi/2$ in the plane, and where the parameters $\alpha \ge 0$ and $\beta \in \mathbb{R}$ satisfy $\alpha^2 + \beta^2 = 1$ and are such that $\beta = 0$ (resp. $\alpha = 0$) corresponds to the parabolic (resp. conservative) case.

The pinning and applied force intensities are parameters which can be tuned, leading to regimes in which one or two forces dominate over the others, or all are of the same order. In [89] no pinning force is considered, and the treated regimes lead to the applied force being of the same order as the interaction. In [84] the pinning and applied forces are chosen to be of the same order, and both dominate the interaction. Finally in [57] in the conservative case, the critical scaling is considered, that is, with all forces of the same order.

Here we consider the situation when the number N_ε of vortices is not fixed but depends on ε, and blows up as $\varepsilon \downarrow 0$, which is a physically more realistic situation in many regimes of applied fields and currents. In the case without pinning and applied current, the mean-field limiting dynamics of $N_\varepsilon \gg 1$ vortices in the parabolic and conservative equations have been rigorously established in a number of settings:

— for the Gross-Pitaevskii equation in the plane, it is shown in [54] in the regime $1 \ll N_\varepsilon \lesssim (\log|\log\varepsilon|)^{1/2}$ that the vorticity of solutions converges to the solution of the incompressible Euler equation in vorticity form, while in [82] it is shown in the regime $|\log\varepsilon| \ll N_\varepsilon \ll \varepsilon^{-1}$ that the current of solutions converges to the solution of the incompressible Euler equation;

— for the parabolic Ginzburg-Landau equation in the plane, the convergence of the vorticity of solutions to the solution of a limiting mean-field equation, first formally derived in [18, 39], is established in [58] in the regime $1 \ll N_\varepsilon \le (\log\log|\log\varepsilon|)^{1/4}$, while the convergence of the current to an appropriate limiting equation is established in [82] in the regime $1 \ll N_\varepsilon \lesssim |\log\varepsilon|$;

— the situation in the remaining regimes remains an open question.

All those results assume that the initial data is suitably "well-prepared". The results of [58] and [54] rely on a direct method and a careful study of the vortex trajectories, while the results of [82] are based on a "modulated energy approach" which we will describe later, and rely on the assumed regularity of the limiting solutions (or equivalently of the initial data).

The goal of the first part of this paper is to adapt the approach of [82] to the setting with pinning and applied force as in [84, 57], but in the whole plane and with $N_\varepsilon \gg 1$ vortices. We treat here the parabolic, conservative and also mixed-flow cases, and obtain the convergence to some limiting fluid-type evolution equations, for which global well-posedness is proved in the companion paper [37]. As described above, different regimes for the intensities of the pinning and applied forces lead to different limiting equations: either a nonlocal transport equation involving the pinning potential h and the applied force F, which in the simplest case takes the following form for the limiting vorticity m,

$$\partial_t \mathrm{m} = \mathrm{div}\,\Big((\alpha - \mathbb{J}\beta)(\nabla h - F - \nabla\triangle^{-1}\mathrm{m})\,\mathrm{m}\Big), \tag{1.2}$$

or a simple linear transport equation with only the pinning and applied forces remaining when these are scaled to be much stronger than the interaction.

The derivation bears several complications compared to the situation of [82], in particular due to the lack of sufficient decay at infinity of the various quantities, and also to the fact that the self-interaction of each vortex varies with its location due to the pinning potential.

We will perform this derivation for a pinning force which varies at the macroscopic scale. The most interesting situation from the modeling viewpoint is however to let the pinning potential oscillate quickly at some mesoscopic scale η_ε, which tends to 0 as $\varepsilon \downarrow 0$ and can have some interplay with the vortex interdistance. In real materials the way the impurities are inserted typically leads them to be uniformly and randomly scattered in the sample. This is well-modeled by a periodic but rapidly oscillating pinning weight $a(x) = \hat{a}_0(x/\eta_\varepsilon)$, or even better by a random pinning weight $\hat{a}_0(x/\eta_\varepsilon, \omega)$ with some good ergodicity properties. One is thus led to the question of combining the mean-field limit for the Ginzburg-Landau evolution equations with an homogenization limit. In other words, can one perform the derivation of the limiting equation as $\varepsilon \downarrow 0$, $N_\varepsilon \uparrow \infty$ and $\eta_\varepsilon \downarrow 0$, and in which regimes does it hold?

While the homogenization of the (static) Ginzburg-Landau functional with pinning weight has been studied in some settings [2, 5, 35], we believe that these questions in the dynamical case are very challenging. They are in fact already very hard for just a finite number of vortices. Studying the limit as $\eta \downarrow 0$ of (1.1) with pinning potential $h(x) = \hat{h}_0(x/\eta)$ with $\hat{h}_0$ periodic or random, is a question of homogenization of a system of coupled ODEs and is notoriously difficult. Note that these difficulties seem to be related to the possible "glassy" properties predicted by physicists for such systems (see e.g. [44]). On the other hand, the case with no interaction term and with F constant is much simpler to analyze, and seems to be known under the term "washboard" in the physics literature. When $F = 0$, the particle is simply attracted towards the local wells of the pinning potential h. Otherwise, the constant applied force $F \neq 0$ can be absorbed into the term $-\nabla h$ by adding to the potential h an affine function, which effectively tilts the potential landscape into a washboard-shaped graph. As will be seen in Section 1.3.1, above some positive value of $|F|$ the tilted potential has no local minimum, leading the particle to fall downwards. In the setting of a superconductor with applied current and with pinning, this corresponds to the critical "depinning current" above which the vortices are depinned from their pinning locations. Note that when the applied force F varies with x at the macroscopic scale (still without interaction term) the situation is much more subtle and only partial results are obtained in [61].

Since our modulated energy method to establish the mean-field limit does not seem well-adapted to include homogenization effects, we will not say much about commuting the limits $\varepsilon \downarrow 0$ and $\eta \downarrow 0$, but instead, in the second part of this paper, we formulate a few partial results in the direction of homogenizing the limiting mean-field equations of the form (1.2) obtained in the first part, and we formulate many open questions which we believe to be interesting both from an applied and a theoretical point of view. This topic is indeed very delicate on its own, with the same kind of difficulties as for the homogenization of the corresponding system of coupled ODEs (1.1), but in the case without interaction and with F constant the problem is considerably simpler and leads to a well-defined limiting stick-slip equation. Finally, in order to model thermal effects, one can replace the transport equations of the type (1.2) by their viscous versions, and we will give a few heuristics in Section 1.3.2 on the corresponding homogenization questions.

Notation. Throughout the paper, C denotes various positive constants which depend on the dimension d, and on various controlled quantities, but do not depend on the parameter ε, and we write $\lesssim$ and $\gtrsim$ for $\leq$ and $\geq$ up to such a constant C. We then write $a \simeq b$ if both $a \lesssim b$ and $a \gtrsim b$ hold. Given sequences $(a_\varepsilon)_\varepsilon, (b_\varepsilon)_\varepsilon \subset \mathbb{R}$, we also set $a_\varepsilon \ll b_\varepsilon$ (or $b_\varepsilon \gg a_\varepsilon$) if $a_\varepsilon/b_\varepsilon$ converges to 0 as the parameter ε goes to 0. Alternatively, we write $a_\varepsilon \leq O(b_\varepsilon)$ if $a_\varepsilon \lesssim b_\varepsilon$, and $a_\varepsilon \leq o(b_\varepsilon)$ if $a_\varepsilon \ll b_\varepsilon$. We add a subscript t to indicate the further dependence on an upper bound on time t, while additional subscripts indicate the dependence on other parameters. A superscript t to a function indicates that this function is evaluated at time t. We let $Q = [-\frac{1}{2}, \frac{1}{2})^2$ denote the unit square, frequently identified with the 2-torus $\mathbb{T}^2$. For any vector field $G = (G_1, G_2)$ on $\mathbb{R}^2$, we denote $G^\perp = (-G_2, G_1)$, $\operatorname{curl} G = \partial_1 G_2 - \partial_2 G_1$, and also as usual $\operatorname{div} G = \partial_1 G_1 + \partial_2 G_2$. We write $\mathbb{J} : \mathbb{R}^2 \to \mathbb{R}^2$ for the rotation of vectors by angle $\pi/2$ in the plane, so that $\mathbb{J}G = G^\perp$. We denote by $B(x, r)$ the ball of radius r centered at x in $\mathbb{R}^2$, and we set $B_r := B(0, r)$ and $B(x) := B(x, 1)$. We use the notation $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$ for $x, y \in \mathbb{R}$. We denote by $\mathrm{L}^p_{\mathrm{uloc}}(\mathbb{R}^2)$ the Banach space of functions that are uniformly locally L^p-integrable on $\mathbb{R}^2$, with norm

$\|f\|_{\mathrm{L}^p_{\mathrm{uloc}}} := \sup_x \|f\|_{\mathrm{L}^p(B(x))}$, and we similarly define the Sobolev spaces $W^{k,p}_{\mathrm{uloc}}(\mathbb{R}^2)$. Given a Banach space X and $t > 0$, we use the notation $\|\cdot\|_{\mathrm{L}^p_t X}$ for the usual norm in $\mathrm{L}^p([0,t];X)$.

1.2 Mean-field limit results

Mesoscopic inhomogeneities in the material are usually modeled by introducing a pinning weight $a : \mathbb{R}^2 \to [0,1]$, which locally lowers the energy penalty associated with the vortices [59, 16] (see also [17]). In the time-dependent Ginzburg-Landau equation, first derived by Schmid [78] and by Gor'kov and Eliashberg [46], and in the simplified version without gauge, the pinning weight appears as follows:

$$(\alpha + i|\log\varepsilon|\beta)\partial_t w_\varepsilon = \triangle w_\varepsilon + \frac{w_\varepsilon}{\varepsilon^2}(a - |w_\varepsilon|^2), \qquad \text{in } \mathbb{R}^+ \times \Omega, \tag{1.3}$$

where Ω is a domain of $\mathbb{R}^2$ and w_ε is the complex-valued order parameter. Here $\alpha \geq 0$, $\beta \in \mathbb{R}$, $\alpha^2 + \beta^2 = 1$, and these parameters α and β allow to consider by the same token the parabolic case ($\alpha = 1$, $\beta = 0$), the Gross-Pitaevskii case ($\alpha = 0$, $\beta = 1$), and the mixed-flow case ($\alpha > 0$, $\beta \in \mathbb{R}$), and are scaled so as to obtain a nontrivial limiting dynamics. The case of the equation with magnetic gauge is briefly discussed in Section 2.2. Since the gauge does not introduce significant mathematical difficulties, we omit it for simplicity in our analysis. In this context, we aim to understand the dynamics of the vortices in the asymptotic regime $\varepsilon \downarrow 0$ as their number N_ε blows up, thus describing the evolution of the density of the corresponding vortex liquid. For simplicity we assume

$$\frac{1}{C} \leq a(x) \leq 1, \qquad \text{for all } x, \tag{1.4}$$

which avoids degenerate situations. Physically one would like to consider a pinning weight a that may vanish, representing normal inclusions [16], however this is much more delicate mathematically (see e.g. [5]).

The equation (1.3) should be supplemented with a boundary condition modeling the inflow of an electric current. Because the presence of the boundary creates mathematical difficulties which we do not know how to overcome (due to the possible entrance and exit of vortices), we take the model studied in [89, 84] and make suitable modifications to consider a version on the whole plane with boundary conditions "at infinity". As in [89, 84], the boundary conditions can be changed into a bulk force term by a suitable change of phase in the unknown function. Dividing also the unknown function by the expected density $\sqrt{a}$, we arrive at the equation

$$\begin{cases} \lambda_\varepsilon(\alpha + i|\log\varepsilon|\beta)\partial_t u_\varepsilon = \triangle u_\varepsilon + \frac{a}{\varepsilon^2}u_\varepsilon(1 - |u_\varepsilon|^2) + \nabla h \cdot \nabla u_\varepsilon + i|\log\varepsilon|F^\perp \cdot \nabla u_\varepsilon + f u_\varepsilon, \\ u_\varepsilon|_{t=0} = u_\varepsilon^\circ, \end{cases} \tag{1.5}$$

with $h = \log a$, $f : \mathbb{R}^2 \to \mathbb{R}$ and $F : \mathbb{R}^2 \to \mathbb{R}^2$, where F becomes an effective applied force corresponding to the applied current. The parameter λ_ε is an appropriate time rescaling to obtain a nontrivial limiting dynamics. Within the derivation of (1.5) from (1.3), the zeroth-order term f takes the following explicit form (but this is largely unimportant, and the scaling in the corresponding bounds (2.1)–(2.2) below may be substantially relaxed),

$$f := \frac{\triangle\sqrt{a}}{\sqrt{a}} - \frac{1}{4}|\log\varepsilon|^2|F|^2. \tag{1.6}$$

The discussion of the derivation of (1.5) from (1.3), as well as that of the boundary conditions and the assumptions at infinity, is postponed to Section 2.1. Setting $F \equiv 0$, $a \equiv 1$, $h \equiv 0$ and $f \equiv 0$, we retrieve the equation studied in [82], and our results will thus be a generalization of those in [82].

The goal is to obtain the convergence of the *supercurrent* defined by

$$j_\varepsilon := \langle \nabla u_\varepsilon, i u_\varepsilon \rangle,$$

where $\langle \cdot, \cdot \rangle$ stands for the scalar product in $\mathbb{C}$ as identified with $\mathbb{R}^2$, that is, $\langle x, y \rangle = \Re(x\bar{y})$ for all $x, y \in \mathbb{C}$. The *vorticity* is derived from the supercurrent by $\mu_\varepsilon := \operatorname{curl} j_\varepsilon$. Note that this also corresponds to the density of vortices, defined as zeros of u_ε weighted by their degrees, in the sense that

$$\mu_\varepsilon \sim 2\pi \sum_i d_i \delta_{x_i}, \qquad \text{as } \varepsilon \downarrow 0, \tag{1.7}$$

with x_i the vortex locations and d_i their degrees (this corresponds to the so-called Jacobian estimates, a notion we will come back to in the course of the paper). We wish to show that $N_\varepsilon^{-1} j_\varepsilon$ converges as $\varepsilon \downarrow 0$ to a velocity field v solving a limiting PDE, which as in [82] is assumed to be regular enough. Note that solutions of the limiting equations are studied in [37] and shown to be global and regular enough if the initial data is.

The method of the proof in [82] is based on a "modulated energy" technique, which originates in the relative entropy method first designed by DiPerna [31] and Dafermos [23, 24] to establish weak-strong stability principles for some hyperbolic systems. Such a relative entropy method was later rediscovered by Yau [90] for the hydrodynamic limit of the Ginzburg-Landau lattice model, was introduced in kinetic theory by Golse [12] for the convergence of suitably scaled solutions of the Boltzmann equation towards solutions of the incompressible Euler equations (see e.g. [71] for the many recent developments on the topic), and first took the form of a modulated *energy* method in the work by Brenier [15] on the quasi-neutral limit of the Vlasov-Poisson system. In the present situation, the method consists in defining a "modulated energy", which without pinning takes the form

$$\frac{1}{2} \int_{\mathbb{R}^2} |\nabla u_\varepsilon - i u_\varepsilon N_\varepsilon \mathrm{v}|^2 + \frac{1}{2\varepsilon^2}(1 - |u_\varepsilon|^2)^2, \tag{1.8}$$

where v denotes the solution of the (postulated) limiting PDE. This modulated energy thus somehow measures the distance between the supercurrent j_ε and the postulated limit $N_\varepsilon \mathrm{v}$ in a way that is well adapted to the energy structure. Under some regularity assumptions on v, it is then proved in [82] that, thanks to the PDE satisfied by v, this quantity (1.8) satisfies a Grönwall relation, so that if it is initially small, more precisely $o(N_\varepsilon^2)$, it remains so, yielding the desired convergence $N_\varepsilon^{-1} j_\varepsilon \to \mathrm{v}$. However, in the regimes where $N_\varepsilon \lesssim |\log \varepsilon|$, the modulated energy cannot be of order $o(N_\varepsilon^2)$, because each vortex carries an energy $\pi|d||\log \varepsilon|$. For that reason (and assuming that all vortices have positive degrees initially), we need to subtract the fixed quantity $\pi N_\varepsilon |\log \varepsilon|$ from (1.8). Note that, while the Ginzburg-Landau energy (that is, (1.8) with $\mathrm{v} \equiv 0$) diverges for configurations u_ε with nonzero degree at infinity,

$$0 \neq \deg(u_\varepsilon) := \lim_{R \uparrow \infty} \int_{\partial B_R} \langle \nabla u_\varepsilon, i u_\varepsilon \rangle \cdot n^\perp,$$

the modulated energy may indeed converge (and does if v has the correct circulation at infinity).

In the present context with pinning weight a, the modulated energy (1.8) should be changed into a weighted one,

$$\frac{1}{2} \int_{\mathbb{R}^2} a \left(|\nabla u_\varepsilon - i u_\varepsilon N_\varepsilon \, \mathrm{v}\,|^2 + \frac{a}{2\varepsilon^2}(1 - |u_\varepsilon|^2)^2 \right). \tag{1.9}$$

This leads to several additional difficulties. First, this energy does usually not remain finite along the flow because ∇h, F and f in (1.5) are only assumed to be bounded (in order to realistically represent at least a fixed applied current circulating through the sample). This leads us to consider a truncated version of (1.9). In the Gross-Pitaevskii case, we have to assume that ∇h, F and f decay sufficiently at infinity in order to guarantee the well-posedness of the mesoscopic model (1.5), and hence a truncation of (1.9) is no longer needed. However, in that case, due to the presence of pinning, the pressure p in the limiting PDE does not belong to L^2, and a different truncation argument then becomes needed in order to deal with this lack of integrability.

Second, for technical reasons, since the pinning potential h depends on ε according to the regime, it is convenient to replace in the modulated energy the map v by some ε-dependent map $\mathrm{v}_\varepsilon : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}^2$ which is better adapted to the ε-dependence of the potential h, and will be shown separately to converge to v. Of course, one may prefer to replace v_ε by its limit v, and directly prove that $N_\varepsilon^{-1} j_\varepsilon$ is close to v in the modulated energy as in [82], which would make the proof a bit shorter. This can be done in some of the considered regimes but not always (e.g. not for the regime (GL_2') below), hence it is more convenient to completely separate the two difficulties, first proving that $N_\varepsilon^{-1} j_\varepsilon$ is close to v_ε by means of a Grönwall argument on the modulated energy, which requires some careful vortex analysis, and then checking that v_ε indeed converges to v, which is a softer consequence of the stability of the limiting equation. In this form, we believe that the proof will appear clearer and more adaptable.

Third, in the present weighted setting, a vortex located at x_0 carries an energy $\pi a(x_0)|\log\varepsilon|$, so what needs to be subtracted from the modulated energy (1.9) is no longer $\pi N_\varepsilon |\log\varepsilon|$ but

$$\pi \sum_i d_i a(x_i)|\log\varepsilon| \sim \frac{|\log\varepsilon|}{2}\int_{\mathbb{R}^2} a\mu_\varepsilon,$$

in view of (1.7). We thus consider the following truncated version of the modulated energy (1.9),

$$\mathcal{E}_{\varepsilon,R} := \int \frac{a\chi_R}{2}\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big), \tag{1.10}$$

as well as the following truncated "modulated energy excess",

$$\mathcal{D}_{\varepsilon,R} := \mathcal{E}_{\varepsilon,R} - \frac{|\log\varepsilon|}{2}\int a\chi_R\mu_\varepsilon = \int \frac{a\chi_R}{2}\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 - |\log\varepsilon|\mu_\varepsilon\Big), \tag{1.11}$$

where for all $r>0$ we set $\chi_r := \chi(\cdot/r)$ for some fixed cut-off function $\chi \in C_c^\infty(\mathbb{R}^2;[0,1])$ with $\chi|_{B_1} \equiv 1$ and $\chi|_{\mathbb{R}^2\setminus B_2} \equiv 0$. In the sequel, all energy integrals are thus truncated as above with the cut-off function χ_R, for some scale $R \gg 1$ to be later suitably chosen as a function of ε. We write $\mathcal{E}_\varepsilon := \mathcal{E}_{\varepsilon,\infty}$ for the corresponding quantity without the cut-off χ_R in the definition (formally $R=\infty$), and also $\mathcal{D}_\varepsilon := \sup_{R\ge 1}\mathcal{D}_{\varepsilon,R}$. Rather than the L^2-norm restricted to the ball B_R centered at the origin, our methods further allow to consider the uniform $\mathrm{L}^2_{\mathrm{loc}}$-norm at the scale R: setting $\chi_R^z := \chi_R(\cdot - z)$ for all $z\in\mathbb{R}^2$, we define

$$\mathcal{E}^*_{\varepsilon,R} := \sup_z \mathcal{E}^z_{\varepsilon,R}\ , \qquad \mathcal{E}^z_{\varepsilon,R} := \int \frac{a\chi_R^z}{2}\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big), \tag{1.12}$$

$$\mathcal{D}^*_{\varepsilon,R} := \sup_z \mathcal{D}^z_{\varepsilon,R}\ , \qquad \mathcal{D}^z_{\varepsilon,R} := \mathcal{E}^z_{\varepsilon,R} - \frac{|\log\varepsilon|}{2}\int a\chi_R^z\mu_\varepsilon, \tag{1.13}$$

where the suprema run over all lattice points $z \in R\mathbb{Z}^2$.

Let us now list our assumptions. For the essential part of the proof, in the dissipative case ($\alpha>0$), it suffices to assume $h\in W^{2,\infty}(\mathbb{R}^2)$ and $F\in W^{1,\infty}(\mathbb{R}^2)^2$ (hence $f\in \mathrm{L}^\infty(\mathbb{R}^2)$ in view of (1.6)). In the Gross-Pitaevskii case, as already explained, we need to restrict to a decaying setting, that is, to further assume $\nabla h, F \in W^{1,p}(\mathbb{R}^2)^2$ for some $p<\infty$, $f\in \mathrm{L}^2(\mathbb{R}^2)$, and additionally $\operatorname{div} F = 0$. Nevertheless, in both cases, in order to ensure strong enough regularity properties of the solution of the limiting equation (as well as the delicate global well-posedness of (1.5) in the Gross-Pitaevskii case, cf. Section 2.3), stronger assumptions on the data are needed and are listed below.

Assumption A. Let $\alpha\ge 0$, $\beta\in\mathbb{R}$, $\alpha^2+\beta^2=1$, $h:\mathbb{R}^2\to\mathbb{R}$, $a:=e^h$, $F:\mathbb{R}^2\to\mathbb{R}^2$, $f:\mathbb{R}^2\to\mathbb{R}$, $u_\varepsilon^\circ:\mathbb{R}^2\to\mathbb{C}$, $\mathrm{v}_\varepsilon^\circ, \mathrm{v}^\circ : \mathbb{R}^2\to\mathbb{R}^2$ for all $\varepsilon>0$. Assume that (1.4) and (1.6) hold, and that the initial data $(u_\varepsilon^\circ, \mathrm{v}_\varepsilon^\circ, \mathrm{v}^\circ)$ are well-prepared in the sense

$$\mathcal{D}_\varepsilon^{*,\circ} := \sup_{R\ge 1}\sup_{z\in\mathbb{R}^2}\int \frac{a\chi_R^z}{2}\Big(|\nabla u_\varepsilon^\circ - iu_\varepsilon^\circ N_\varepsilon \mathrm{v}_\varepsilon^\circ|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon^\circ|^2)^2 - |\log\varepsilon|\operatorname{curl}\langle\nabla u_\varepsilon^\circ, iu_\varepsilon^\circ\rangle\Big) \ll N_\varepsilon^2, \tag{1.14}$$

with $\mathrm{v}_\varepsilon^\circ \to \mathrm{v}^\circ$ in $\mathrm{L}^2_{\mathrm{uloc}}(\mathbb{R}^2)^2$, and with $\operatorname{curl} \mathrm{v}_\varepsilon^\circ$, $\operatorname{curl} \mathrm{v}^\circ \in \mathcal{P}(\mathbb{R}^2)$. Assume that $\mathrm{v}_\varepsilon^\circ$ and v° are bounded in $W^{1,q}(\mathbb{R}^2)^2$ for all $q > 2$.

(a) *Dissipative case ($\alpha > 0$):*
For some $s > 0$, assume that $u_\varepsilon^\circ \in H^1_{\mathrm{uloc}}(\mathbb{R}^2;\mathbb{C})$, that $h \in W^{s+3,\infty}(\mathbb{R}^2)$, $F \in W^{s+2,\infty}(\mathbb{R}^2)^2$ (hence $f \in W^{1,\infty}(\mathbb{R}^2)$ in view of (1.6)), that $\mathrm{v}_\varepsilon^\circ$, v° are bounded in $W^{s+2,\infty}(\mathbb{R}^2)^2$, and that $\operatorname{curl} \mathrm{v}_\varepsilon^\circ$, $\operatorname{curl} \mathrm{v}^\circ$, $\operatorname{div}(a\mathrm{v}_\varepsilon^\circ)$ are bounded in $H^{s+1} \cap W^{s+1,\infty}(\mathbb{R}^2)$.

(b) *Gross-Pitaevskii case ($\alpha = 0$):*
Assume that $u_\varepsilon^\circ \in U + H^2(\mathbb{R}^2;\mathbb{C})$ for some reference map $U \in \mathrm{L}^\infty(\mathbb{R}^2;\mathbb{C})$ with $\nabla^2 U \in H^1(\mathbb{R}^2;\mathbb{C})$, $\nabla|U| \in \mathrm{L}^2(\mathbb{R}^2)$, $1-|U|^2 \in \mathrm{L}^2(\mathbb{R}^2)$, and $\nabla U \in \mathrm{L}^p(\mathbb{R}^2;\mathbb{C})$ for all $p > 2$ (typically we may choose U smooth and equal to $e^{iN_\varepsilon\theta}$ in polar coordinates outside a ball at the origin). Assume that $h \in W^{3,\infty}(\mathbb{R}^2)$, $\nabla h \in H^2(\mathbb{R}^2)^2$, $F \in H^3 \cap W^{3,\infty}(\mathbb{R}^2)^2$, $f \in H^2 \cap W^{2,\infty}(\mathbb{R}^2)$, and that we have $\operatorname{div} F = 0$ pointwise, and $a(x) \to 1$ uniformly as $|x| \uparrow \infty$. Assume that $\mathrm{v}_\varepsilon^\circ$, v° are bounded in $W^{2,\infty}(\mathbb{R}^2)^2$, and that $\operatorname{curl} \mathrm{v}_\varepsilon^\circ$, $\operatorname{curl} \mathrm{v}^\circ$ are bounded in $H^1(\mathbb{R}^2)$.

We distinguish between the following three main (critically scaled) regimes, in which the relative strengths of the pinning, the applied forces and the interaction emerge.

(GL_1) Weighted mixed-flow case, small number of vortices:
$\alpha > 0$, $N_\varepsilon \ll |\log\varepsilon|$, $\lambda_\varepsilon = \frac{N_\varepsilon}{|\log\varepsilon|}$, $F = \lambda_\varepsilon \hat F$, $h = \lambda_\varepsilon \hat h$ (hence $a = \hat a^{\lambda_\varepsilon}$);

(GL_2) Weighted mixed-flow case, critical number of vortices:
$\alpha > 0$, $N_\varepsilon \simeq |\log\varepsilon|$, $\frac{N_\varepsilon}{|\log\varepsilon|} \to \lambda \in (0,\infty)$, $\lambda_\varepsilon = 1$, $F = \hat F$, $h = \hat h$ (hence $a = \hat a$);

(GP) Weighted Gross-Pitaevskii case, large number of vortices:
$\alpha = 0$, $\beta = 1$, $N_\varepsilon \gg |\log\varepsilon|$, $\lambda_\varepsilon = \frac{N_\varepsilon}{|\log\varepsilon|}$, $F = \lambda_\varepsilon \hat F$, $h = \hat h$ (hence $a = \hat a$);

where $\hat h$ and $\hat F$ are independent of ε, and $\hat h \le 0$ is bounded below. Note that, just as in [82], it is not clear what happens in the Gross-Pitaevskii case with fewer (but still unboundedly many) vortices, nor in the dissipative case with more vortices (cf. Remarks 1.2–1.4).

Let us intuitively justify the choice of the above scalings. From energy considerations, we expect the pinning, the applied force, and the interaction to be of order $N_\varepsilon|\log\varepsilon||\nabla h|$, $N_\varepsilon|\log\varepsilon||F|$, and N_ε^2, respectively. The critical scaling (such that pinning, applied force and interactions are all of the same order) should thus amount to choosing both ∇h and F of order $N_\varepsilon/|\log\varepsilon|$. However, the non-degeneracy condition (1.4) for the pinning weight $a = e^h$ imposes for the pinning potential $h \le 0$ to remain uniformly bounded in ε, hence the particular non-critical choice in (GP) (with h of order 1 rather than $\lambda_\varepsilon \gg 1$).

In the dissipative case, we may also consider sub- or supercritical scalings, for which the pinning either dominates, or is dominated by the interaction. In these cases, the limiting equations are considerably simplified.

(GL_1') (GL_1) with subcritically scaled oscillating pinning, very weak interaction:
$\alpha > 0$, $N_\varepsilon \ll |\log\varepsilon|$, $\lambda_\varepsilon = 1$, $F = \hat F$, $h = \hat h$ (hence $a = \hat a$);

(GL_2') (GL_1) with subcritically scaled oscillating pinning, weak interaction:
$\alpha > 0$, $N_\varepsilon \ll |\log\varepsilon|$, $\frac{N_\varepsilon}{|\log\varepsilon|} \ll \lambda_\varepsilon \ll 1$, $F = \lambda_\varepsilon \hat F$, $h = \lambda_\varepsilon \hat h$ (hence $a = \hat a^{\lambda_\varepsilon}$);

(GL_3') (GL_1) with supercritically scaled oscillating pinning, strong interaction:
$\alpha > 0$, $N_\varepsilon \ll |\log\varepsilon|$, $\lambda_\varepsilon = \frac{N_\varepsilon}{|\log\varepsilon|}$, $F = \lambda_\varepsilon \hat F$, $h = \lambda_\varepsilon' \hat h$ (hence $a = \hat a^{\lambda_\varepsilon'}$), $\lambda_\varepsilon' \ll \lambda_\varepsilon$;

(GL_4') (GL_2) with supercritically scaled oscillating pinning, strong interaction:
$\alpha > 0$, $N_\varepsilon \simeq |\log\varepsilon|$, $\frac{N_\varepsilon}{|\log\varepsilon|} \to \lambda \in (0,\infty)$, $\lambda_\varepsilon = 1$, $F = \hat F$, $h = \lambda_\varepsilon' \hat h$, $\lambda_\varepsilon' \ll 1$;

where again $\hat h$ and $\hat F$ are independent of ε, with $\hat h \le 0$ bounded below. Since in the present work we are mostly interested in pinning effects, we may focus on the subcritical regimes (GL_1') and (GL_2'), while for the two supercritical regimes the pinning effects vanish in the limiting equation and the situation is thus much easier and closer to [82]. For simplicity, subscripts "ε" are systematically dropped from the data a, h, F, f, the precise dependence being chosen as above.

We are now in position to state our main mean-field results. As in [82] the mean-field limiting equations are fluid-like equations with an incompressibility condition (hence the existence of a pressure p) which can be lost when the number of vortices becomes large enough. We begin with the dissipative case, and consider both critical regimes (GL_1) and (GL_2), as well as the subcritical regimes (GL_1') and (GL_2'). Note that the results are slightly finer in the purely parabolic case. In the regimes (GL_1) and (GL_2'), the weight a naturally disappears from the incompressibility condition $\operatorname{div} \mathrm{v} = 0$ due to the assumption $a = \hat{a}^{\lambda_\varepsilon} \to 1$ as $\varepsilon \downarrow 0$. Although all the proofs in this paper are quantitative, we only give qualitative statements to simplify the exposition.

Theorem 1.1 (Dissipative case)**.** *Let Assumption A(a) hold, with the initial data $(u_\varepsilon^\circ, \mathrm{v}_\varepsilon^\circ, \mathrm{v}^\circ)$ satisfying the well-preparedness condition (1.14). For all $\varepsilon > 0$, let $u_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^1_{\mathrm{uloc}}(\mathbb{R}^2;\mathbb{C}))$ denote the unique global solution of (1.5) on $\mathbb{R}^+ \times \mathbb{R}^2$. Then, the following hold for the supercurrent density $j_\varepsilon := \langle \nabla u_\varepsilon, iu_\varepsilon\rangle$.*

(i) Regime (GL_1) with $\log|\log\varepsilon| \ll N_\varepsilon \ll |\log\varepsilon|$, and $\operatorname{div}(a\mathrm{v}_\varepsilon^\circ) = \operatorname{div} \mathrm{v}^\circ = 0$:
We have $N_\varepsilon^{-1} j_\varepsilon \to \mathrm{v}$ in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^1_{\mathrm{uloc}}(\mathbb{R}^2)^2)$ as $\varepsilon \downarrow 0$, where v is the unique global (smooth) solution of

$$\partial_t \mathrm{v} = \nabla \mathrm{p} + (\alpha - \mathbb{J}\beta)(\nabla^\perp \hat{h} - \hat{F}^\perp - 2\mathrm{v}) \operatorname{curl} \mathrm{v}, \qquad \operatorname{div} \mathrm{v} = 0, \qquad \mathrm{v}|_{t=0} = \mathrm{v}^\circ. \tag{1.15}$$

In the parabolic case $\beta = 0$, the same conclusion also holds for $1 \ll N_\varepsilon \lesssim \log|\log\varepsilon|$.

(ii) Regime (GL_2) with $N_\varepsilon/|\log\varepsilon| \to \lambda \in (0,\infty)$, and $\mathrm{v}_\varepsilon^\circ = \mathrm{v}^\circ$:
For some $T > 0$, we have $N_\varepsilon^{-1} j_\varepsilon \to \mathrm{v}$ in $\mathrm{L}^\infty_{\mathrm{loc}}([0,T); \mathrm{L}^1_{\mathrm{uloc}}(\mathbb{R}^2)^2)$ as $\varepsilon \downarrow 0$, where v is the unique local (smooth) solution of

$$\partial_t \mathrm{v} = \alpha^{-1} \nabla(\hat{a}^{-1} \operatorname{div}(\hat{a}\mathrm{v})) + (\alpha - \mathbb{J}\beta)(\nabla^\perp \hat{h} - \hat{F}^\perp - 2\lambda\mathrm{v}) \operatorname{curl} \mathrm{v}, \quad \mathrm{v}|_{t=0} = \mathrm{v}^\circ, \tag{1.16}$$

on $[0,T) \times \mathbb{R}^2$. In the parabolic case $\beta = 0$, this solution v can be extended globally, and the above holds with $T = \infty$.

(iii) Regime (GL_1') with $\log|\log\varepsilon| \ll N_\varepsilon \ll |\log\varepsilon|$, and $\mathrm{v}_\varepsilon^\circ = \mathrm{v}^\circ$:
We have $N_\varepsilon^{-1} j_\varepsilon \to \mathrm{v}$ in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^1_{\mathrm{uloc}}(\mathbb{R}^2)^2)$ as $\varepsilon \downarrow 0$, where v is the unique global (smooth) solution of

$$\partial_t \mathrm{v} = \alpha^{-1} \nabla(\hat{a}^{-1} \operatorname{div}(\hat{a}\mathrm{v})) + (\alpha - \mathbb{J}\beta)(\nabla^\perp \hat{h} - \hat{F}^\perp) \operatorname{curl} \mathrm{v}, \qquad \mathrm{v}|_{t=0} = \mathrm{v}^\circ. \tag{1.17}$$

(iv) Regime (GL_2') with $\log|\log\varepsilon| \ll N_\varepsilon \ll |\log\varepsilon|$, and $\operatorname{div}(a\mathrm{v}_\varepsilon^\circ) = \operatorname{div} \mathrm{v}^\circ = 0$:
We have $N_\varepsilon^{-1} j_\varepsilon \to \mathrm{v}$ in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^1_{\mathrm{uloc}}(\mathbb{R}^2)^2)$ as $\varepsilon \downarrow 0$, where v is the unique global (smooth) solution of

$$\partial_t \mathrm{v} = \nabla \mathrm{p} + (\alpha - \mathbb{J}\beta)(\nabla^\perp \hat{h} - \hat{F}^\perp) \operatorname{curl} \mathrm{v}, \qquad \operatorname{div} \mathrm{v} = 0, \qquad \mathrm{v}|_{t=0} = \mathrm{v}^\circ. \tag{1.18}$$

In the parabolic case $\beta = 0$ with $N_\varepsilon/|\log\varepsilon| \ll \lambda_\varepsilon \lesssim e^{o(N_\varepsilon)}/|\log\varepsilon|$, the same conclusion also holds for $1 \ll N_\varepsilon \lesssim \log|\log\varepsilon|$.

Remark 1.2. It is not clear how to treat the regime $N_\varepsilon \gg |\log\varepsilon|$ (with $\lambda_\varepsilon = N_\varepsilon/|\log\varepsilon|$, $F = \lambda_\varepsilon \hat{F}$, $h = \hat{h}$) by modulated energy methods in the dissipative case. The corresponding mean-field equation is formally expected to take the following degenerate form,

$$\partial_t \mathrm{v} = (\alpha - \mathbb{J}\beta)(-\hat{F}^\perp - 2\mathrm{v}) \operatorname{curl} \mathrm{v}, \qquad \mathrm{v}|_{t=0} = \mathrm{v}^\circ,$$

for which local well-posedness is obtained in [37].

We now turn to the Gross-Pitaevskii case in the (supercritical) regime (GP). Note that as $N_\varepsilon \gg |\log \varepsilon|$ the well-preparedness condition (1.14) can be simplified. The pinning force ∇h is naturally absent from the limiting equation since in the regime (GP) the interaction and the applied force dominate, but the weight a $(= \hat a)$ nevertheless remains in the incompressibility condition – in the weighted space L^2_a – $\operatorname{div}(a\mathrm{v}) = 0$ since it is of order 1.

Theorem 1.3 (Gross-Pitaevskii case)**.** *Let Assumption A(b) hold, with $\mathrm{v}_\varepsilon^\circ = \mathrm{v}^\circ$, and with the well-preparedness condition (1.14) for the initial data $(u_\varepsilon^\circ, \mathrm{v}_\varepsilon^\circ, \mathrm{v}^\circ)$ replaced by*

$$\mathcal{E}_\varepsilon^\circ := \int \frac{a}{2}\Big(|\nabla u_\varepsilon^\circ - iu_\varepsilon^\circ N_\varepsilon \mathrm{v}^\circ|^2 + \frac{a}{2\varepsilon^2}(1 - |u_\varepsilon^\circ|^2)^2\Big) \ll N_\varepsilon^2.$$

For all $\varepsilon > 0$, let $u_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; U + H^2(\mathbb{R}^2;\mathbb{C}))$ denote the unique global solution of (1.5) on $\mathbb{R}^+ \times \mathbb{R}^2$. Then, in the regime (GP) with $|\log\varepsilon| \ll N_\varepsilon \ll \varepsilon^{-1}$, we have $N_\varepsilon^{-1} j_\varepsilon \to \mathrm{v}$ in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; (\mathrm{L}^1 + \mathrm{L}^2)(\mathbb{R}^2)^2)$ as $\varepsilon \downarrow 0$, where v is the unique global (smooth) solution of

$$\partial_t \mathrm{v} = \nabla \mathrm{p} + (-\hat F + 2\mathrm{v}^\perp)\operatorname{curl}\mathrm{v}, \qquad \operatorname{div}(\hat a \mathrm{v}) = 0, \qquad \mathrm{v}^t|_{t=0} = \mathrm{v}^\circ\,. \tag{1.19}$$

Remark 1.4. The Gross-Pitaevskii model for vortices with $N_\varepsilon \ll |\log\varepsilon|$ formally behaves like the conservative flow for a particle system with Coulomb pairwise interactions. However note that no modulated energy proof is known for the mean-field limit of such a simplified discrete particle system [36] (the only known proof is by compactness [80]), although it should be easier than for the complete Gross-Pitaevskii model. We believe that the approach in [58, 54] can be adapted to this case, but it would anyway be limited to a regime with a very small (unbounded) number of vortices $N_\varepsilon \gg 1$. In contrast, the regime $N_\varepsilon \gg |\log\varepsilon|$ treated here is quite different in nature and should probably not be paralleled with a true particle system.

The structure of the mean-field equations (1.15)–(1.19) is more transparent when expressed in terms of the limiting vorticity $\mathrm{m} := \operatorname{curl}\mathrm{v}$. In the case of (1.15) (and similarly for (1.18) and (1.19)), the vorticity m satisfies a nonlinear nonlocal transport equation,

$$\begin{cases} \partial_t \mathrm{m} = \operatorname{div}\big((\alpha - \mathbb{J}\beta)(\nabla \hat h - \hat F + 2\mathrm{v}^\perp)\,\mathrm{m}\big), \\ \operatorname{curl}\mathrm{v} = \mathrm{m}, \quad \operatorname{div}\mathrm{v} = 0, \end{cases} \tag{1.20}$$

while in the case of (1.16) (and similarly for (1.17)) the vorticity m satisfies a similar equation coupled with a transport-diffusion equation for the divergence $\mathrm{d} := \operatorname{div}(\hat a\mathrm{v})$,

$$\begin{cases} \partial_t \mathrm{m} = \operatorname{div}\big((\alpha - \mathbb{J}\beta)(\nabla \hat h - \hat F + 2\lambda\mathrm{v}^\perp)\,\mathrm{m}\big), \\ \partial_t \mathrm{d} - \alpha^{-1}\triangle\,\mathrm{d} + \alpha^{-1}\operatorname{div}(\mathrm{d}\,\nabla \hat h) = \operatorname{div}\,\big((\alpha - \mathbb{J}\beta)(\nabla^\perp \hat h - \hat F^\perp - 2\lambda\mathrm{v})\hat a\mathrm{m}\big), \\ \operatorname{curl}\mathrm{v} = \mathrm{m}, \quad \operatorname{div}(\hat a\mathrm{v}) = \mathrm{d}\,. \end{cases} \tag{1.21}$$

A detailed study of these families of equations is given in the companion paper [37], including global existence results for rough initial data. While the limiting vorticity m satisfies strictly different equations in the critical regimes (GL_1) and (GL_2), we observe that it satisfies just the same equation in both subcritical regimes (GL_1') and (GL_2'), that is a simple linear transport equation.

The proofs of Theorems 1.1 and 1.3 follow the outline of [82], and rely on all the tools for vortex analysis developed over the years: lower bounds via the ball construction, "Jacobian estimate", "product estimate". In addition to the problems at infinity created by the non-decay of the forcing F that we want to allow, the presence of the pinning weight introduces additional technical difficulties, as always in the analysis of Ginzburg-Landau. The fact that the energy of a vortex depends on its location makes it more difficult to a priori control the total number of vortices, and requires localized estimates, in particular localized ball constructions. Adapting the required tools and analysis to this setting is done in Section 5.

1.3 Homogenization results and questions

As explained above, the most interesting situation from the modeling viewpoint is to let the pinning potential h vary quickly at the mesoscale $\eta_\varepsilon \ll 1$, thus coupling the mean-field limit for the vortex density with an homogenization limit. More precisely, we set

$$\hat{h}(x) := \eta_\varepsilon \hat{h}^0(x, x/\eta_\varepsilon), \tag{1.22}$$

for some $\hat{h}^0$ independent of ε, and we will refer to η_ε as the "pin separation". For simplicity, we assume that $\hat{h}^0$ is periodic in its second variable. Since in the Gross-Pitaevskii case we are anyway limited to less interesting subcritical regimes, we focus attention on the dissipative case.

1.3.1 Small pin separation limit and stick-slip models

As explained in Section 8.3, our methods only allow to treat a diagonal regime, that is, when the pin separation η_ε tends very slowly to 0, in which case the homogenization limit can simply be performed *after* the mean-field limit. The other regimes are left as an open question.

Corollary 1.5. *Let the same assumptions hold as in Theorem 1.1. In the regime* (GL_2)*, we further restrict to the parabolic case $\beta = 0$. Then there exists a sequence $\eta_{\varepsilon,0} \ll 1$ (depending on all the data of the problem) such that for all $\eta_{\varepsilon,0} \ll \eta_\varepsilon \ll 1$, choosing the fast oscillating pinning potential* (1.22)*, the same conclusions hold as in Theorem 1.1 in the form $N_\varepsilon^{-1} j_\varepsilon - \bar{\mathrm{v}}_\varepsilon \to 0$, where $\bar{\mathrm{v}}_\varepsilon$ is now the unique global (smooth) solution of the corresponding equations* (1.15)–(1.18) *with $\nabla \hat{h}(x)$ replaced by $\nabla_2 \hat{h}^0(x, x/\eta_\varepsilon)$.*

The above result thus reduces in a diagonal regime the understanding of the limiting behavior of the rescaled supercurrent $N_\varepsilon^{-1} j_\varepsilon$ to that of the solution $\bar{\mathrm{v}}_\varepsilon$ of the mean-field equations (1.15)–(1.18) with fast oscillating pinning, that is, a (periodic) homogenization problem for the mean-field equations. In more general regimes, only two minor rigorous results are obtained:

(a) For very small forcing $\|F\|_{\mathrm{L}^\infty} \ll \|\nabla h\|_{\mathrm{L}^\infty}$, in the subcritical regimes (GL_1') and (GL_2'), the vorticity is shown to remain "stuck" in the limit, that is, to converge at all times to its initial data (cf. Proposition 8.12). This is a very particular case of the pinning phenomenon evidenced below in the diagonal regime.

(b) In a short timescale of order $O(\eta_\varepsilon)$, the vorticity is shown to concentrate in each (mesoscopic) periodicity cell onto the invariant measure associated with the initial vector field (cf. Proposition 8.2). This mesoscopic initial-boundary layer result is in clear agreement with the description of the dynamics on larger timescales obtained below in the diagonal regime, where the transport is indeed shown to happen "along" the invariant measures.

Subcritical regimes. In the subcritical regimes (GL_1') and (GL_2'), the nonlinear interaction term vanishes (cf. (1.17)–(1.18)): in terms of the vorticity $\bar{\mathrm{m}}_\varepsilon$ we are thus left with a (periodic) homogenization problem for a simple *linear* transport equation, but with a compressible velocity field. Such questions were first investigated in 2D by Menon [61], and are still partially open. The situation is however much simpler if the pinning potential $\hat{h}^0(x, x/\eta_\varepsilon) := \tilde{h}^0(x/\eta_\varepsilon)$ is independent of the macroscopic variable, and if the forcing is a constant vector $\hat{F} := F \in \mathbb{R}^2$, that is, the so-called "washboard model". The homogenization result is then a particular case of the nonlinear setting considered in [27] (see also [38] for the incompressible case, and [42, 26] for the linear Hamiltonian case), but in the present framework a more precise characterization of the asymptotic behavior of $\bar{\mathrm{m}}_\varepsilon$ is possible (cf. Theorem 8.7). In the simplest situation, the result is summarized as follows.

Proposition 1.6 (Subcritical regimes)**.** *Let $\bar{\mathrm{v}}_\varepsilon$ denote the unique global (smooth) solution of* (1.17) *or* (1.18) *with $\nabla\hat{h}(x)$ replaced by $\nabla\tilde{h}^0(x/\eta_\varepsilon)$, for $\tilde{h}^0 \in C^2_{\mathrm{per}}(Q)$ (independent of ε) and $\eta_\varepsilon \ll 1$, and with $\hat{F} := F \in \mathbb{R}^2$ a constant vector. Consider the periodic vector field*

$$\Gamma^F := (\alpha - \mathbb{J}\beta)(\nabla\tilde{h}^0 - F) \quad : \quad Q \to \mathbb{R}^2,$$

and assume that the associated dynamics on the 2-torus Q has a unique stable invariant measure $\mu^F \in \mathcal{P}_{\mathrm{per}}(Q)$. Define the averaged vector

$$\Gamma^F_{\mathrm{hom}} := \int_Q \Gamma^F d\mu^F.$$

Then we have $\bar{\mathrm{m}}_\varepsilon := \operatorname{curl} \bar{\mathrm{v}}_\varepsilon \overset{*}{\rightharpoonup} \bar{\mathrm{m}}$ in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{P}(\mathbb{R}^2))$, where $\bar{\mathrm{m}}$ is the unique solution of the constant-coefficient transport equation

$$\partial_t \bar{\mathrm{m}} = \operatorname{div}(\Gamma^F_{\mathrm{hom}} \bar{\mathrm{m}}), \qquad \bar{\mathrm{m}}|_{t=0} = \operatorname{curl} \mathrm{v}^\circ .$$

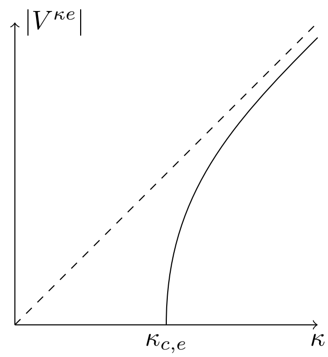


Figure 1 – Typical forcing-velocity characteristics exhibiting a stick-slip velocity law.

This result describes a so-called stick-slip velocity law: On the one hand, for F close enough to 0 the invariant measure μ^F is concentrated at a fixed point, hence the corresponding velocity field is $V^F := -\Gamma^F_{\mathrm{hom}} = 0$, that is, the vorticity gets stuck, as the vortices are trapped in local wells of the pinning potential. On the other hand, for F large enough the measure μ^F becomes non-trivial, hence we have $V^F \neq 0$, that is, the vorticity is transported, but at a reduced speed due to the attraction by the local wells of the pinning potential. We further show that the velocity law $F \mapsto V^F := -\Gamma^F_{\mathrm{hom}}$ is not smooth at the depinning threshold, but typically has a square-root behavior (cf. Proposition 8.10), denoting $\kappa := |F|$,

$$|V^{\kappa e}| = C(1 + o(1))(\kappa - \kappa_{c,e})^{1/2}, \qquad \text{as} \quad 0 < \kappa - \kappa_{c,e} \ll 1, \tag{1.23}$$

where $e \in \mathbb{S}^1$ is some direction, and where $\kappa_{c,e} e$ ($\kappa_{c,e} \geq 0$) is the critical depinning threshold in the direction e. However, no general such result is obtained (cf. open question in Remark 8.11(a)). For very large $|F| \gg 1$, we naturally find $V^F \sim (\alpha - \mathbb{J}\beta)F$, that is, the system flows as if there were no disorder. The typical response of the system in this stick-slip velocity law is plotted in Figure 1. For more detail, we refer to Section 8.5. Note that a similar frictional stick-slip dynamics is observed for very different physical processes (see e.g. the Barkhausen effect for the magnetization of a domain under an applied field [47]).

Critical regimes. In the critical regimes (GL$_1$) and (GL$_2$), the nonlinear interaction term can no longer be neglected (cf. (1.15)–(1.16)). A purely formal 2-scale expansion yields the following heuristics for the asymptotic behavior of $\bar{\mathrm{v}}_\varepsilon$. Note that a rigorous justification of this homogenization limit seems particularly challenging due to the nonlinear nonlocal character of the mean-field equations and to their instability as $\eta_\varepsilon \downarrow 0$, and moreover the well-posedness of the formal limiting equations (1.24)–(1.25) below is unclear (since the vector field $\Gamma_{\mathrm{hom}}[\bar{\mathrm{v}}]$ is in general not Lipschitz continuous even for smooth $\bar{\mathrm{v}}$). Making good sense of the formal limiting equations and justifying the limit are thus left as open questions. We refer to Section 8.4 and Remark 8.5 for detail.

Heuristics 1.7 (Critical regimes — formal asymptotic)**.** *For $w : \mathbb{R}^2 \to \mathbb{R}^2$, consider the periodic vector field*

$$-\Gamma_x[w] := -(\alpha - \mathbb{J}\beta)\big(\nabla_2 \hat{h}^0(x,\cdot) - \hat{F}(x) + 2w^\perp(x)\big) \quad : \quad Q \to \mathbb{R}^2,$$

and assume that the associated dynamics on the 2-torus Q has a unique stable invariant measure $\mu_x[w] \in \mathcal{P}_{\mathrm{per}}(Q)$. We then define the averaged vector field

$$\Gamma_{\mathrm{hom}}[w](x) := \int_Q \Gamma_x[w](y) d\mu_x[w](y).$$

(i) Regime (GL_1) with fast oscillating pinning (1.22):
Let $\bar{\mathrm{v}}_\varepsilon$ denote the unique global (smooth) solution of (1.15) with $\nabla\hat{h}(x)$ replaced by $\nabla_2\hat{h}^0(x, x/\eta_\varepsilon)$, $\eta_\varepsilon \ll 1$, and with $\hat{h}^0$ independent of ε. Then we expect $\operatorname{curl}\bar{\mathrm{v}}_\varepsilon \overset{*}{\rightharpoonup} \bar{\mathrm{m}}$ in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{P}(\mathbb{R}^2))$, where $\bar{\mathrm{m}}$ satisfies

$$\partial_t \bar{\mathrm{m}} = \operatorname{div}\big(\Xi_{\mathrm{hom}}[\bar{\mathrm{m}}]\,\bar{\mathrm{m}}\big), \qquad \bar{\mathrm{m}}|_{t=0} = \operatorname{curl} \mathrm{v}^\circ, \tag{1.24}$$

where the homogenized velocity is given by the following formula,

$$\Xi_{\mathrm{hom}}[\bar{\mathrm{m}}](x) := \Gamma_{\mathrm{hom}}[\nabla^\perp\triangle^{-1}\bar{\mathrm{m}}](x).$$

Similarly, $\bar{\mathrm{v}}_\varepsilon \overset{*}{\rightharpoonup} \bar{\mathrm{v}} := \nabla^\perp\triangle^{-1}\bar{\mathrm{m}}$ in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2))$, where $\bar{\mathrm{v}}$ thus satisfies

$$\partial_t \bar{\mathrm{v}} = \nabla\bar{\mathrm{p}} + \Gamma_{\mathrm{hom}}[\bar{\mathrm{v}}]^\perp \operatorname{curl}\bar{\mathrm{v}}, \qquad \operatorname{div}\,\bar{\mathrm{v}} = 0, \qquad \bar{\mathrm{v}}|_{t=0} = \mathrm{v}^\circ\,.$$

More precisely, for all $\tau > 0$, we expect

$$\int_0^\tau \Big(\operatorname{curl}\bar{\mathrm{v}}_\varepsilon^t(x) - \bar{\mathrm{m}}^t(x)\,\mu_x[\nabla^\perp\triangle^{-1}\bar{\mathrm{m}}^t](x/\eta_\varepsilon)\Big)dt \to 0,$$

in the strong sense of measures.

(ii) Regime (GL_2) in the parabolic case $\beta = 0$, with fast oscillating pinning (1.22):
Let $\beta = 0$, and let $\bar{\mathrm{v}}_\varepsilon$ denote the unique global (smooth) solution of (1.16) with $\nabla\hat{h}(x)$ replaced by $\nabla_2\hat{h}^0(x, x/\eta_\varepsilon)$, $\eta_\varepsilon \ll 1$, and with $\hat{h}^0$ independent of ε. Then we expect $\operatorname{curl}\bar{\mathrm{v}}_\varepsilon \overset{*}{\rightharpoonup} \bar{\mathrm{m}}$ in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{P}(\mathbb{R}^2))$ and $\operatorname{div}(\hat{a}\bar{\mathrm{v}}_\varepsilon) \rightharpoonup \bar{\mathrm{d}}$ in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^2)$, where $\bar{\mathrm{m}}$ and $\bar{\mathrm{d}}$ satisfy

$$\begin{aligned} \partial_t \bar{\mathrm{m}} &= \operatorname{div}\big(\Xi_{\mathrm{hom}}[\bar{\mathrm{m}}, \bar{\mathrm{d}}]\,\bar{\mathrm{m}}\big), \qquad \bar{\mathrm{m}}|_{t=0} = \operatorname{curl}\,\mathrm{v}^\circ, \\ \partial_t \bar{\mathrm{d}} &= \alpha^{-1}\triangle\bar{\mathrm{d}} + \operatorname{div}\big(\Xi_{\mathrm{hom}}[\bar{\mathrm{m}}, \bar{\mathrm{d}}]^\perp\,\bar{\mathrm{m}}\big), \qquad \bar{\mathrm{d}}|_{t=0} = \operatorname{div}(\hat{a}\mathrm{v}^\circ), \end{aligned} \tag{1.25}$$

where the homogenized velocity is given by the following formula,

$$\Xi_{\mathrm{hom}}[\bar{\mathrm{m}}, \bar{\mathrm{d}}](x) := \Gamma_{\mathrm{hom}}[\nabla^\perp\triangle^{-1}\bar{\mathrm{m}} + \nabla\triangle^{-1}\bar{\mathrm{d}}](x).$$

Similarly, $\bar{\mathrm{v}}_\varepsilon \overset{*}{\rightharpoonup} \bar{\mathrm{v}} := \nabla^\perp\triangle^{-1}\bar{\mathrm{m}} + \nabla\triangle^{-1}\bar{\mathrm{d}}$ in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2))$, where $\bar{\mathrm{v}}$ thus satisfies

$$\partial_t \bar{\mathrm{v}} = \alpha^{-1}\nabla\operatorname{div}\,\bar{\mathrm{v}} + \Gamma_{\mathrm{hom}}[\bar{\mathrm{v}}]^\perp \operatorname{curl}\bar{\mathrm{v}}, \qquad \bar{\mathrm{v}}|_{t=0} = \mathrm{v}^\circ\,.$$

More precisely, for all $\tau > 0$, we expect

$$\int_0^\tau \Big(\operatorname{curl}\bar{\mathrm{v}}_\varepsilon^t(x) - \bar{\mathrm{m}}^t(x)\,\mu_x[\nabla^\perp\triangle^{-1}\bar{\mathrm{m}}^t + \nabla\triangle^{-1}\bar{\mathrm{d}}^t](x/\eta_\varepsilon)\Big)dt \to 0,$$

in the strong sense of measures.

Due to the competition between the pinning potential and the vortex interaction, the dynamical properties of the limiting $\bar{\mathrm{v}}$ are expected to change dramatically with respect to the subcritical regimes: the interacting vortices are now expected to move as a coherent elastic object in an heterogeneous medium, yielding very particular glassy properties [44, 70]. To describe the dynamics, we again consider the forcing-velocity curve. Assume that the forcing $\hat{F} := F \in \mathbb{R}^2$ is a constant vector, let $\bar{\mathrm{v}}^F := \bar{\mathrm{v}}$ denote as above the corresponding limit of $\bar{\mathrm{v}}_\varepsilon$ as $\varepsilon \downarrow 0$, and set $\bar{\mathrm{m}}^F := \operatorname{curl}\bar{\mathrm{v}}^F$. Formally, the mean velocity is then defined by

$$V^F := \lim_{t\uparrow\infty}\frac{1}{t}\int x\,d\bar{\mathrm{m}}^{F,t}(x). \tag{1.26}$$

Intuitively, for F close enough to 0, the above heuristics predicts that the vorticity $\bar{\mathrm{m}}^F$ should spread due to the vortex repulsion, until the interaction force $\bar{\mathrm{v}}^F$ becomes small enough that the invariant measure $\mu_x^F[\bar{\mathrm{v}}^F]$ remains concentrated at a fixed point of the dynamics generated by $-\Gamma_x^F[\bar{\mathrm{v}}^F]$, in which case $\Gamma_{\mathrm{hom}}^F[\bar{\mathrm{v}}^F] = 0$ holds. We therefore expect, just as in the subcritical regimes, to find $V^F = 0$ for F close enough to 0, $V^F \neq 0$ for F large enough, and $V^F \sim \alpha F - \beta F^\perp$ for very large $|F| \gg 1$ (cf. Figure 1). Nevertheless, the precise picture is expected to become very different at the depinning threshold: the velocity law $F \mapsto V^F$ should still be non-smooth at this threshold, of the form

$$|V^{\kappa e}| = C(1+o(1))(\kappa-\kappa_{c,e})^\zeta, \qquad \text{as} \quad 0 < \kappa - \kappa_{c,e} \ll 1,$$

in some direction $e \in \mathbb{S}^1$, but the value of the depinning threshold $\kappa_{c,e} > 0$ and of the depinning exponent $\zeta \in (0,1)$ are expected to differ completely from the case without interaction (1.23) and to be related to the glassy properties of the system, as predicted in the physics literature [64, 67, 20] (see also [44, Section 5]). A rigorous justification of this whole description is left as an open question.

Since the vortices are elastically coupled by the interaction, the problem is formally analogous to that of understanding the motion of general elastic systems in disordered media, which is the framework considered in the above-cited physics papers. In this spirit, a considerable attention has been devoted in the physics community to the simpler Quenched Edwards-Wilkinson model for elastic interface motion in disordered media [55, 14]. Note that for this interface model some rigorous mathematical understanding is available: the pinning of the interface at low forcing is proved in [32] in dimension $d \geq 2$, while the (ballistic) motion of the interface at large forcing is obtained in [22, 33] in dimension $d = 2$, and more recently in [11, 34] for various related discrete models in any dimension $d \geq 2$. These questions are also related (although again for different models) to the recent rigorous homogenization results for the forced mean curvature equation and for more general geometric Hamilton-Jacobi equations [6].

1.3.2 System with temperature

Stochastic variants of the Ginzburg-Landau equation have been introduced to model the effect of thermal noise [49, 77, 28, 29, 43]. Although we do not study here the mean-field limit problem for such models, for a finite number N of vortices, in the limit $\varepsilon \downarrow 0$, we expect the thermal noise to act on the vortices as N independent Brownian motions: more precisely, in the regime (GL$_1$), the limiting trajectories $(x_i)_{i=1}^N$ of the N vortices are expected to satisfy the following system of coupled SDEs (see e.g. [39, Section III.B]),

$$dx_i = (\alpha - \mathbb{J}\beta)\big(N^{-1}\nabla_{x_i}W_N(x_1,\ldots,x_N) - \nabla\hat h(x_i) + \hat F(x_i)\big)dt + \sqrt{2T}dB_i^t, \qquad 1\le i\le N, \tag{1.27}$$
$$W_N(x_1,\ldots,x_N) := -\pi\sum_{i\neq j}^N \log|x_i - x_j|,$$

where $B_1,\ldots,B_N$ are N independent 2D Brownian motions. Such macroscopic phenomenological models, where the thermal noise acts via random Langevin kicks, are abundantly used by physicists [10, 44, 70].

In the case of a large number of vortices $N_\varepsilon \gg 1$, in the regime (GL_1), it is then natural to postulate that a good phenomenological model for the limiting supercurrent $\mathrm{v} := \lim_\varepsilon N_\varepsilon^{-1} j_\varepsilon$ is given by the (deterministic) mean-field limit of the particle system (1.27), that is, the following version of (1.15) with viscosity,

$$\partial_t \mathrm{v} = \nabla \mathrm{p} + (\alpha - \mathbb{J}\beta)(\nabla^\perp \hat h - \hat F^\perp - 2\mathrm{v})\mathrm{curl}\ \mathrm{v} + T\triangle \mathrm{v}, \qquad \mathrm{div}\ \mathrm{v} = 0, \qquad \mathrm{v}|_{t=0} = \mathrm{v}^\circ, \tag{1.28}$$

while in the regime (GL_2) a natural model for the limit v is rather given by the following version of (1.16) with viscosity,

$$\partial_t \mathrm{v} = \alpha^{-1}\nabla(\hat a^{-1}\,\mathrm{div}\,(\hat a \mathrm{v})) + (\alpha - \mathbb{J}\beta)(\nabla^\perp \hat h - \hat F^\perp - 2\lambda \mathrm{v})\mathrm{curl}\ \mathrm{v} + T\triangle \mathrm{v}, \quad \mathrm{v}|_{t=0} = \mathrm{v}^\circ\,. \tag{1.29}$$

In the regimes (GL_1') and (GL_2'), these equations should be replaced by their versions without interaction term. Note that in [41] the mean-field limit of the particle system (1.27) has been rigorously proved to coincide with (1.28), although the modulated energy method seems to fail [36].

In this viscous context, we may now consider the homogenization limit of the phenomenological mean-field models (1.28)–(1.29) with fast oscillating pinning (1.22), or equivalently, with $\nabla \hat h(x)$ replaced by $\nabla_2 \hat h^0(x, x/\eta_\varepsilon)$. We denote by $\bar{\mathrm{v}}_\varepsilon$ the unique (smooth) solution of the corresponding equation. We naturally restrict attention to the critical scaling for the temperature, that is, $T := \eta_\varepsilon T_0$ for some fixed $T_0 > 0$.

Remark 1.8. On the one hand, for temperatures $T \ll \eta_\varepsilon$, the viscous term in equations (1.28)–(1.29) is expected to have no effect in the limit, yielding the same asymptotic behavior for $T = 0$. On the other hand, for $T \gg \eta_\varepsilon$, the viscous term is so strong that the energy barriers are instantaneously overcome by the dynamics: for $T = \kappa_\varepsilon T_0$ with $\eta_\varepsilon \ll \kappa_\varepsilon \ll 1$, the limit $\bar{\mathrm{v}}$ of the solution $\bar{\mathrm{v}}_\varepsilon$ of (1.28) or (1.29) with oscillating pinning is expected to satisfy respectively (as suggested by a formal 2-scale expansion)

$$\partial_t \bar{\mathrm{v}} = \nabla \bar{\mathrm{p}} - (\alpha - \mathbb{J}\beta)(\hat F^\perp + 2\bar{\mathrm{v}})\mathrm{curl}\,\bar{\mathrm{v}}, \qquad \mathrm{div}\ \bar{\mathrm{v}} = 0, \qquad \bar{\mathrm{v}}|_{t=0} = \mathrm{v}^\circ,$$
$$\text{or} \qquad \partial_t \bar{\mathrm{v}} = \alpha^{-1}\nabla(\mathrm{div}\ \bar{\mathrm{v}}) - (\alpha - \mathbb{J}\beta)(\hat F^\perp + 2\lambda\bar{\mathrm{v}})\mathrm{curl}\,\bar{\mathrm{v}}, \qquad \bar{\mathrm{v}}|_{t=0} = \mathrm{v}^\circ,$$

while for $T = T_0$ of order 1 the limit $\bar{\mathrm{v}}$ should satisfy respectively

$$\partial_t \bar{\mathrm{v}} = \nabla \bar{\mathrm{p}} - (\alpha - \mathbb{J}\beta)(\hat F^\perp + 2\bar{\mathrm{v}})\mathrm{curl}\,\bar{\mathrm{v}} + T_0\triangle\bar{\mathrm{v}}, \qquad \mathrm{div}\ \bar{\mathrm{v}} = 0, \qquad \bar{\mathrm{v}}|_{t=0} = \mathrm{v}^\circ,$$
$$\text{or} \qquad \partial_t \bar{\mathrm{v}} = \alpha^{-1}\nabla(\mathrm{div}\ \bar{\mathrm{v}}) - (\alpha - \mathbb{J}\beta)(\hat F^\perp + 2\lambda\bar{\mathrm{v}})\mathrm{curl}\,\bar{\mathrm{v}} + T_0\triangle\bar{\mathrm{v}}, \qquad \bar{\mathrm{v}}|_{t=0} = \mathrm{v}^\circ\,.$$

It is thus indeed natural to rather restrict attention to the less trivial case of the critically scaled temperature $T \simeq \eta_\varepsilon$ (say $T := \eta_\varepsilon T_0$ for some fixed $T_0 > 0$).

Subcritical regimes. In the subcritical regimes (GL_1') and (GL_2'), the mean-field equations take the form (1.28)–(1.29) without interaction term; hence, in terms of the vorticity $\bar{\mathrm{m}}_\varepsilon := \mathrm{curl}\,\bar{\mathrm{v}}_\varepsilon$, with oscillating pinning, and with critically scaled temperature $T = \eta_\varepsilon T_0$, $T_0 > 0$, the equation takes the form

$$\partial_t \bar{\mathrm{m}}_\varepsilon = \mathrm{div}\ \big((\alpha - \mathbb{J}\beta)(\nabla_2 \hat h^0(\cdot, \cdot/\eta_\varepsilon) - \hat F)\bar{\mathrm{m}}_\varepsilon\big) + \eta_\varepsilon T_0 \triangle \bar{\mathrm{m}}_\varepsilon, \qquad \bar{\mathrm{m}}_\varepsilon|_{t=0} = \mathrm{curl}\ \mathrm{v}^\circ\,. \tag{1.30}$$

The limit $\eta_\varepsilon \downarrow 0$ of this equation is a very particular case of homogenization of a parabolic equation with vanishing viscosity, as studied by Dalibard [25]. Alternatively, using Nguetseng's 2-scale compactness theorem (in the form of Lemma 8.9, as in the proof of Theorem 8.7), we easily obtain the following.

Proposition 1.9 (Subcritical regimes with temperature). *Let $\bar{\mathrm{m}}_\varepsilon$ be as above, and assume that $\hat{h}^0 \in C_b(\mathbb{R}^2; C^1_{\mathrm{per}}(Q))$, and $\hat{F} \in C_b(\mathbb{R}^2)$. Let $\tilde{\mu}^{T_0} \in \mathrm{L}^\infty(\mathbb{R}^2; H^1_{\mathrm{per}}(Q)/\mathbb{R})$ denote the unique solution of the following cell problem,*

$$T_0 \triangle_y \tilde{\mu}^{T_0}(x,y) + \mathrm{div}_y\big((\alpha - \mathbb{J}\beta)(\nabla_2 \hat{h}^0(x,y) - \hat{F}(x))(1 + \tilde{\mu}^{T_0}(x,y))\big) = 0, \tag{1.31}$$

and define the following averaged vector field,

$$\Gamma^{T_0}_{\mathrm{hom}}(x) := \int_Q (\alpha - \mathbb{J}\beta)(\nabla_2 \hat{h}^0(x,y) - \hat{F}(x))(1 + \tilde{\mu}^{T_0}(x,y))dy. \tag{1.32}$$

Then we have $\bar{\mathrm{m}}_\varepsilon \xrightarrow{*} \bar{\mathrm{m}}$ in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{P}(\mathbb{R}^2))$, where $\bar{\mathrm{m}}$ is the unique solution of the transport equation

$$\partial_t \bar{\mathrm{m}} = \mathrm{div}\,(\Gamma^{T_0}_{\mathrm{hom}} \bar{\mathrm{m}}), \qquad \bar{\mathrm{m}}|_{t=0} = \mathrm{curl}\ \mathrm{v}^\circ\,.$$

Note that this result is very similar to that of Proposition 1.6, except that here the invariant measure is replaced by its viscous version (1.31). In order to describe the dynamical properties of this limiting model, we again investigate the behavior of the typical forcing-velocity curve: we consider a constant forcing vector $\hat{F} := F \in \mathbb{R}^2$, we assume that $\hat{h}^0(x, x/\eta_\varepsilon) := \tilde{h}^0(x/\eta_\varepsilon)$ is independent of the macroscopic variable, we denote by $\Gamma^{F,T_0}_{\mathrm{hom}} \in \mathbb{R}^2$ the corresponding averaged vector (1.32), and we investigate the behavior of the velocity law $F \mapsto V^{F,T_0} := -\Gamma^{F,T_0}_{\mathrm{hom}}$. For large $|F|$, the picture is essentially the same as in the case without temperature $T_0 = 0$. However, since the viscous invariant measure $1 + \tilde{\mu}^{F,T_0} \in \mathcal{P}(Q)$ always has maximal support in the cell Q, we find $V^{F,T_0} \neq 0$ for all $F \neq 0$, that is, in the presence of temperature $T_0 > 0$ the mass is always transported (at a reduced speed) and cannot get stuck forever in the local wells of the pinning potential. The precise behavior of V^{F,T_0} for F close to 0 is then of particular interest. Heuristically, the forcing $F \neq 0$ tilts the energy landscape, and the energy barriers of size $\mathrm{osc}\,\tilde{h}^0 := \max \tilde{h}^0 - \min \tilde{h}^0$ are then overcome by thermal activation even for small $F \neq 0$. The velocity law for this so-called thermally assisted flux flow is then expected to satisfy the classical Arrhenius law from statistical thermodynamics (see e.g. [44, Section 5.1]),

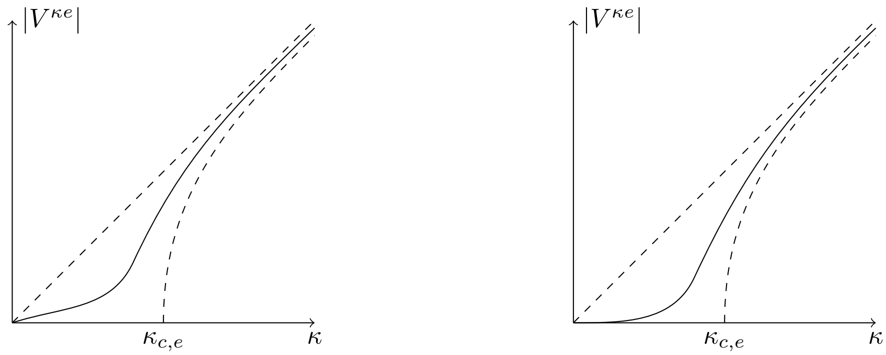
$$V^{F,T_0} = C(1 + o(1)) \exp\Big(-\frac{C}{T_0}\,\mathrm{osc}\,\tilde{h}^0\Big)F, \qquad \text{as} \quad T_0 \ll 1 \quad \text{and} \quad |F| \ll 1, \tag{1.33}$$

that is, the response should be linear, but exponentially small as a function of T_0. More precise versions of this asymptotic result, which is related (via characteristics) to the Eyring-Kramers formula, are proved to hold in any dimension in [13, 48, 8]. Note that for the corresponding problem in dimension 1 (with $\beta = 0$) the averaged vector V^{F,T_0} can be explicitly computed, and the asymptotic law (1.33) is easily checked by hand. The typical forcing-velocity characteristics are plotted in Figure 2(a).

Critical regimes. In the critical regimes (GL_1) and (GL_2), the nonlinear interaction term can no longer be neglected, and we need to consider the homogenization limit of the complete mean-field models (1.28)–(1.29), with $\nabla \hat{h}(x)$ replaced by $\nabla_2 \hat{h}^0(x, x/\eta_\varepsilon)$, and with critically scaled temperature $T := \eta_\varepsilon T_0$, $T_0 > 0$. In spite of the vanishing viscosity term, the rigorous justification of this homogenization limit remains very challenging due to the nonlinear nonlocal character of the mean-field models and to their instability as $\eta_\varepsilon \downarrow 0$. A purely formal 2-scale expansion yields the following heuristics for the asymptotic behavior of $\bar{\mathrm{v}}_\varepsilon$. Note that this coincides with Heuristics 1.7 except that here the invariant measures are replaced by viscous versions. Justifying the limit is again left as an open question. We refer to Section 8.4 and Remark 8.6 for detail.

Heuristics 1.10 (Critical regimes with temperature — formal asymptotics). *For all $w : \mathbb{R}^2 \to \mathbb{R}^2$, consider the periodic vector field*

$$\Gamma_x[w] := (\alpha - \mathbb{J}\beta)(\nabla_2 \hat{h}^0(x,\cdot) - \hat{F}(x) + 2w^\perp(x)) \quad : \quad Q \to \mathbb{R}^2,$$



(a) Subcritical regimes: (linear) ohmic velocity law in the low-forcing limit.

(b) Critical regimes: (nonlinear) creep velocity law in the low-forcing limit.

Figure 2 – Typical forcing-velocity characteristics in the presence of (low) temperature.

denote by $\tilde\mu_x^{T_0}[w] \in \mathcal{P}_{\rm per}(Q)$ *the unique solution of the following equation on the* 2*-torus* Q*,*

$$T_0 \triangle \tilde\mu_x^{T_0}[w] + \operatorname{div}\left(\Gamma_x[w]\tilde\mu_x^{T_0}[w]\right) = 0,$$

and define the averaged vector field

$$\Gamma_{\rm hom}^{T_0}[w](x) := \int_Q \Gamma_x[w](y) d\tilde\mu_x^{T_0}[w](y).$$

Let $\bar{\rm v}_\varepsilon$ *denote the unique global (smooth) solution of* (1.28) *or* (1.29) *with* $\nabla \hat h(x)$ *replaced by* $\nabla_2 \hat h^0(x, x/\eta_\varepsilon)$*, and with* $T := \eta_\varepsilon T_0$*,* $\eta_\varepsilon \ll 1$*, with* $\hat h^0$ *and* $T_0 > 0$ *independent of* ε*. Then the same asymptotic results should hold as in Heuristics 1.7, but with* $\Gamma_{\rm hom}[\cdot]$ *replaced by its better-behaved viscous version* $\Gamma_{\rm hom}^{T_0}[\cdot]$*.*

Noting that the viscous invariant measures $\tilde\mu_x^{T_0}[w]$ depend smoothly on w — unlike the situation without temperature —, the local well-posedness of the limiting equations for $\bar{\rm v}$ is now easily obtained. Again we are interested in the mean velocity law $F \mapsto V^{F,T_0}$ (defined as in (1.26)). The overall picture is essentially the same as in the subcritical regimes. However, as in the case without temperature, due to the competition between the pinning potential and the vortex interaction, the precise dynamical properties of $\bar{\rm v}$ are expected to change dramatically: the interacting vortices now move as a coherent whole, satisfying glassy properties [44]. The main manifestation of this difference is visible in the low-forcing low-temperature limit ($|F|, T_0 \ll 1$), where the linear Arrhenius law (1.33) is now expected to break down, being replaced by the following so-called creep law, with stretched exponential dependence in the imposed forcing,

$$V^{F,T_0} = C(1+o(1)) \exp\Big(-\frac{C}{T_0 F^\mu}\Big),$$

for some creep exponent $\mu > 0$. This was first predicted by physicists for related elastic interface motion models [65, 51] and then adapted to vortex systems [40, 66, 45, 19, 20] (see also [44, Section 5] and references therein). The typical forcing-velocity curves are plotted in Figure 2(b). This particular glassy dynamical behavior is more generally expected to hold for any elastic object (here, a system of interacting vortices) that fluctuates in a heterogeneous medium, but even for simpler models no rigorous derivation is available. For an attempt at a mathematical approach to creep laws, we refer to [3]. Note that the crucial influence of the interactions on the dynamics is interestingly already exemplified in a simplified 1D model in [39, Section IV].

1.3.3 Infinite mobility limit and Bean's model

A further asymptotic limit may be considered in order to reduce the above limiting equations to simpler laws: let us assume that the forcing $\hat F$ is time-dependent, but varies on a much larger timescale than the vortex motion. More precisely, let us consider the following rescaling of the mean-field equations (1.28)–(1.29) for $\bar{\mathrm{v}}_\varepsilon$ with oscillating pinning potential and with critically scaled temperature $T := \eta_\varepsilon T_0$: in the regime ($\mathrm{GL}_1$),

$$\eta_\varepsilon\partial_t\bar{\mathrm{v}}_\varepsilon = \nabla\bar{\mathrm{p}}_\varepsilon + (\alpha-\mathbb{J}\beta)(\nabla_2^\perp \hat h^0(\cdot,\cdot/\eta_\varepsilon) - \hat F^\perp - 2\bar{\mathrm{v}}_\varepsilon)\operatorname{curl}\bar{\mathrm{v}}_\varepsilon + \eta_\varepsilon T_0\triangle\bar{\mathrm{v}}_\varepsilon, \qquad \operatorname{div}\bar{\mathrm{v}}_\varepsilon = 0, \qquad \bar{\mathrm{v}}_\varepsilon|_{t=0} = \mathrm{v}^\circ,$$

and in the regime (GL_2),

$$\eta_\varepsilon\partial_t\bar{\mathrm{v}}_\varepsilon = \alpha^{-1}\nabla(\hat a^{-1}\operatorname{div}(\hat a\bar{\mathrm{v}}_\varepsilon)) + (\alpha-\mathbb{J}\beta)(\nabla_2^\perp \hat h^0(\cdot,\cdot/\eta_\varepsilon) - \hat F^\perp - 2\bar{\mathrm{v}}_\varepsilon)\operatorname{curl}\bar{\mathrm{v}}_\varepsilon + \eta_\varepsilon T_0\triangle\bar{\mathrm{v}}_\varepsilon, \qquad \bar{\mathrm{v}}_\varepsilon|_{t=0} = \mathrm{v}^\circ,$$

while in the subcritical regimes (GL_1')–(GL_2') we consider the corresponding equations without interaction term. In the case without temperature ($T_0 = 0$), in the timescale of variation of the forcing $\hat F$, we may heuristically replace the velocity law plotted in Figure 1 by the simplified law pictured in Figure 3, meaning that the vortices have infinite mobility beyond the depinning threshold, hence rearrange themselves instantaneously. Such rate-independent limiting models are known as the Bean or the Kim-Anderson models; we refer to [17, Sections 6.3–6.4] and [79] for more detail. In the subcritical regimes (GL_1')–(GL_2'), for the model without interaction and without temperature ($T_0 = 0$), the convergence to a suitable rate-independent process is proved in any dimension in [87], while an approach to the corresponding case with temperature $T_0 > 0$ is proposed in [88]. Rigorously treating the critical regimes with interaction is much more delicate, and is not pursued here.

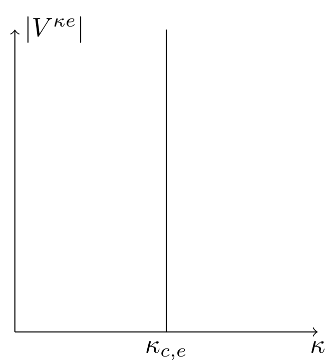


Figure 3 – In the Bean and Kim-Anderson models, the exact velocity law typically given by Figure 1 is replaced by this simplified law.

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Part I
Mean-field limits

2 Discussion of the model and well-posedness

For future reference, note that in each of the above regimes, using the explicit choice (1.6) of the zeroth-order term f, we have the following scalings,

(a) *Dissipative case:* we have

$$\|\nabla h\|_{\mathrm{L}^\infty} \lesssim 1 \wedge \lambda_\varepsilon, \qquad \|\nabla^2 h\|_{\mathrm{L}^\infty} \lesssim \eta_\varepsilon^{-1}(1 \wedge \lambda_\varepsilon), \qquad \|F\|_{W^{1,\infty}} \lesssim \lambda_\varepsilon,$$
$$\|f\|_{\mathrm{L}^\infty} \lesssim \eta_\varepsilon^{-1}(1 \wedge \lambda_\varepsilon) + |\log \varepsilon|^2 \lambda_\varepsilon^2, \qquad \|\nabla f\|_{\mathrm{L}^\infty} \lesssim \eta_\varepsilon^{-2}(1 \wedge \lambda_\varepsilon) + |\log \varepsilon|^2 \lambda_\varepsilon^2,$$

hence in the case $\eta_\varepsilon = 1$,

$$\|\nabla h\|_{W^{1,\infty}} \lesssim 1 \wedge \lambda_\varepsilon, \qquad \|F\|_{W^{1,\infty}} \lesssim \lambda_\varepsilon, \qquad \|f\|_{W^{1,\infty}} \lesssim 1 \wedge \lambda_\varepsilon + |\log \varepsilon|^2 \lambda_\varepsilon^2, \tag{2.1}$$

(b) *Gross-Pitaevskii case:* we have in the case $\eta_\varepsilon = 1$,

$$\|\nabla h\|_{H^1 \cap W^{1,\infty}} \lesssim 1, \qquad \|F\|_{H^1 \cap W^{1,\infty}} \lesssim \lambda_\varepsilon, \qquad \|f\|_{H^1 \cap W^{1,\infty}} \lesssim 1 + |\log \varepsilon|^2 \lambda_\varepsilon^2. \tag{2.2}$$

2.1 Derivation of the modified Ginzburg-Landau equation

In this section we derive (1.5). We start from the equations considered in [89, 84], where the applied current is modeled by a term appearing on the boundary of a bounded domain Ω,

$$\begin{cases} \lambda_\varepsilon(\alpha + i|\log \varepsilon|\beta)\partial_t w_\varepsilon = \triangle w_\varepsilon + \frac{w_\varepsilon}{\varepsilon^2}(a - |w_\varepsilon|^2), & \text{in } \mathbb{R}^+ \times \Omega, \\ n \cdot \nabla w_\varepsilon = i w_\varepsilon |\log \varepsilon| n \cdot J_{\mathrm{ex}}, & \text{on } \mathbb{R}^+ \times \partial\Omega, \\ w_\varepsilon|_{t=0} = w_\varepsilon^\circ \end{cases} \tag{2.3}$$

where n is the outer unit normal. As in [89, 84], we may modify the rescaled order parameter $w_\varepsilon/\sqrt{a}$ in order to turn the Neumann boundary condition into an homogeneous one, which then makes the imposed current J_{ex} appear directly in the equation. For that purpose, we assume that $a = 1$ at the boundary $\partial\Omega$, and that the total incoming current equals the total outgoing current, that is, $\int_{\partial\Omega} n \cdot J_{\mathrm{ex}} = 0$. We then have $\int_{\partial\Omega} an \cdot J_{\mathrm{ex}} = 0$, so that there exists a unique solution $\psi \in H^1(\Omega)$ of

$$\begin{cases} \operatorname{div}(a\nabla\psi) = 0, & \text{in } \Omega, \\ n \cdot \nabla\psi = n \cdot J_{\mathrm{ex}}, & \text{on } \partial\Omega. \end{cases}$$

A straightforward computation shows that the transformed order parameter $u_\varepsilon := e^{-i|\log \varepsilon|\psi} w_\varepsilon/\sqrt{a}$ satisfies

$$\begin{cases} \lambda_\varepsilon(\alpha + i|\log \varepsilon|\beta)\partial_t u_\varepsilon = \triangle u_\varepsilon + \frac{a u_\varepsilon}{\varepsilon^2}(1 - |u_\varepsilon|^2) + \nabla h \cdot \nabla u_\varepsilon + i|\log \varepsilon| F^\perp \cdot \nabla u_\varepsilon + f u_\varepsilon, & \text{in } \mathbb{R}^+ \times \Omega, \\ n \cdot \nabla(u_\varepsilon \sqrt{a}) = 0, & \text{on } \mathbb{R}^+ \times \partial\Omega, \\ u_\varepsilon|_{t=0} = u_\varepsilon^\circ, \end{cases} \tag{2.4}$$

where we have set

$$h := \log a, \qquad F := -2\nabla^\perp \psi, \qquad \text{and} \qquad f := \frac{\triangle\sqrt{a}}{\sqrt{a}} - \frac{1}{4}|\log \varepsilon|^2 |F|^2. \tag{2.5}$$

Note that the vector field F satisfies $\operatorname{div} F = \operatorname{curl}(aF) = 0$. In order to avoid delicate boundary issues,[1] a natural approach consists in sending the boundary $\partial\Omega$ to infinity and study the corresponding problem on the whole of $\mathbb{R}^2$. The assumption $a|_{\partial\Omega} = 1$ is now replaced by the assumption that

$$a(x) \to 1 \quad (\text{that is, } h(x) \to 0), \qquad \text{and} \qquad \nabla h(x) \to 0, \qquad \text{uniformly as } |x| \uparrow \infty,$$

while F, f are simply assumed to be bounded. Noting that $2\nabla\sqrt{a} = \sqrt{a}\nabla h \to 0$ holds by assumption at infinity, the Neumann boundary condition then formally translates into $\frac{x}{|x|} \cdot \nabla u_\varepsilon \to 0$ at infinity. Further imposing the natural condition $|u_\varepsilon| \to 1$ at infinity, we look for a global solution $u_\varepsilon : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{C}$ of the corresponding equation (2.4) with fixed total degree $D_\varepsilon \in \mathbb{Z}$, and with

$$|u_\varepsilon| \to 1, \qquad \frac{x}{|x|} \cdot \nabla u_\varepsilon \to 0, \qquad \text{as } |x| \uparrow \infty, \qquad \text{and} \qquad \deg u_\varepsilon = D_\varepsilon.$$

In the dissipative case $\alpha > 0$, global existence and uniqueness of a solution $u_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^1_{\mathrm{uloc}}(\mathbb{R}^2; \mathbb{C}))$ is proved in Appendix A, as well as additional regularity, but, due to the possibly complicated advection structure at infinity caused by the non-decaying fields F, f, it is unclear whether the above properties at infinity are satisfied. In particular, it is not even clear whether the total degree of the constructed solution u_ε is well-defined. This difficulty originates in the possibility of instantaneous creation of many vortex dipoles at infinity for fixed $\varepsilon > 0$ due to forcing and pinning effects, although these dipoles are shown to necessarily disappear at infinity in the limit $\varepsilon \downarrow 0$ e.g. as a consequence of our mean-field results. Anyway, since a more precise description of u_ε at infinity is irrelevant for our purposes, it is not pursued here. Note that the global existence and uniqueness for u_ε in the uniformly locally integrable class is proved even without any decay assumption on h.

For simplicity, we may further truncate the forcing F, f at infinity, thus focusing on the local behavior of the solution near the origin. In the Gross-Pitaevskii case, our results are limited to this decaying setting. Note that then at least one of the conditions $\operatorname{div} F = \operatorname{curl}(aF) = 0$ must be relaxed: we may for instance rather truncate ψ and define F via formula (2.5), so that the condition $\operatorname{div} F = 0$ is preserved. Since there is no advection at infinity in this setting, we prove existence and uniqueness of a solution u_ε in an affine space $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; U_\varepsilon + H^1(\mathbb{R}^2; \mathbb{C}))$, for some fixed smooth non-decaying "reference map" U_ε satisfying $|U_\varepsilon| \to 1$ and $\frac{x}{|x|} \cdot \nabla U_\varepsilon \to 0$ as $|x| \uparrow \infty$. Given $D_\varepsilon \in \mathbb{Z}$, we typically choose $U_\varepsilon := U_{D_\varepsilon}$ smooth and equal to $e^{iD_\varepsilon\theta}$ (in polar coordinates) outside a neighborhood of the origin, which imposes for u_ε a fixed total degree equal to D_ε.

Remark 2.1. Rather than normalizing the original order parameter w_ε by the expected density $\sqrt{a}$, another natural choice was proposed by Lassoued and Mironescu [60], and consists in normalizing w_ε by a minimizer γ_ε of the weighted Ginzburg-Landau energy, that is, a nonvanishing solution of

$$\begin{cases} -\triangle\gamma_\varepsilon = \frac{\gamma_\varepsilon}{\varepsilon^2}(a - |\gamma_\varepsilon|^2), & \text{in } \Omega, \\ n \cdot \nabla\gamma_\varepsilon = 0, & \text{on } \partial\Omega, \end{cases}$$

and setting $\tilde{u}_\varepsilon := e^{-i|\log\varepsilon|\psi} w_\varepsilon / \gamma_\varepsilon$ with ψ as before. This new order parameter $\tilde{u}_\varepsilon$ satisfies

$$\lambda_\varepsilon(\alpha + i|\log\varepsilon|\beta)\partial_t\tilde{u}_\varepsilon = \triangle\tilde{u}_\varepsilon + \frac{\gamma_\varepsilon^2\tilde{u}_\varepsilon}{\varepsilon^2}(1 - |\tilde{u}_\varepsilon|^2) + \nabla\tilde{h} \cdot \nabla\tilde{u}_\varepsilon + i|\log\varepsilon|\tilde{F}^\perp \cdot \nabla\tilde{u}_\varepsilon + \tilde{f}\tilde{u}_\varepsilon,$$

in terms of $\tilde{h} := \log\gamma_\varepsilon^2$, $\tilde{F} := -2\nabla^\perp\psi$, and $\tilde{f} := -\frac{1}{4}|F|^2$. We are thus again reduced to a very similar framework as the one above, and the results could easily be adapted.

1. Another way of avoiding boundary issues would consist in rather considering the equation on the 2-torus. Nevertheless, the total degree of the map u_ε then necessarily vanishes, and hence, in order to describe a non-trivial vorticity with distinguished sign, we would have no other choice than working with the complete Ginzburg-Landau model with gauge. Working with the gauge actually does not change anything deep, but makes all computations even heavier, which we wanted to avoid for clarity purposes.

2.2 Case with gauge

In the dissipative case, it is interesting to make the computations also in the gauged case, which is the true physical model for superconductors. The evolution equation (2.3) is then replaced by the following, as first derived by Schmid [78] and by Gor'kov and Eliashberg [46], here written in the mixed-flow case, with strong (critically scaled) imposed current $|\log\varepsilon|J_{\mathrm{ex}}:\partial\Omega\to\mathbb{R}^2$ and imposed magnetic field $|\log\varepsilon|H_{\mathrm{ex}}:\partial\Omega\to\mathbb{R}$ at the boundary, and with a non-uniform pinning weight $a:\mathbb{R}^2\to[0,1]$,

$$\begin{cases}\lambda_\varepsilon(\alpha+i|\log\varepsilon|\beta)(\partial_t w_\varepsilon-iw_\varepsilon\Psi_\varepsilon)=\nabla^2_{B_\varepsilon}w_\varepsilon+\frac{w_\varepsilon}{\varepsilon^2}(a-|w_\varepsilon|^2), & \text{in } \mathbb{R}^+\times\Omega,\\ \sigma(\partial_t B_\varepsilon-\nabla\Psi_\varepsilon)=\nabla^\perp\mathrm{curl}\,B_\varepsilon+\langle iw_\varepsilon,\nabla_{B_\varepsilon}w_\varepsilon\rangle, & \text{in } \mathbb{R}^+\times\Omega,\\ \mathrm{curl}\,B_\varepsilon=|\log\varepsilon|H_{\mathrm{ex}}, & \text{on } \mathbb{R}^+\times\partial\Omega,\\ n\cdot\nabla_{B_\varepsilon}w_\varepsilon=iw_\varepsilon|\log\varepsilon|n\cdot J_{\mathrm{ex}}, & \text{on } \mathbb{R}^+\times\partial\Omega,\\ w_\varepsilon|_{t=0}=w_\varepsilon^\circ,\end{cases}$$

where $B_\varepsilon:\mathbb{R}^+\times\mathbb{R}^2\to\mathbb{R}^2$ now represents the gauge of the magnetic field $\mathrm{curl}\,B_\varepsilon$, where $\Psi_\varepsilon:\mathbb{R}^+\times\mathbb{R}^2\to\mathbb{R}$ is the gauge of the electric field $-\partial_t B_\varepsilon+\nabla\Psi_\varepsilon$, where $\nabla_{B_\varepsilon}:=\nabla-iB_\varepsilon$ is the usual covariant derivative, and where the real parameter $\sigma\ge 0$ characterizes the relaxation time of the magnetic field. We are then interested in the asymptotic behavior of the supercurrent density $\langle\nabla_{B_\varepsilon}(w_\varepsilon/\sqrt{a}),i(w_\varepsilon/\sqrt{a})\rangle$, naturally obtained after rescaling the order parameter w_ε by the pinning weight. However, as in [89, 84], it is useful to further modify the rescaled order parameter $w_\varepsilon/\sqrt{a}$ in order to turn the boundary conditions into homogeneous ones, which then makes the imposed current and magnetic field J_{ex} and H_{ex} appear directly in the equation. Further, for simplicity, in order to avoid boundary issues, under similar assumptions on a as in Section 2.1, we may formally send the boundary $\partial\Omega$ to infinity and study the corresponding problem on the whole of $\mathbb{R}^2$. Without explicitly describing the transformation (which includes a choice of the gauge Ψ_ε; we refer to [84, Section 2] for detail), the transformed couple $(u_\varepsilon,A_\varepsilon)$ replacing the triplet $(w_\varepsilon,B_\varepsilon,\Psi_\varepsilon)$ then satisfies the following equation,

$$\begin{cases}\lambda_\varepsilon(\alpha+i|\log\varepsilon|\beta)\partial_t u_\varepsilon=\nabla^2_{A_\varepsilon}u_\varepsilon+\frac{au_\varepsilon}{\varepsilon^2}(1-|u_\varepsilon|^2)+\nabla h\cdot\nabla_{A_\varepsilon}u_\varepsilon+i|\log\varepsilon|F^\perp\cdot\nabla_{A_\varepsilon}u_\varepsilon+fu_\varepsilon, & \text{in } \mathbb{R}^+\times\Omega,\\ \sigma\partial_t A_\varepsilon=\nabla^\perp\mathrm{curl}\,A_\varepsilon+a\langle iu_\varepsilon,\nabla_{A_\varepsilon}u_\varepsilon\rangle-\frac12|\log\varepsilon|aF^\perp(1-|u_\varepsilon|^2), & \text{in } \mathbb{R}^+\times\Omega,\\ u_\varepsilon|_{t=0}=u_\varepsilon^\circ,\end{cases}$$

where $h:=\log a$, and where F and f are given explicitly in terms of a, J_{ex} and H_{ex}. Natural quantities associated with this transformed model are the gauge-invariant supercurrent and vorticity,

$$j_\varepsilon:=\langle\nabla_{A_\varepsilon}u_\varepsilon,iu_\varepsilon\rangle,\qquad \mu_\varepsilon:=\mathrm{curl}\,(j_\varepsilon+A_\varepsilon),$$

and the electric field

$$E_\varepsilon:=-\partial_t A_\varepsilon.$$

We believe that the derivation of mean-field limit results from this gauged version of the model (1.5) does not cause any major difficulty, and can be achieved following the kind of computations performed in [82, Appendix C]. Formally, the corresponding results to Theorem 1.1 are the convergences

$$\frac{j_\varepsilon}{N_\varepsilon}\to\mathrm{v},\qquad \frac{\mu_\varepsilon}{N_\varepsilon}\to\mathrm{m}:=\mathrm{curl}\,\mathrm{v}+\mathrm{H},\qquad \frac{\mathrm{curl}\,A_\varepsilon}{N_\varepsilon}\to\mathrm{H},\qquad \frac{E_\varepsilon}{N_\varepsilon}\to\mathrm{E},$$

where the limiting triplet (v,H,E) satisfies, in the regime (GL_1),

$$\begin{cases}\partial_t\mathrm{v}-\mathrm{E}=\nabla\mathrm{p}+(\alpha-\mathbb{J}\beta)(\nabla^\perp\hat h-\hat F^\perp-2\mathrm{v})\,\mathrm{m},\\ \mathrm{div}\,\mathrm{v}=0,\\ -\sigma\mathrm{E}=\mathrm{v}+\nabla^\perp\mathrm{H},\\ \partial_t\mathrm{H}=-\mathrm{curl}\,\mathrm{E},\end{cases}\tag{2.6}$$

or in the regime (GL_2)

$$\begin{cases}\partial_t \mathrm{v} - \mathrm{E} = \alpha^{-1}\nabla(\hat a^{-1}\operatorname{div}(\hat a \mathrm{v})) + (\alpha - \mathbb{J}\beta)(\nabla^\perp \hat h - \hat F^\perp - 2\lambda \mathrm{v})\,\mathrm{m},\\ -\sigma \mathrm{E} = \mathrm{v} + \nabla^\perp \mathrm{H},\\ \partial_t \mathrm{H} = -\operatorname{curl}\,\mathrm{E},\end{cases} \tag{2.7}$$

while in the subcritical regimes (GL_1')–(GL_2') the equations are obtained from the above by removing the nonlinear interaction terms vm. The structure of these equations is maybe more transparent at the level of the vorticity $\mathrm{m} := \operatorname{curl}\,\mathrm{v} + \mathrm{H}$: the system (2.6) takes the form

$$\begin{cases}\partial_t \mathrm{m} = \operatorname{div}\,\big((\alpha - \mathbb{J}\beta)(\nabla \hat h - \hat F + 2\mathrm{v}^\perp)\,\mathrm{m}\big),\\ \sigma\partial_t \mathrm{H} - \triangle\,\mathrm{H} + \mathrm{H} = \mathrm{m},\\ \operatorname{div}\,\mathrm{v} = 0, \quad \operatorname{curl}\,\mathrm{v} = \mathrm{m} - H\end{cases}$$

while (2.7) becomes for $\sigma > 0$,

$$\begin{cases}\partial_t \mathrm{m} = \operatorname{div}\,\big((\alpha - \mathbb{J}\beta)(\nabla \hat h - \hat F + 2\mathrm{v}^\perp)\,\mathrm{m}\big),\\ \partial_t \mathrm{d} - \alpha^{-1}\triangle\,\mathrm{d} + \alpha^{-1}\operatorname{div}(\mathrm{d}\,\nabla \hat h) + \frac{1}{\sigma}\,\mathrm{d} = -\frac{1}{\sigma}\hat a \nabla \hat h\cdot\nabla^\perp H + \operatorname{div}\,\big((\alpha - \mathbb{J}\beta)(\nabla^\perp \hat h - \hat F^\perp - 2\lambda \mathrm{v})\hat a \mathrm{m}\big),\\ \sigma\partial_t \mathrm{H} - \triangle\,\mathrm{H} + \mathrm{H} = \mathrm{m},\\ \operatorname{div}(\hat a \mathrm{v}) = \mathrm{d}, \quad \operatorname{curl}\,\mathrm{v} = \mathrm{m} - H,\end{cases}$$

that is a transport equation for m, coupled with a linear heat equation for H, and in the case (2.7) further coupled with a transport-diffusion equation for the divergence $\mathrm{d} := \operatorname{div}(\hat a \mathrm{v})$. For simplicity, we focus in this work on the model without gauge (1.5).

2.3 Well-posedness for the modified Ginzburg-Landau equation

In this section, we address global well-posedness for equation (1.5), both in the dissipative ($\alpha > 0$) and in the Gross-Pitaevskii ($\alpha = 0$) regimes. In the dissipative regime, a well-posedness result for (1.5) in the space $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^1_{\mathrm{uloc}}(\mathbb{R}^2;\mathbb{C}))$ is obtained in the general non-decaying setting, but no precise description of the solution is obtained in that case, due to a possibly subtle advection force at infinity. In particular, it is not even clear to us whether the total degree of the constructed solution is well-defined. In the decaying setting, in contrast, we do not allow any advection at infinity. As is classical since the work of Bethuel and Smets [9] (see also [63]), we then consider the existence of a solution u_ε of (1.5) in the space $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; U_\varepsilon + H^1(\mathbb{R}^2;\mathbb{C}))$ for some "reference map" U_ε, which is typically chosen smooth and equal (in polar coordinates) to $e^{iD_\varepsilon\theta}$ outside a ball at the origin, for some given $D_\varepsilon \in \mathbb{Z}$. Such a choice $U_\varepsilon = U_{D_\varepsilon}$ imposes a fixed total degree D_ε at infinity. More generally, we may consider the following set of "admissible" reference maps

$$E_1(\mathbb{R}^2) := \{U \in \mathrm{L}^\infty(\mathbb{R}^2;\mathbb{C}) : \nabla^2 U \in H^1(\mathbb{R}^2;\mathbb{C}), \nabla|U| \in \mathrm{L}^2(\mathbb{R}^2), 1 - |U|^2 \in \mathrm{L}^2(\mathbb{R}^2), \nabla U \in \mathrm{L}^p(\mathbb{R}^2;\mathbb{C})\ \forall p > 2\}.$$

Our global well-posedness results are summarized in the following; finer results and detailed proofs are given in Appendix A, including additional regularity statements.

Proposition 2.2 (Well-posedness for (1.5))**.**

(i) Dissipative case $\alpha > 0$, $\beta \in \mathbb{R}$ (general setting):
Let $h \in W^{1,\infty}(\mathbb{R}^2)$, $a := e^h$, $F \in \mathrm{L}^\infty(\mathbb{R}^2)^2$, $f \in \mathrm{L}^\infty(\mathbb{R}^2)$, and $u_\varepsilon^\circ \in H^1_{\mathrm{uloc}}(\mathbb{R}^2;\mathbb{C})$. Then there exists a unique global solution $u_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^1_{\mathrm{uloc}}(\mathbb{R}^2;\mathbb{C}))$ of (1.5) on $\mathbb{R}^+ \times \mathbb{R}^2$ with initial data u_ε°, and this solution satisfies $\partial_t u_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^2_{\mathrm{uloc}}(\mathbb{R}^2;\mathbb{C}))$.

(ii) Gross-Pitaevskii case $\alpha = 0$, $\beta \in \mathbb{R}$ (decaying setting):
Let $h \in W^{3,\infty}(\mathbb{R}^2)$, $\nabla h \in H^2(\mathbb{R}^2)^2$, $a := e^h$, $F \in H^3 \cap W^{3,\infty}(\mathbb{R}^2)^2$ with $\operatorname{div} F = 0$, $f \in H^2 \cap W^{2,\infty}(\mathbb{R}^2)$, and $u_\varepsilon^\circ \in U + H^2(\mathbb{R}^2;\mathbb{C})$ for some $U \in E_1(\mathbb{R}^2)$. Then there exists a unique global solution $u_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; U + H^2(\mathbb{R}^2;\mathbb{C}))$ of (1.5) on $\mathbb{R}^+ \times \mathbb{R}^2$ with initial data u_ε°, and this solution satisfies $\partial_t u_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^2(\mathbb{R}^2;\mathbb{C}))$.

Proof. Item (i) follows from Proposition A.2. We turn to item (ii). By Proposition A.1(ii), the assumptions in the above statement ensure the existence of a unique global solution $u_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; U + H^2(\mathbb{R}^2;\mathbb{C}))$. This directly implies that $\triangle u_\varepsilon$, $\nabla h \cdot \nabla u_\varepsilon$, $F^\perp \cdot \nabla u_\varepsilon$, and $f u_\varepsilon$ belong to $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^2(\mathbb{R}^2;\mathbb{C}))$. Using the Sobolev embedding of $H^1(\mathbb{R}^2)$ into $\mathrm{L}^4 \cap \mathrm{L}^6(\mathbb{R}^2)$, and decomposing $u_\varepsilon(1-|u_\varepsilon|^2)$ in terms of $u_\varepsilon = U + \hat u_\varepsilon$ with $\hat u_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^2(\mathbb{R}^2;\mathbb{C}))$, we further deduce that $u_\varepsilon(1-|u_\varepsilon|^2)$ belongs to $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^2(\mathbb{R}^2;\mathbb{C}))$. Inserting this into equation (1.5) yields the claimed integrability of $\partial_t u_\varepsilon$. ☐

Although a detailed proof of this well-posedness statement is included in Appendix A, we close this section with some comments on the strategy. In the dissipative case with decaying h, F, f, the arguments by [9, 63] are easily adapted to the present context with both pinning and forcing. The Gross-Pitaevskii regime is however more delicate, and we then use the structure of the equation to make a change of variables that usefully transforms the first-order terms into zeroth-order ones. The additional regularity assumptions in item (ii) above are precisely needed for this transformation to be well-behaved. Finally, the general result stated in item (i) for the dissipative case with non-decaying h, F, f, is deduced from the corresponding result with decaying h, F, f by a careful approximation argument in the space $H^1_{\mathrm{uloc}}(\mathbb{R}^2;\mathbb{C})$.

3 Preliminaries on the limiting equations

The limiting equations that we derive are all of the form

$$\partial_t \mathrm{v}_\varepsilon = \nabla \mathrm{p}_\varepsilon + \Gamma_\varepsilon \operatorname{curl} \mathrm{v}_\varepsilon, \qquad \mathrm{v}_\varepsilon|_{t=0} = \mathrm{v}_\varepsilon^\circ, \tag{3.1}$$

for some smooth pressure $\mathrm{p}_\varepsilon : \mathbb{R}^2 \to \mathbb{R}$, and some smooth vector field $\Gamma_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2$. The pressure will either be proportional to $a^{-1} \operatorname{div}(a\mathrm{v}_\varepsilon)$, or be the Lagrange multiplier associated with the constraint $\operatorname{div}(a\mathrm{v}_\varepsilon) = 0$. Before Sections 6–7, we only manipulate these quantities abstractly. In order for all our computations to be licit, we then need to work under the following integrability and smoothness assumptions.

Assumption B.

(a) *Dissipative case ($\alpha > 0$):* There exists some $T > 0$ such that for all $\varepsilon > 0$, all $t \in [0, T)$, and all $q > 2$,

$$\|(\mathrm{v}_\varepsilon^t, \nabla \mathrm{v}_\varepsilon^t)\|_{(\mathrm{L}^2+\mathrm{L}^q)\cap \mathrm{L}^\infty} \lesssim_{t,q} 1, \quad \|\operatorname{curl} \mathrm{v}_\varepsilon^t\|_{\mathrm{L}^1\cap\mathrm{L}^\infty} \lesssim_t 1, \quad \|\operatorname{div}(a\mathrm{v}_\varepsilon^t)\|_{\mathrm{L}^2\cap\mathrm{L}^\infty} \lesssim_t 1,$$
$$\|\mathrm{p}_\varepsilon^t\|_{\mathrm{L}^2\cap\mathrm{L}^\infty} \lesssim_t \lambda_\varepsilon^{-1/2}, \quad \|\nabla \mathrm{p}_\varepsilon\|_{\mathrm{L}^2_t\mathrm{L}^2} \lesssim_t 1,$$
$$\|\partial_t \mathrm{v}_\varepsilon^t\|_{\mathrm{L}^2\cap\mathrm{L}^\infty} \lesssim_t \lambda_\varepsilon^{-1/2}, \quad \|\partial_t \mathrm{v}_\varepsilon\|_{\mathrm{L}^2_t\mathrm{L}^2} \lesssim_t 1, \quad \|\partial_t \mathrm{p}_\varepsilon^t\|_{\mathrm{L}^2_t\mathrm{L}^2} \lesssim_t \lambda_\varepsilon^{-1},$$
$$\|\Gamma_\varepsilon^t\|_{W^{1,\infty}} \lesssim_t 1, \quad \|\partial_t \Gamma_\varepsilon\|_{\mathrm{L}^2_t\mathrm{L}^2} \lesssim_t 1.$$

(b) *Gross-Pitaevskii case ($\alpha = 0$):* There exists some $T > 0$ such that for all $\varepsilon > 0$, all $t \in [0, T)$, and all $q > 2$,

$$\|(\mathrm{v}_\varepsilon^t, \nabla \mathrm{v}_\varepsilon^t)\|_{(\mathrm{L}^2+\mathrm{L}^q)\cap \mathrm{L}^\infty} \lesssim_{t,q} 1, \quad \|\operatorname{curl} \mathrm{v}_\varepsilon^t\|_{\mathrm{L}^1\cap\mathrm{L}^\infty} \lesssim_t 1$$
$$\|\mathrm{p}_\varepsilon^t\|_{\mathrm{L}^q\cap\mathrm{L}^\infty} \lesssim_{t,q} 1, \quad \|\nabla \mathrm{p}_\varepsilon^t\|_{\mathrm{L}^2\cap\mathrm{L}^\infty} \lesssim_t 1, \quad \|\partial_t \mathrm{v}_\varepsilon^t\|_{\mathrm{L}^2} \lesssim_t 1, \quad \|\partial_t \mathrm{p}_\varepsilon^t\|_{\mathrm{L}^q} \lesssim_{t,q} 1,$$
$$\|\Gamma_\varepsilon^t\|_{W^{1,\infty}} \lesssim_t 1, \quad \|\partial_t \Gamma_\varepsilon^t\|_{\mathrm{L}^2} \lesssim_t 1.$$

In the dissipative case of Theorem 1.1 the rescaled supercurrent density $N_\varepsilon^{-1} j_\varepsilon$ is shown in Section 6 to remain close to the solution v_ε of the following equation

$$\begin{gathered}\partial_t \mathrm{v}_\varepsilon = \nabla \mathrm{p}_\varepsilon + \Gamma_\varepsilon \mathrm{curl}\ \mathrm{v}_\varepsilon, \qquad \mathrm{v}_\varepsilon|_{t=0} = \mathrm{v}_\varepsilon^\circ, \\ \Gamma_\varepsilon := \lambda_\varepsilon^{-1}(\alpha - \mathbb{J}\beta)\Big(\nabla^\perp h - F^\perp - \frac{2N_\varepsilon}{|\log \varepsilon|}\,\mathrm{v}_\varepsilon\Big), \qquad \mathrm{p}_\varepsilon := (\lambda_\varepsilon \alpha a)^{-1}\,\mathrm{div}\,(a\mathrm{v}_\varepsilon),\end{gathered} \tag{3.2}$$

while in the Gross-Pitaevskii case of Theorem 1.3 the rescaled supercurrent density $N_\varepsilon^{-1} j_\varepsilon$ is shown in Section 7 to remain close to the solution v_ε of the following equation

$$\begin{gathered}\partial_t \mathrm{v}_\varepsilon = \nabla \mathrm{p}_\varepsilon + \Gamma_\varepsilon \mathrm{curl}\ \mathrm{v}_\varepsilon, \qquad \mathrm{div}\,(a\mathrm{v}_\varepsilon) = 0, \qquad \mathrm{v}_\varepsilon|_{t=0} = \mathrm{v}_\varepsilon^\circ, \\ \Gamma_\varepsilon := -\lambda_\varepsilon^{-1}\Big(\nabla^\perp h - F^\perp - \frac{2N_\varepsilon}{|\log \varepsilon|}\,\mathrm{v}_\varepsilon\Big)^\perp.\end{gathered} \tag{3.3}$$

In the present section, we show that the solutions v_ε of the above equations (3.2)–(3.3) exist and satisfy all the properties of Assumption B. Using the choice of the scalings for $\lambda_\varepsilon, h, F$ in each regime, we further show how to pass to the limit $\varepsilon \downarrow 0$ in these equations, which is needed to conclude the proofs of Theorems 1.1 and 1.3. Note that in the regimes (GL_1) and (GL_2'), as a consequence of the choice $\lambda_\varepsilon \downarrow 0$, we expect the solution v_ε of (3.2) to converge to the solution v of some incompressible equation with the constraint $\mathrm{div}\ \mathrm{v} = 0$. We thus naturally refer to (GL_1), (GL_2') and (GP) as the *incompressible regimes*, and to (GL_2) and (GL_1') as the *compressible regimes*.

3.1 Dissipative case

3.1.1 Properties of solutions to (3.2)

It is instructive to examine the vorticity formulation of the equation (3.2) for v_ε. Setting $\mathrm{m}_\varepsilon := \mathrm{curl}\ \mathrm{v}_\varepsilon$ and $\mathrm{d}_\varepsilon := \mathrm{div}\,(a\mathrm{v}_\varepsilon)$, equation (3.2) may be rewritten as a nonlinear nonlocal transport equation for the vorticity m_ε, coupled with a transport-diffusion equation for the divergence d_ε,

$$\begin{cases}\partial_t \mathrm{m}_\varepsilon = -\,\mathrm{div}\,(\Gamma_\varepsilon^\perp \mathrm{m}_\varepsilon), \quad \mathrm{m}_\varepsilon|_{t=0} = \mathrm{curl}\ \mathrm{v}_\varepsilon^\circ, \\ \partial_t \mathrm{d}_\varepsilon - (\alpha\lambda_\varepsilon)^{-1} \triangle \mathrm{d}_\varepsilon + (\alpha\lambda_\varepsilon)^{-1}\,\mathrm{div}\,(\mathrm{d}_\varepsilon \nabla h) = \mathrm{div}\,(a\Gamma_\varepsilon \mathrm{m}_\varepsilon), \quad \mathrm{d}_\varepsilon|_{t=0} = \mathrm{div}\,(a\mathrm{v}_\varepsilon^\circ), \\ \mathrm{curl}\ \mathrm{v}_\varepsilon = \mathrm{m}_\varepsilon, \quad \mathrm{div}\,(a\mathrm{v}_\varepsilon) = \mathrm{d}_\varepsilon\,.\end{cases} \tag{3.4}$$

A detailed study of this kind of equations is given in the companion paper [37], including global existence results for vortex-sheet initial data. The following proposition in particular states that a solution v_ε always exists and satisfies the various properties of Assumption B(a), under suitable regularity assumptions on the initial data $\mathrm{v}_\varepsilon^\circ$. Compared with [37], this result however requires some more work in the incompressible cases $\lambda_\varepsilon \downarrow 0$, as it is then needed to make clear the link with the limiting incompressible equations, in particular in order to obtain global existence in the mixed-flow case.

Proposition 3.1. *Let $h : \mathbb{R}^2 \to \mathbb{R}$, $a := e^h$, $F : \mathbb{R}^2 \to \mathbb{R}^2$, and let $\mathrm{v}_\varepsilon^\circ : \mathbb{R}^2 \to \mathbb{R}^2$ be bounded in $W^{1,q}(\mathbb{R}^2)^2$ for all $q > 2$, and satisfy* $\mathrm{curl}\ \mathrm{v}_\varepsilon^\circ \in \mathcal{P}(\mathbb{R}^2)$. *For some $s > 0$, assume that $h \in W^{s+3,\infty}(\mathbb{R}^2)$, $F \in W^{s+2,\infty}(\mathbb{R}^2)^2$, that $\mathrm{v}_\varepsilon^\circ$ is bounded in $W^{s+2,\infty}(\mathbb{R}^2)^2$, and that* $\mathrm{curl}\ \mathrm{v}_\varepsilon^\circ$ *and* $\mathrm{div}\,(a\mathrm{v}_\varepsilon^\circ)$ *are bounded in $H^{s+1} \cap W^{s+1,\infty}(\mathbb{R}^2)$.*

(i) Compressible regimes $\lambda_\varepsilon \simeq 1$ (that is, (GL_2)–(GL_1')):
There exist $T > 0$ (independent of ε) and a unique (local) solution $\mathrm{v}_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T); \mathrm{v}_\varepsilon^\circ + H^2 \cap W^{2,\infty}(\mathbb{R}^2)^2)$ of (3.2) on $[0,T) \times \mathbb{R}^2$. Moreover, all the properties of Assumption B(a) are satisfied, that is, for all $t \in [0,T)$, and all $q > 2$,

$$\begin{gathered}\|(\mathrm{v}_\varepsilon^t, \nabla \mathrm{v}_\varepsilon^t)\|_{(\mathrm{L}^2+\mathrm{L}^q)\cap \mathrm{L}^\infty} \lesssim_{t,q} 1, \quad \|\mathrm{curl}\ \mathrm{v}_\varepsilon^t\|_{\mathrm{L}^1\cap\mathrm{L}^\infty} \lesssim_t 1, \quad \|\mathrm{div}\,(a\mathrm{v}_\varepsilon^t)\|_{\mathrm{L}^2\cap\mathrm{L}^\infty} \lesssim_t 1, \\ \|\mathrm{p}_\varepsilon^t\|_{\mathrm{L}^2\cap\mathrm{L}^\infty} \lesssim_t 1, \quad \|\nabla \mathrm{p}_\varepsilon^t\|_{\mathrm{L}^2} \lesssim_t 1, \quad \|\partial_t \mathrm{v}_\varepsilon^t\|_{\mathrm{L}^2\cap\mathrm{L}^\infty} \lesssim_t 1, \quad \|\partial_t \mathrm{p}_\varepsilon^t\|_{\mathrm{L}^2_t \mathrm{L}^2} \lesssim_t 1.\end{gathered}$$

In the parabolic case $\beta = 0$, the solution v_ε can be extended globally, $T = \infty$. In the small-interaction regime (GL_1'), in the mixed-flow case $\beta \neq 0$, the existence time T can be taken arbitrarily large for $\varepsilon > 0$ small enough.

(ii) Incompressible regimes $\lambda_\varepsilon \ll 1$ (that is, (GL_1)–(GL_2')):
Further assume $\operatorname{div}(a\mathrm{v}_\varepsilon^\circ) = 0$. *There exist $T > 0$ (independent of ε) and a unique (local) solution $\mathrm{v}_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T); \mathrm{v}_\varepsilon^\circ + H^2 \cap W^{2,\infty}(\mathbb{R}^2)^2)$ of (3.2) on $\mathbb{R}^+ \times \mathbb{R}^2$. Moreover, all the properties of Assumption B(a) are satisfied, that is, for all $t \in [0,T)$, and all $q > 2$,*

$$\|(\mathrm{v}_\varepsilon^t, \nabla \mathrm{v}_\varepsilon^t)\|_{(\mathrm{L}^2+\mathrm{L}^q)\cap \mathrm{L}^\infty} \lesssim_{t,q} 1, \quad \|\operatorname{curl}\, \mathrm{v}_\varepsilon^t\|_{\mathrm{L}^1\cap\mathrm{L}^\infty} \lesssim_t 1, \quad \|\operatorname{div}(a\mathrm{v}_\varepsilon^t)\|_{\mathrm{L}^2\cap\mathrm{L}^\infty} \lesssim_t 1,$$
$$\|\mathrm{p}_\varepsilon^t\|_{\mathrm{L}^2\cap\mathrm{L}^\infty} \lesssim_t \lambda_\varepsilon^{-1/2}, \quad \|\nabla \mathrm{p}_\varepsilon\|_{\mathrm{L}^2_t\mathrm{L}^2} \lesssim_t 1, \quad \|\partial_t \mathrm{v}_\varepsilon^t\|_{\mathrm{L}^2\cap\mathrm{L}^\infty} \lesssim_t \lambda_\varepsilon^{-1/2}, \quad \|\partial_t \mathrm{v}_\varepsilon\|_{\mathrm{L}^2_t\mathrm{L}^2} \lesssim_t 1, \quad \|\partial_t \mathrm{p}_\varepsilon^t\|_{\mathrm{L}^2_t\mathrm{L}^2} \lesssim_t \lambda_\varepsilon^{-1}.$$

In the parabolic case $\beta = 0$, this solution v_ε can be extended globally, i.e. $T = \infty$. In the mixed-flow case $\beta \neq 0$, the existence time T can be taken arbitrarily large for $\varepsilon > 0$ small enough.

Proof. Item (i) is proved in Step 1 below, while the proof of (ii) is split into three further steps. The proof of the global existence for the regime (GL_1'), also stated in (i), is postponed to the last step.

Step 1: compressible regimes. Let $s > 0$ be non-integer. The assumption $\|\hat h\|_{W^{s+3,\infty}}, \|\hat F\|_{W^{s+2,\infty}} \lesssim 1$ yields $\|\lambda_\varepsilon^{-1}(\nabla^\perp h - F^\perp)\|_{W^{s+2,\infty}} \lesssim 1$ in the considered regimes, and also $\lambda_\varepsilon^{-1} N_\varepsilon/|\log \varepsilon| \lesssim 1$ and $\lambda_\varepsilon \simeq 1$. Further using the assumptions on the initial data $\mathrm{v}_\varepsilon^\circ$, the results in [37] imply that in each of the compressible regimes (GL_2)–(GL_1') there exists a unique (local) solution $\mathrm{v}_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T); \mathrm{v}_\varepsilon^\circ + H^2 \cap W^{2,\infty}(\mathbb{R}^2)^2)$ of (3.2) on $[0,T) \times \mathbb{R}^2$ with initial data $\mathrm{v}_\varepsilon^\circ$, for some $T \gtrsim 1$. Moreover, it is shown in [37] that this solution satisfies for all $t \in [0,T)$,

$$\|\mathrm{v}_\varepsilon^t - \mathrm{v}_\varepsilon^\circ\|_{H^1\cap W^{1,\infty}} \lesssim_t 1, \qquad \|\mathrm{m}_\varepsilon^t\|_{H^1\cap W^{1,\infty}} \lesssim_t 1, \qquad \|\mathrm{d}_\varepsilon^t\|_{\mathrm{L}^2\cap\mathrm{L}^\infty} \lesssim_t 1, \qquad \int \mathrm{m}_\varepsilon^t = 1, \qquad \mathrm{m}_\varepsilon^t \geq 0. \quad (3.5)$$

Note that in the parabolic case $\beta = 0$ the results in [37] actually give a global solution, that is, $T = \infty$. We claim that all the desired properties of v_ε follow from (3.5). Combining (3.5) with the assumption that $\mathrm{v}_\varepsilon^\circ$ is bounded in $W^{1,q}(\mathbb{R}^2)^2$ for all $q > 2$, we obtain

$$\|(\mathrm{v}_\varepsilon^t, \nabla \mathrm{v}_\varepsilon^t)\|_{(\mathrm{L}^2+\mathrm{L}^q)\cap\mathrm{L}^\infty} \lesssim_{t,q} 1.$$

Using the choice (3.2) in the form $\mathrm{p}_\varepsilon = (\lambda_\varepsilon \alpha a)^{-1}\,\mathrm{d}_\varepsilon$ with $\lambda_\varepsilon \simeq 1$, where the divergence $\mathrm{d}_\varepsilon = \operatorname{div}(a\mathrm{v}_\varepsilon)$ satisfies the transport-diffusion equation (3.4), the a priori estimates in [37, Lemma 2.3] give

$$\|\mathrm{p}_\varepsilon^t\|_{H^1\cap W^{1,\infty}} \lesssim \|\mathrm{d}_\varepsilon^t\|_{H^1\cap W^{1,\infty}} \lesssim_t \|\mathrm{d}_\varepsilon^\circ\|_{H^1\cap W^{1,\infty}} + \|a\Gamma_\varepsilon^t \mathrm{m}_\varepsilon^t\|_{H^1\cap W^{1,\infty}} \lesssim_t 1,$$

where the last bound follows from (3.5). inserting this information into (3.2), we conclude that

$$\|\partial_t \mathrm{v}_\varepsilon^t\|_{\mathrm{L}^2\cap\mathrm{L}^\infty} \lesssim \|\nabla \mathrm{p}_\varepsilon^t\|_{\mathrm{L}^2\cap\mathrm{L}^\infty} + \|a\Gamma_\varepsilon^t \mathrm{m}_\varepsilon^t\|_{\mathrm{L}^2\cap\mathrm{L}^\infty} \lesssim_t 1.$$

Testing the transport-diffusion equation $\partial_t \mathrm{d}_\varepsilon - (\lambda_\varepsilon\alpha)^{-1}(\triangle \mathrm{d}_\varepsilon - \operatorname{div}(\mathrm{d}_\varepsilon \nabla h)) = \operatorname{div}(a\Gamma_\varepsilon \mathrm{m}_\varepsilon)$ against $\partial_t \mathrm{d}_\varepsilon$ yields

$$\int |\partial_t \mathrm{d}_\varepsilon|^2 + \frac{1}{2}(\lambda_\varepsilon\alpha)^{-1}\partial_t \int |\nabla \mathrm{d}_\varepsilon|^2 = -\int \partial_t \mathrm{d}_\varepsilon \operatorname{div}\big((\lambda_\varepsilon\alpha)^{-1}\,\mathrm{d}_\varepsilon \nabla h - a\Gamma_\varepsilon \mathrm{m}_\varepsilon\big),$$

and hence, integrating in time, with $\lambda_\varepsilon \simeq 1$,

$$\begin{aligned}\|\partial_t \mathrm{d}_\varepsilon\|^2_{\mathrm{L}^2_t\mathrm{L}^2} + \frac{1}{2}(\lambda_\varepsilon\alpha)^{-1}\|\nabla \mathrm{d}_\varepsilon\|^2_{\mathrm{L}^2} &\lesssim \|\nabla \mathrm{d}_\varepsilon^\circ\|^2_{\mathrm{L}^2} + \|\partial_t \mathrm{d}_\varepsilon\|_{\mathrm{L}^2_t\mathrm{L}^2}(\|\mathrm{d}_\varepsilon\|_{\mathrm{L}^2_t H^1} + \|a\Gamma_\varepsilon\|_{\mathrm{L}^\infty_t W^{1,\infty}}\|\mathrm{m}_\varepsilon\|_{\mathrm{L}^2_t H^1})\\ &\lesssim_t 1 + \|\partial_t \mathrm{d}_\varepsilon\|_{\mathrm{L}^2_t\mathrm{L}^2}.\end{aligned}$$

Absorbing the last right-hand side term, we conclude

$$\|\partial_t \mathrm{p}_\varepsilon\|_{\mathrm{L}^2_t \mathrm{L}^2} \lesssim \|\partial_t \mathrm{d}_\varepsilon\|_{\mathrm{L}^2_t \mathrm{L}^2} \lesssim_t 1. \tag{3.6}$$

All the stated estimates follow.

Step 2: estimates for transport-diffusion equations with large diffusivity. In the incompressible regimes (GL_1) and (GL_2'), the conclusion does not follow as in Step 1, because the corresponding choice $\mathrm{p}_\varepsilon = (\lambda_\varepsilon \alpha a)^{-1} \operatorname{div}(a\mathrm{v}_\varepsilon)$ contains the prefactor $(\lambda_\varepsilon\alpha)^{-1} \gg 1$. In particular, the equation (3.4) for the divergence $\mathrm{d}_\varepsilon := \operatorname{div}(a\mathrm{v}_\varepsilon)$ now takes the form

$$\begin{aligned}\partial_t \mathrm{d}_\varepsilon - (\lambda_\varepsilon\alpha)^{-1}\triangle \mathrm{d}_\varepsilon + \alpha^{-1}\operatorname{div}(\mathrm{d}_\varepsilon \nabla \hat h) &= \operatorname{div}(a\Gamma_\varepsilon \mathrm{m}_\varepsilon) \\ &= \operatorname{div}\bigg((\alpha - \mathbb{J}\beta)\Big(\nabla^\perp \hat h - \hat F^\perp - \frac{2\lambda_\varepsilon^{-1} N_\varepsilon}{|\log\varepsilon|}\,\mathrm{v}_\varepsilon\Big) a\mathrm{m}_\varepsilon\bigg),\end{aligned} \tag{3.7}$$

with a large prefactor $(\lambda_\varepsilon\alpha)^{-1} \gg 1$ in front of the Laplacian, and with initial data $\mathrm{d}_\varepsilon^\circ := \operatorname{div}(a\mathrm{v}_\varepsilon^\circ) = 0$. In this step, we consider the model transport-diffusion equation

$$\partial_t w - \nu\triangle w + \operatorname{div}(w\nabla\hat h) = \operatorname{div} g, \qquad w|_{t=0} = 0,$$

with large diffusivity $\nu \gg 1$. A direct adaptation of [37, Lemma 2.3] gives the following bounds, for any $\nu \gtrsim 1$, using that the initial condition is chosen to be zero,

(a) for all $s \ge 0$, $t \ge 0$, for some constant C depending only on an upper bound on s and $\|\nabla h\|_{W^{s,\infty}}$,

$$\|w^t\|_{H^s} + \nu^{1/2}\|\nabla w\|_{\mathrm{L}^2_t H^s} \le C(t/\nu)^{1/2} e^{Ct/\nu}\|g\|_{\mathrm{L}^\infty_t H^s} \le Ct^{1/2}e^{Ct}\|g\|_{\mathrm{L}^\infty_t H^s};$$

(b) for some constant C depending only on an upper bound on $\|\nabla h\|_{\mathrm{L}^\infty}$,

$$\|w^t\|_{\dot H^{-1}} \le Ce^{Ct}\|g\|_{\mathrm{L}^2_t \mathrm{L}^2};$$

(c) for all $1 \le p, q \le \infty$, $t \ge 0$, for some constant C depending only on an upper bound on $\|\nabla h\|_{\mathrm{L}^\infty}$,

$$\|w\|_{\mathrm{L}^p_t \mathrm{L}^q} \le C(t/\nu)^{1/2} e^{C(t/\nu)^2}\|g\|_{\mathrm{L}^p_t \mathrm{L}^q} \le Ct^{1/2}e^{Ct^2}\|g\|_{\mathrm{L}^p_t \mathrm{L}^q}.$$

In particular, the same bounds as in [37, Lemma 2.3] hold uniformly with respect to the large diffusivity $\nu \gg 1$. Further adapting the proof of (3.6) in Step 1, we easily obtain

(d) for some constant C depending only on an upper bound on $\|\nabla h\|_{W^{1,\infty}}$,

$$\|\partial_t w\|_{\mathrm{L}^2_t \mathrm{L}^2} \le Ct^{1/2}e^{Ct}\|g\|_{\mathrm{L}^\infty_t H^1}.$$

Step 3: incompressible regimes. In the vorticity formulation (3.4), the large prefactor $(\lambda_\varepsilon\alpha)^{-1} \gg 1$ does not affect the equation for the vorticity m_ε, but only the equation for the divergence d_ε, which now takes the form (3.7). However, for the choice $\mathrm{d}_\varepsilon^\circ = 0$, the result of Step 2 ensures that the estimates for d_ε used in [37] hold uniformly with respect the large prefactor. Hence, as in Step 1, using the assumptions on the initial data, the results in [37] imply that in the incompressible regimes (GL_1) and (GL_2') there exists a unique (local) solution $\mathrm{v}_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T); \mathrm{v}^\circ + H^2 \cap W^{2,\infty}(\mathbb{R}^2)^2)$ of (3.2) on $[0,T) \times \mathbb{R}^2$ with initial data v°, for some $T \gtrsim 1$. Moreover, it is shown in [37] that this solution satisfies for all $t \in [0,T)$,

$$\|\mathrm{v}_\varepsilon^t - \mathrm{v}_\varepsilon^\circ\|_{H^1\cap W^{1,\infty}} \lesssim_t 1, \quad \|\mathrm{m}_\varepsilon^t\|_{H^1\cap W^{1,\infty}} \lesssim_t 1, \quad \|\mathrm{d}_\varepsilon^t\|_{\mathrm{L}^2\cap\mathrm{L}^\infty} \lesssim_t 1, \quad \int \mathrm{m}_\varepsilon^t = 1, \quad \mathrm{m}_\varepsilon^t \ge 0. \tag{3.8}$$

Note that in the parabolic case $\beta = 0$ the results in [37] actually give a global solution, that is, $T = \infty$. We claim that all the desired properties of v_ε follow from (3.8). By definition (3.2), we find $\|\Gamma_\varepsilon^t\|_{W^{1,\infty}} \lesssim_t 1$. Combining (3.5) with the assumption that $\mathrm{v}_\varepsilon^\circ$ is bounded in $W^{1,q}(\mathbb{R}^2)^2$ for all $q > 2$, we obtain

$$\|(\mathrm{v}_\varepsilon^t, \nabla \mathrm{v}_\varepsilon^t)\|_{(\mathrm{L}^2 + \mathrm{L}^q) \cap \mathrm{L}^\infty} \lesssim_{t,q} 1.$$

Using (3.2) in the form $\mathrm{p}_\varepsilon = (\lambda_\varepsilon \alpha a)^{-1} \, \mathrm{d}_\varepsilon$, and applying items (a)–(c) of Step 2, we find

$$\|\mathrm{p}_\varepsilon^t\|_{H^1 \cap W^{1,\infty}} \lesssim \lambda_\varepsilon^{-1} \|\mathrm{d}_\varepsilon^t\|_{H^1 \cap W^{1,\infty}} \lesssim_t \lambda_\varepsilon^{-1/2} \|a \Gamma_\varepsilon \mathrm{m}_\varepsilon\|_{\mathrm{L}_t^\infty(H^1 \cap W^{1,\infty})} \lesssim_t \lambda_\varepsilon^{-1/2},$$

where the last bound follows from (3.8). Similarly, item (a) of Step 2 yields

$$\|\nabla \mathrm{p}_\varepsilon^t\|_{\mathrm{L}_t^2 \mathrm{L}^2} \lesssim \lambda_\varepsilon^{-1} \|\nabla \mathrm{d}_\varepsilon^t\|_{\mathrm{L}_t^2 \mathrm{L}^2} \lesssim_t \|a \Gamma_\varepsilon \mathrm{m}_\varepsilon\|_{\mathrm{L}_t^\infty \mathrm{L}^2} \lesssim_t 1.$$

inserting this information into (3.2), we deduce

$$\|\partial_t \mathrm{v}_\varepsilon^t\|_{\mathrm{L}^2 \cap \mathrm{L}^\infty} \lesssim \|\nabla \mathrm{p}_\varepsilon^t\|_{\mathrm{L}^2 \cap \mathrm{L}^\infty} + \|\Gamma_\varepsilon^t\|_{\mathrm{L}^\infty} \|\mathrm{m}_\varepsilon^t\|_{\mathrm{L}^2 \cap \mathrm{L}^\infty} \lesssim_t \lambda_\varepsilon^{-1/2},$$

and similarly

$$\|\partial_t \mathrm{v}_\varepsilon\|_{\mathrm{L}_t^2 \mathrm{L}^2} \lesssim \|\nabla \mathrm{p}_\varepsilon\|_{\mathrm{L}_t^2 \mathrm{L}^2} + \|\Gamma_\varepsilon\|_{\mathrm{L}_t^\infty \mathrm{L}^\infty} \|\mathrm{m}_\varepsilon\|_{\mathrm{L}_t^2 \mathrm{L}^2} \lesssim_t 1.$$

Finally, item (d) of Step 2 yields

$$\|\partial_t \mathrm{p}_\varepsilon\|_{\mathrm{L}_t^2 \mathrm{L}^2} \lesssim \lambda_\varepsilon^{-1} \|\partial_t \mathrm{d}_\varepsilon\|_{\mathrm{L}_t^2 \mathrm{L}^2} \lesssim_t \lambda_\varepsilon^{-1} \|a \Gamma_\varepsilon \mathrm{m}_\varepsilon\|_{\mathrm{L}_t^\infty H^1} \lesssim_t \lambda_\varepsilon^{-1}.$$

All the stated estimates follow.

Step 4: global existence in the (mixed-flow) incompressible regimes. Using [37, Lemma 4.1(iii)], we find

$$\|\mathrm{v}_\varepsilon^t - \mathrm{v}_\varepsilon^\circ\|_{\mathrm{L}^2} \lesssim_t 1.$$

Arguing as in [37, Step 1 of the proof of Lemma 4.5], using the above estimate, as well as $\int |\mathrm{m}_\varepsilon^t| = 1$ for all t, we easily obtain

$$\|\mathrm{v}_\varepsilon^t\|_{\mathrm{L}^\infty} \lesssim_t 1 + \|\mathrm{m}_\varepsilon^t\|_{\mathrm{L}^\infty}^{1/2} \log^{1/2}(2 + \|\mathrm{m}_\varepsilon^t\|_{\mathrm{L}^\infty}) + \|\mathrm{div}\,(\mathrm{v}_\varepsilon^t - \mathrm{v}_\varepsilon^\circ)\|_{\mathrm{L}^2} \log^{1/2}(2 + \|\mathrm{div}\,(\mathrm{v}_\varepsilon^t - \mathrm{v}_\varepsilon^\circ)\|_{\mathrm{L}^2 \cap \mathrm{L}^\infty}). \quad (3.9)$$

On the other hand, item (a) of Step 2 yields

$$\begin{aligned}\|\mathrm{d}_\varepsilon^t\|_{\mathrm{L}^2} \lesssim_t \lambda_\varepsilon^{1/2} \|a \Gamma_\varepsilon \mathrm{m}_\varepsilon\|_{\mathrm{L}_t^\infty \mathrm{L}^2} &\lesssim_t \lambda_\varepsilon^{1/2} \|\mathrm{v}_\varepsilon - \mathrm{v}^\circ\|_{\mathrm{L}_t^\infty \mathrm{L}^2} \|\mathrm{m}_\varepsilon\|_{\mathrm{L}_t^\infty \mathrm{L}^\infty} + \lambda_\varepsilon^{1/2} \|\mathrm{m}_\varepsilon\|_{\mathrm{L}_t^\infty \mathrm{L}^2} \\ &\lesssim_t \lambda_\varepsilon^{1/2} \|\mathrm{m}_\varepsilon\|_{\mathrm{L}_t^\infty \mathrm{L}^\infty} + \lambda_\varepsilon^{1/2} \|\mathrm{m}_\varepsilon\|_{\mathrm{L}_t^\infty \mathrm{L}^\infty}^{1/2},\end{aligned}$$

hence, in terms of $\mathrm{div}\,(\mathrm{v}_\varepsilon - \mathrm{v}_\varepsilon^\circ) = a^{-1} \mathrm{d}_\varepsilon - \nabla h \cdot (\mathrm{v}_\varepsilon - \mathrm{v}_\varepsilon^\circ)$,

$$\|\mathrm{div}\,(\mathrm{v}_\varepsilon^t - \mathrm{v}_\varepsilon^\circ)\|_{\mathrm{L}^2} \lesssim_t \lambda_\varepsilon^{1/2} (1 + \|\mathrm{m}_\varepsilon\|_{\mathrm{L}_t^\infty \mathrm{L}^\infty}).$$

inserting this into (3.9), we find

$$\|\mathrm{v}_\varepsilon^t\|_{\mathrm{L}^\infty} \lesssim_t (1 + \|\mathrm{m}_\varepsilon\|_{\mathrm{L}_t^\infty \mathrm{L}^\infty}) \log^{1/2}(2 + \|\mathrm{m}_\varepsilon\|_{\mathrm{L}_t^\infty \mathrm{L}^\infty} + \|\mathrm{div}\ \mathrm{v}_\varepsilon^t\|_{\mathrm{L}^\infty}). \quad (3.10)$$

Item (c) of Step 2 yields

$$\|\mathrm{d}_\varepsilon^t\|_{\mathrm{L}^\infty} \lesssim_t \lambda_\varepsilon^{1/2} \|a \Gamma_\varepsilon \mathrm{m}_\varepsilon\|_{\mathrm{L}_t^\infty \mathrm{L}^\infty} \lesssim \lambda_\varepsilon^{1/2} (1 + \|\mathrm{v}_\varepsilon\|_{\mathrm{L}_t^\infty \mathrm{L}^\infty}) \|\mathrm{m}_\varepsilon\|_{\mathrm{L}_t^\infty \mathrm{L}^\infty},$$

or alternatively, for $\operatorname{div} \mathrm{v}_\varepsilon = a^{-1}\mathrm{d}_\varepsilon - \nabla h\cdot \mathrm{v}_\varepsilon$,

$$\|\operatorname{div}\,\mathrm{v}_\varepsilon^t\|_{\mathrm{L}^\infty} \lesssim_t \lambda_\varepsilon^{1/2}(1+\|\mathrm{v}_\varepsilon\|_{\mathrm{L}^\infty_t\mathrm{L}^\infty})(1+\|\mathrm{m}_\varepsilon\|_{\mathrm{L}^\infty_t\mathrm{L}^\infty}).$$

Combining with (3.10) yields

$$\|\operatorname{div}\,\mathrm{v}_\varepsilon^t\|_{\mathrm{L}^\infty} \lesssim_t \lambda_\varepsilon^{1/2}(1+\|\mathrm{m}_\varepsilon\|^2_{\mathrm{L}^\infty_t\mathrm{L}^\infty})\log^{1/2}(2+\|\mathrm{m}_\varepsilon\|_{\mathrm{L}^\infty_t\mathrm{L}^\infty}+\|\operatorname{div}\,\mathrm{v}_\varepsilon^t\|_{\mathrm{L}^\infty}),$$

and hence, using $\lambda_\varepsilon \ll 1$ and the inequality $a\log b \le b + a\log a$ for $a,b\ge 0$, in order to absorb the term $\|\operatorname{div}\,\mathrm{v}_\varepsilon^t\|_{\mathrm{L}^\infty}$ appearing in the right-hand side,

$$\|\operatorname{div}\,\mathrm{v}_\varepsilon^t\|_{\mathrm{L}^\infty} \lesssim_t \lambda_\varepsilon^{1/2}(1+\|\mathrm{m}_\varepsilon\|^2_{\mathrm{L}^\infty_t\mathrm{L}^\infty})\log(2+\|\mathrm{m}_\varepsilon\|_{\mathrm{L}^\infty_t\mathrm{L}^\infty}),$$

so that (3.10) finally takes the form

$$\|\mathrm{v}_\varepsilon^t\|_{\mathrm{L}^\infty} \lesssim_t (1+\|\mathrm{m}_\varepsilon\|_{\mathrm{L}^\infty_t\mathrm{L}^\infty})\log^{1/2}(2+\|\mathrm{m}_\varepsilon\|_{\mathrm{L}^\infty_t\mathrm{L}^\infty}).$$

In particular, we have proved the following estimates,

$$\|\mathrm{v}_\varepsilon^t\|_{\mathrm{L}^\infty} \lesssim_t (1+\|\mathrm{m}_\varepsilon\|^2_{\mathrm{L}^\infty_t\mathrm{L}^\infty}),\qquad\text{and}\qquad \|\mathrm{d}_\varepsilon^t\|_{\mathrm{L}^\infty} \lesssim_t \lambda_\varepsilon^{1/2}(1+\|\mathrm{m}_\varepsilon\|^3_{\mathrm{L}^\infty_t\mathrm{L}^\infty}).$$

The result in [37, Lemma 4.3(i)] then gives the following bound on the vorticity m_ε,

$$\|\mathrm{m}_\varepsilon^t\|_{\mathrm{L}^\infty} \lesssim \exp\Big[Ct\big(1+\|\mathrm{d}_\varepsilon\|_{\mathrm{L}^\infty_t\mathrm{L}^\infty}+\lambda_\varepsilon\|\mathrm{v}_\varepsilon\|_{\mathrm{L}^\infty_t\mathrm{L}^\infty}\big)\Big] \lesssim \exp\big[Ct\lambda_\varepsilon^{1/2}(1+\|\mathrm{m}_\varepsilon\|^3_{\mathrm{L}^\infty_t\mathrm{L}^\infty})\big].$$

As $\lambda_\varepsilon \ll 1$, this bound easily implies that for all $T>0$ there exists some $\varepsilon_0(T)$ such that for all $0<\varepsilon<\varepsilon_0(T)$ the vorticity m_ε^t (if it exists) remains bounded in $\mathrm{L}^\infty(\mathbb{R}^2)$ for all $t\in[0,T]$. Then repeating the arguments in [37, Sections 4.2–4.3], this a priori bound on the vorticity allows to deduce existence and uniqueness of a solution on the whole time interval $[0,T]$.

Step 5: global existence in the (mixed-flow) compressible regime (GL_1'). Just as in (3.9), we obtain the bounds $\|\mathrm{v}_\varepsilon^t-\mathrm{v}_\varepsilon^\circ\|_{\mathrm{L}^2}\lesssim_t 1$ and

$$\|\mathrm{v}_\varepsilon^t\|_{\mathrm{L}^\infty} \lesssim_t 1+\|\mathrm{m}_\varepsilon^t\|^{1/2}_{\mathrm{L}^\infty}\log^{1/2}(2+\|\mathrm{m}_\varepsilon^t\|_{\mathrm{L}^\infty})+\|\operatorname{div}(\mathrm{v}_\varepsilon^t-\mathrm{v}_\varepsilon^\circ)\|_{\mathrm{L}^2}\log^{1/2}(2+\|\operatorname{div}(\mathrm{v}_\varepsilon^t-\mathrm{v}_\varepsilon^\circ)\|_{\mathrm{L}^2\cap\mathrm{L}^\infty}).\quad(3.11)$$

On the other hand, considering the equation (3.4) satisfied by d_ε, the a priori estimates in [37, Lemma 2.3] yield

$$\|\mathrm{d}_\varepsilon^t\|_{\mathrm{L}^2} \lesssim_t 1+\|a\Gamma_\varepsilon\mathrm{m}_\varepsilon\|_{\mathrm{L}^\infty_t\mathrm{L}^2} \lesssim_t 1+\|\mathrm{m}_\varepsilon\|_{\mathrm{L}^\infty_t\mathrm{L}^2}+\|\mathrm{m}_\varepsilon\|_{\mathrm{L}^\infty_t\mathrm{L}^\infty}\|\mathrm{v}_\varepsilon-\mathrm{v}_\varepsilon^\circ\|_{\mathrm{L}^\infty_t\mathrm{L}^2} \lesssim_t 1+\|\mathrm{m}_\varepsilon\|_{\mathrm{L}^\infty_t\mathrm{L}^\infty},$$

and also

$$\|\mathrm{d}_\varepsilon^t\|_{\mathrm{L}^\infty} \lesssim_t 1+\|a\Gamma_\varepsilon\mathrm{m}_\varepsilon\|_{\mathrm{L}^\infty_t\mathrm{L}^\infty} \lesssim_t 1+\|\mathrm{m}_\varepsilon\|_{\mathrm{L}^\infty_t\mathrm{L}^\infty}(1+\|\mathrm{v}_\varepsilon\|_{\mathrm{L}^\infty_t\mathrm{L}^\infty}).$$

As by definition $\operatorname{div}(\mathrm{v}_\varepsilon^t-\mathrm{v}_\varepsilon^\circ)=a^{-1}(\mathrm{d}_\varepsilon^t-\mathrm{d}_\varepsilon^\circ)-\nabla h\cdot(\mathrm{v}_\varepsilon^t-\mathrm{v}_\varepsilon^\circ)$, the above estimates take the following form,

$$\begin{aligned}\|\operatorname{div}(\mathrm{v}_\varepsilon^t-\mathrm{v}_\varepsilon^\circ)\|_{\mathrm{L}^2} &\lesssim_t 1+\|\mathrm{m}_\varepsilon\|_{\mathrm{L}^\infty_t\mathrm{L}^\infty},\\ \|\operatorname{div}\,\mathrm{v}_\varepsilon^t\|_{\mathrm{L}^\infty} &\lesssim_t (1+\|\mathrm{m}_\varepsilon\|_{\mathrm{L}^\infty_t\mathrm{L}^\infty})(1+\|\mathrm{v}_\varepsilon\|_{\mathrm{L}^\infty_t\mathrm{L}^\infty}).\end{aligned}\quad(3.12)$$

Combining these estimates with (3.11) yields

$$\|\mathrm{v}_\varepsilon^t\|_{\mathrm{L}^\infty} \lesssim_t 1+\|\mathrm{m}_\varepsilon^t\|^{1/2}_{\mathrm{L}^\infty}\log^{1/2}(2+\|\mathrm{m}_\varepsilon^t\|_{\mathrm{L}^\infty})+(1+\|\mathrm{m}_\varepsilon\|_{\mathrm{L}^\infty_t\mathrm{L}^\infty})\log^{1/2}\big((1+\|\mathrm{m}_\varepsilon\|_{\mathrm{L}^\infty_t\mathrm{L}^\infty})(1+\|\mathrm{v}_\varepsilon\|_{\mathrm{L}^\infty_t\mathrm{L}^\infty})\big),$$

and hence, using the inequality $a\log b \le b + a\log a$ for $a, b \ge 0$, in order to absorb the term $\|\mathrm{v}_\varepsilon\|_{\mathrm{L}^\infty_t \mathrm{L}^\infty}$ appearing in the right-hand side,

$$\|\mathrm{v}_\varepsilon\|_{\mathrm{L}^\infty_t \mathrm{L}^\infty} \lesssim_t (1 + \|\mathrm{m}_\varepsilon\|_{\mathrm{L}^\infty_t \mathrm{L}^\infty}) \log(1 + \|\mathrm{m}_\varepsilon\|_{\mathrm{L}^\infty_t \mathrm{L}^\infty}),$$

so that (3.12) finally takes the form,

$$\|\mathrm{div}\ \mathrm{v}_\varepsilon\|_{\mathrm{L}^\infty_t \mathrm{L}^\infty} \lesssim_t (1 + \|\mathrm{m}_\varepsilon\|_{\mathrm{L}^\infty_t \mathrm{L}^\infty})^2 \log(1 + \|\mathrm{m}_\varepsilon\|_{\mathrm{L}^\infty_t \mathrm{L}^\infty}).$$

The result in [37, Lemma 4.3(i)] then gives the following bound on the vorticity m_ε, in the considered regime (GL_1'),

$$\|\mathrm{m}_\varepsilon^t\|_{\mathrm{L}^\infty} \lesssim \exp\left[Ct\Big(1 + \frac{N_\varepsilon}{|\log\varepsilon|}\|(\mathrm{v}_\varepsilon, \mathrm{div}\ \mathrm{v}_\varepsilon)\|_{\mathrm{L}^\infty_t \mathrm{L}^\infty}\Big)\right] \lesssim_t \exp\left(\frac{CtN_\varepsilon}{|\log\varepsilon|}\|\mathrm{m}_\varepsilon\|^3_{\mathrm{L}^\infty_t \mathrm{L}^\infty}\right).$$

As $N_\varepsilon/|\log\varepsilon| \ll 1$, this bound easily implies that for all $T > 0$ there exists some $\varepsilon_0(T)$ such that for all $0 < \varepsilon < \varepsilon_0(T)$ the vorticity m_ε^t (if it exists) remains bounded in $\mathrm{L}^\infty(\mathbb{R}^2)$ for all $t \in [0, T]$. Then repeating the arguments in [37, Sections 4.2–4.3], existence and uniqueness of a solution on the whole time interval $[0, T]$ follows from this a priori bound. ☐

3.1.2 Passing to the limit in (3.2)

We now show how to pass to the limit in (3.2) as $\varepsilon \to 0$, which is easily achieved e.g. by a Grönwall type argument for the L^2-distance between v_ε and its limit.

Lemma 3.2. *Let $\alpha > 0$, $\beta \in \mathbb{R}$, let $h : \mathbb{R}^2 \to \mathbb{R}$, $a := e^h$, $F : \mathbb{R}^2 \to \mathbb{R}^2$, let $\mathrm{v}_\varepsilon : [0, T) \times \mathbb{R}^2 \to \mathbb{R}^2$ be a solution of (3.2) as in Proposition 3.1, for some $T > 0$, and assume that $\mathrm{v}_\varepsilon^\circ \to \mathrm{v}^\circ$ in $\mathrm{L}^2_{\mathrm{uloc}}(\mathbb{R}^2)^2$. Then,*

(i) in the regime (GL_1), we have $\mathrm{v}_\varepsilon \to \mathrm{v}$ in $\mathrm{L}^\infty_{\mathrm{loc}}([0, T); \mathrm{L}^2_{\mathrm{uloc}}(\mathbb{R}^2)^2)$ as $\varepsilon \downarrow 0$, where v is the unique solution (in the space $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{v}^\circ + \mathrm{L}^2(\mathbb{R}^2)^2)$ with $\mathrm{curl}\ \mathrm{v} \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^1 \cap \mathrm{L}^\infty(\mathbb{R}^2))$) of

$$\partial_t \mathrm{v} = \nabla \mathrm{p} + (\alpha - \mathbb{J}\beta)(\nabla^\perp \hat{h} - \hat{F}^\perp - 2\mathrm{v})\mathrm{curl}\ \mathrm{v}, \qquad \mathrm{div}\ \mathrm{v} = 0, \qquad \mathrm{v}|_{t=0} = \mathrm{v}^\circ; \tag{3.13}$$

(ii) in the regime (GL_2), with $N_\varepsilon/|\log\varepsilon| \to \lambda \in (0, \infty)$ and $\mathrm{v}_\varepsilon^\circ = \mathrm{v}^\circ$, we have $\mathrm{v}_\varepsilon \to \mathrm{v}$ in $\mathrm{L}^\infty_{\mathrm{loc}}([0, T); \mathrm{L}^2(\mathbb{R}^2)^2)$ as $\varepsilon \downarrow 0$, where v is the unique solution (in the space $\mathrm{L}^\infty_{\mathrm{loc}}([0, T); \mathrm{v}^\circ + \mathrm{L}^2 \cap W^{1,\infty}(\mathbb{R}^2)^2)$ with $\mathrm{curl}\ \mathrm{v} \in \mathrm{L}^\infty_{\mathrm{loc}}([0, T); \mathrm{L}^1(\mathbb{R}^2))$ and $\mathrm{div}\,(\hat{a}\mathrm{v}) \in \mathrm{L}^\infty_{\mathrm{loc}}([0, T); \mathrm{L}^2(\mathbb{R}^2))$) of

$$\partial_t \mathrm{v} = \alpha^{-1}\nabla(\hat{a}^{-1}\,\mathrm{div}\,(\hat{a}\mathrm{v})) + (\alpha - \mathbb{J}\beta)(\nabla^\perp \hat{h} - \hat{F}^\perp - 2\lambda\mathrm{v})\mathrm{curl}\ \mathrm{v}, \qquad \mathrm{v}|_{t=0} = \mathrm{v}^\circ; \tag{3.14}$$

(iii) in the regime (GL_1'), with $\mathrm{v}_\varepsilon^\circ = \mathrm{v}^\circ$, we have $\mathrm{v}_\varepsilon \to \mathrm{v}$ in $\mathrm{L}^\infty_{\mathrm{loc}}([0, T); \mathrm{L}^2(\mathbb{R}^2)^2)$ as $\varepsilon \downarrow 0$, where v is the unique solution (in the space $\mathrm{L}^\infty_{\mathrm{loc}}([0, T); \mathrm{v}^\circ + \mathrm{L}^2(\mathbb{R}^2)^2)$ with $\mathrm{curl}\ \mathrm{v} \in \mathrm{L}^\infty_{\mathrm{loc}}([0, T); \mathrm{L}^1(\mathbb{R}^2))$ and $\mathrm{div}\,(\hat{a}\mathrm{v}) \in \mathrm{L}^\infty_{\mathrm{loc}}([0, T); \mathrm{L}^2(\mathbb{R}^2))$) of

$$\partial_t \mathrm{v} = \alpha^{-1}\nabla(\hat{a}^{-1}\,\mathrm{div}\,(\hat{a}\mathrm{v})) + (\alpha - \mathbb{J}\beta)(\nabla^\perp \hat{h} - \hat{F}^\perp)\mathrm{curl}\ \mathrm{v}, \qquad \mathrm{v}|_{t=0} = \mathrm{v}^\circ; \tag{3.15}$$

(iv) in the regime (GL_2'), we have $\mathrm{v}_\varepsilon \to \mathrm{v}$ in $\mathrm{L}^\infty_{\mathrm{loc}}([0, T); \mathrm{L}^2_{\mathrm{uloc}}(\mathbb{R}^2)^2)$ as $\varepsilon \downarrow 0$, where v is the unique solution (in the space $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{v}^\circ + \mathrm{L}^2(\mathbb{R}^2)^2)$ with $\mathrm{curl}\ \mathrm{v} \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^1 \cap \mathrm{L}^\infty(\mathbb{R}^2))$) of

$$\partial_t \mathrm{v} = \nabla \mathrm{p} + (\alpha - \mathbb{J}\beta)(\nabla^\perp \hat{h} - \hat{F}^\perp)\mathrm{curl}\ \mathrm{v}, \qquad \mathrm{div}\ \mathrm{v} = 0, \qquad \mathrm{v}|_{t=0} = \mathrm{v}^\circ\,. \tag{3.16}$$

Proof. We treat each of the four regimes separately. For $R \geq 1$, we denote by $\xi_R^z(x) := e^{-|x-z|/R}$ the exponential cut-off at the scale R centered at $z \in R\mathbb{Z}^2$.

Step 1: regime (GL_1). Using the choice of the scalings for $\lambda_\varepsilon, h, F$ in the regime (GL_1), equation (3.2) takes the following form, with $\lambda_\varepsilon = N_\varepsilon/|\log \varepsilon| \ll 1$, and explicitly setting $a_\varepsilon := \hat{a}^{\lambda_\varepsilon}$,

$$\partial_t \mathrm{v}_\varepsilon = \nabla \mathrm{p}_\varepsilon + (\alpha - \mathbb{J}\beta)(\nabla^\perp \hat{h} - \hat{F}^\perp - 2\mathrm{v}_\varepsilon)\mathrm{curl}\ \mathrm{v}_\varepsilon, \qquad \mathrm{p}_\varepsilon := (\lambda_\varepsilon \alpha a_\varepsilon)^{-1} \operatorname{div}(a_\varepsilon \mathrm{v}_\varepsilon),$$

with initial data $\mathrm{v}_\varepsilon|_{t=0} = \mathrm{v}_\varepsilon^\circ \to \mathrm{v}^\circ$ in $\mathrm{L}^2_{\mathrm{uloc}}(\mathbb{R}^2)^2$. As $\lambda_\varepsilon \to 0$, it is then formally clear from the vorticity formulation of this equation that v_ε should converge to the solution v of (3.13).

The existence and uniqueness of a solution $\mathrm{v} \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{v}^\circ + \mathrm{L}^2(\mathbb{R}^2)^2)$ of (3.13) with curl $\mathrm{v} \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^1 \cap \mathrm{L}^\infty(\mathbb{R}^2))$ are proved in [37]. Moreover, the following estimates hold for all $t \geq 0$ and $R, \theta > 0$,

$$\|\mathrm{v}^t\|_{W^{1,\infty}} \lesssim_t 1, \qquad \|(\mathrm{v}^t, \mathrm{p}^t)\|_{\mathrm{L}^2(B_R)} \lesssim_{t,\theta} R^\theta. \tag{3.17}$$

The above bounds for v follow from the results in [37] with $\mathrm{v}^\circ \in W^{s+1,\infty}(\mathbb{R}^2)^2$ for some $s > 0$, and with $\mathrm{v}^\circ \in \mathrm{L}^q(\mathbb{R}^2)^2$ for all $q > 2$. It remains to prove the bound on the pressure p. Taking the divergence of both sides of equation (3.13), we obtain the following equation for the pressure p^t, for all $t \geq 0$,

$$-\triangle \mathrm{p}^t = \operatorname{div}\big((\alpha - \mathbb{J}\beta)(\nabla^\perp \hat{h} - \hat{F}^\perp - 2\mathrm{v}^t)\mathrm{curl}\ \mathrm{v}^t\big).$$

By Riesz potential theory, we deduce for all $2 < q < \infty$,

$$\|\mathrm{p}^t\|_{\mathrm{L}^q} \lesssim_q (1 + \|\mathrm{v}^t\|_{\mathrm{L}^\infty})\|\mathrm{curl}\ \mathrm{v}^t\|_{\mathrm{L}^{2q/(2+q)}} \lesssim_t 1,$$

and the bound on the pressure p^t in (3.17) follows.

Now we turn to the Grönwall argument for proving the convergence $\mathrm{v}_\varepsilon \to \mathrm{v}$ in $\mathrm{L}^\infty_{\mathrm{loc}}([0,T); \mathrm{L}^2_{\mathrm{uloc}}(\mathbb{R}^2)^2)$. Using the equations for $\mathrm{v}_\varepsilon, \mathrm{v}$, we find

$$\begin{aligned}\partial_t \int a_\varepsilon \xi_R^z |\mathrm{v}_\varepsilon - \mathrm{v}|^2 = 2\int a_\varepsilon \xi_R^z (\mathrm{v}_\varepsilon - \mathrm{v}) \cdot \nabla(\mathrm{p}_\varepsilon - \mathrm{p}) - 4\alpha \int a_\varepsilon \xi_R^z |\mathrm{v}_\varepsilon - \mathrm{v}|^2 \mathrm{curl}\ \mathrm{v}_\varepsilon \\ + 2\int a_\varepsilon \xi_R^z (\alpha - \mathbb{J}\beta)(\nabla^\perp \hat{h} - \hat{F}^\perp - 2\mathrm{v}) \cdot (\mathrm{v}_\varepsilon - \mathrm{v})\mathrm{curl}\,(\mathrm{v}_\varepsilon - \mathrm{v}).\end{aligned} \tag{3.18}$$

Integrating by parts in the first term, decomposing

$$\operatorname{div}(a_\varepsilon \xi_R^z (\mathrm{v}_\varepsilon - \mathrm{v})) = a_\varepsilon \nabla \xi_R^z \cdot (\mathrm{v}_\varepsilon - \mathrm{v}) + \lambda_\varepsilon \alpha a_\varepsilon \xi_R^z \mathrm{p}_\varepsilon - \lambda_\varepsilon a_\varepsilon \xi_R^z \nabla \hat{h} \cdot \mathrm{v},$$

noting that the second right-hand side term in (3.18) is nonpositive, and using the following weighted Delort-type identity (as e.g. in [37])

$$\begin{aligned}(\mathrm{v}_\varepsilon - \mathrm{v})\mathrm{curl}\,(\mathrm{v}_\varepsilon - \mathrm{v}) &= a_\varepsilon^{-1}(\mathrm{v}_\varepsilon - \mathrm{v})^\perp \operatorname{div}(a_\varepsilon(\mathrm{v}_\varepsilon - \mathrm{v})) - \frac{1}{2} a_\varepsilon^{-1}|\mathrm{v}_\varepsilon - \mathrm{v}|^2 \nabla^\perp a_\varepsilon - a_\varepsilon^{-1}(\operatorname{div}(a_\varepsilon S_{\mathrm{v}_\varepsilon - \mathrm{v}}))^\perp \\ &= \lambda_\varepsilon \alpha \mathrm{p}_\varepsilon (\mathrm{v}_\varepsilon - \mathrm{v})^\perp - \lambda_\varepsilon (\nabla \hat{h} \cdot \mathrm{v})(\mathrm{v}_\varepsilon - \mathrm{v})^\perp - \frac{\lambda_\varepsilon}{2}|\mathrm{v}_\varepsilon - \mathrm{v}|^2 \nabla^\perp \hat{h} - a_\varepsilon^{-1}(\operatorname{div}(a_\varepsilon S_{\mathrm{v}_\varepsilon - \mathrm{v}}))^\perp,\end{aligned} \tag{3.19}$$

in terms of the stress-energy tensor $S_w := w \otimes w - \frac{1}{2}|w|^2\,\mathrm{Id}$, we deduce

$$\partial_t \int a_\varepsilon \xi_R^z |\mathrm{v}_\varepsilon - \mathrm{v}|^2 \le -2\int a_\varepsilon(\mathrm{p}_\varepsilon - \mathrm{p})\nabla\xi_R^z\cdot(\mathrm{v}_\varepsilon - \mathrm{v}) - 2\lambda_\varepsilon\alpha\int a_\varepsilon\xi_R^z\,\mathrm{p}_\varepsilon(\mathrm{p}_\varepsilon - \mathrm{p}) + 2\lambda_\varepsilon\int a_\varepsilon\xi_R^z(\mathrm{p}_\varepsilon - \mathrm{p})\,\mathrm{v}\cdot\nabla\hat h$$
$$+ 2\lambda_\varepsilon\alpha\int a_\varepsilon\xi_R^z\mathrm{p}_\varepsilon(\alpha - \mathbb{J}\beta)(\nabla^\perp\hat h - \hat F^\perp - 2\mathrm{v})\cdot(\mathrm{v}_\varepsilon - \mathrm{v})^\perp$$
$$- 2\lambda_\varepsilon\int a_\varepsilon\xi_R^z(\nabla\hat h\cdot\mathrm{v})(\alpha - \mathbb{J}\beta)(\nabla^\perp\hat h - \hat F^\perp - 2\mathrm{v})\cdot(\mathrm{v}_\varepsilon - \mathrm{v})^\perp$$
$$- \lambda_\varepsilon\int a_\varepsilon\xi_R^z|\mathrm{v}_\varepsilon - \mathrm{v}|^2(\alpha - \mathbb{J}\beta)(\nabla^\perp\hat h - \hat F^\perp - 2\mathrm{v})\cdot\nabla^\perp\hat h$$
$$- 2\int a_\varepsilon S_{\mathrm{v}_\varepsilon - \mathrm{v}} : \nabla\Big(\xi_R^z(\alpha\mathbb{J} + \beta)(\nabla^\perp\hat h - \hat F^\perp - 2\mathrm{v})\Big),$$

and hence, using (3.17) in the form $\|\mathrm{v}^t\|_{W^{1,\infty}} \lesssim 1$, the assumption $\|(\nabla\hat h, \hat F)\|_{W^{1,\infty}} \lesssim 1$, the property $|\nabla\xi_R^z| \lesssim R^{-1}\xi_R^z$ of the exponential cut-off, and the pointwise estimate $|S_w| \lesssim |w|^2$, we obtain

$$\partial_t\int a_\varepsilon\xi_R^z|\mathrm{v}_\varepsilon - \mathrm{v}|^2 \le (R^{-2} - \lambda_\varepsilon\alpha)\int a_\varepsilon\xi_R^z|\mathrm{p}_\varepsilon|^2 + C_t(R^{-2} + \lambda_\varepsilon)\int a_\varepsilon\xi_R^z(|\mathrm{p}|^2 + |\mathrm{v}|^2) + C_t\int a_\varepsilon\xi_R^z|\mathrm{v}_\varepsilon - \mathrm{v}|^2.$$

Choosing $R = \lambda_\varepsilon^{-n}$ for some $n \ge 1$, we obtain $R^{-2} \ll \lambda_\varepsilon$, and hence, for ε small enough, using (3.17) to estimate the second term, we obtain

$$\partial_t\int a_\varepsilon\xi_R^z|\mathrm{v}_\varepsilon - \mathrm{v}|^2 \lesssim_{t,\theta} R^{2\theta}(R^{-2} + \lambda_\varepsilon) + \int a_\varepsilon\xi_R^z|\mathrm{v}_\varepsilon - \mathrm{v}|^2 \lesssim \lambda_\varepsilon^{1-2n\theta} + \int a_\varepsilon\xi_R^z|\mathrm{v}_\varepsilon - \mathrm{v}|^2.$$

For $\theta > 0$ small enough, the conclusion follows from the Grönwall inequality.

Step 2: regime (GL$_2$). Using the choice of the scalings for $\lambda_\varepsilon, h, F$ in the regime (GL$_2$), equation (3.2) takes the following form,

$$\partial_t\mathrm{v}_\varepsilon = \alpha^{-1}\nabla(\hat a^{-1}\,\mathrm{div}\,(\hat a\mathrm{v}_\varepsilon)) + \bigg((\alpha - \mathbb{J}\beta)\Big(\nabla^\perp\hat h - \hat F^\perp - \frac{2N_\varepsilon}{|\log\varepsilon|}\mathrm{v}_\varepsilon\Big)\bigg)\mathrm{curl}\ \mathrm{v}_\varepsilon,$$

with initial data $\mathrm{v}_\varepsilon|_{t=0} = \mathrm{v}^\circ$. As $N_\varepsilon/|\log\varepsilon| \to \lambda \in (0,\infty)$, it is formally clear that v_ε should converge to the solution v of equation (3.14). Note that the existence and uniqueness of the solution v are given by Proposition 3.1 just as for v_ε, and yields in particular the following bounds for all $t \in [0,T)$,

$$\|(\mathrm{v}^t, \mathrm{v}_\varepsilon^t)\|_{W^{1,\infty}} \lesssim_t 1, \qquad \|\mathrm{curl}\ \mathrm{v}^t\|_{\mathrm{L}^1} \lesssim_t 1. \tag{3.20}$$

Using the equations for $\mathrm{v}_\varepsilon, \mathrm{v}$, we find

$$\partial_t\int\hat a\xi_R^z|\mathrm{v}_\varepsilon - \mathrm{v}|^2 = 2\alpha^{-1}\int\hat a\xi_R^z(\mathrm{v}_\varepsilon - \mathrm{v})\cdot\nabla(\hat a^{-1}\,\mathrm{div}\,(\hat a(\mathrm{v}_\varepsilon - \mathrm{v}))) - \frac{4\alpha N_\varepsilon}{|\log\varepsilon|}\int\hat a\xi_R^z|\mathrm{v}_\varepsilon - \mathrm{v}|^2\mathrm{curl}\ \mathrm{v}_\varepsilon$$
$$+ 2\int\hat a\xi_R^z\bigg((\alpha - \mathbb{J}\beta)\Big(\nabla^\perp\hat h - \hat F^\perp - \frac{2N_\varepsilon}{|\log\varepsilon|}\mathrm{v}\Big)\bigg)\cdot(\mathrm{v}_\varepsilon - \mathrm{v})(\mathrm{curl}\ \mathrm{v}_\varepsilon - \mathrm{curl}\ \mathrm{v})$$
$$- 4\Big(\frac{N_\varepsilon}{|\log\varepsilon|} - \lambda\Big)\int\hat a\xi_R^z(\mathrm{v}_\varepsilon - \mathrm{v})\cdot(\alpha - \mathbb{J}\beta)\,\mathrm{v}\,\mathrm{curl}\ \mathrm{v}\,.$$

Integrating by parts, using the weighted Delort-type identity (3.19) in the form

$$(\mathrm{v}_\varepsilon - \mathrm{v})\mathrm{curl}\,(\mathrm{v}_\varepsilon - \mathrm{v}) = \hat a^{-1}(\mathrm{v}_\varepsilon - \mathrm{v})^\perp\,\mathrm{div}\,(\hat a(\mathrm{v}_\varepsilon - \mathrm{v})) - \frac{1}{2}|\mathrm{v}_\varepsilon - \mathrm{v}|^2\nabla^\perp\hat h - \hat a^{-1}(\mathrm{div}\,(\hat a S_{\mathrm{v}_\varepsilon - \mathrm{v}}))^\perp,$$

using the properties (3.20) of $\mathrm{v}, \mathrm{v}_\varepsilon$, and the assumption $\|(\nabla\hat h, \hat F)\|_{W^{1,\infty}} \lesssim 1$, and simplifying the terms as in Step 1, we easily obtain

$$\partial_t \int \hat a \xi_R^z |\mathrm{v}_\varepsilon - \mathrm{v}|^2 \le -2\alpha^{-1} \int \hat a^{-1} \xi_R^z |\operatorname{div}(\hat a(\mathrm{v}_\varepsilon - \mathrm{v}))|^2 \\ + C_t \int \xi_R^z |\mathrm{v}_\varepsilon - \mathrm{v}| |\operatorname{div}(\hat a(\mathrm{v}_\varepsilon - \mathrm{v}))| + C_t \int \hat a \xi_R^z |\mathrm{v}_\varepsilon - \mathrm{v}|^2 + C_t \Big| \frac{N_\varepsilon}{|\log\varepsilon|} - \lambda \Big|,$$

hence $\partial_t \int \hat a \xi_R^z |\mathrm{v}_\varepsilon - \mathrm{v}|^2 \lesssim C_t \int \hat a \xi_R^z |\mathrm{v}_\varepsilon - \mathrm{v}|^2 + o_t(1)$, and the conclusion now follows from the Grönwall inequality, letting $R \uparrow \infty$.

Step 3: regime (GL_1'). Using the choice of the scalings for $\lambda_\varepsilon, h, F$ in the regime (GL_1'), equation (3.2) takes the following form

$$\partial_t \mathrm{v}_\varepsilon = \alpha^{-1}\nabla(\hat a^{-1}\operatorname{div}(\hat a \mathrm{v}_\varepsilon)) + (\alpha - \mathbb{J}\beta)\Big(\nabla^\perp \hat h - \hat F^\perp - \frac{2N_\varepsilon}{|\log\varepsilon|}\mathrm{v}_\varepsilon\Big)\operatorname{curl} \mathrm{v}_\varepsilon,$$

with initial data $\mathrm{v}_\varepsilon|_{t=0} = \mathrm{v}^\circ$. As by assumption $N_\varepsilon/|\log\varepsilon| \to 0$, it is formally clear that v_ε should converge to the solution v of equation (3.15) as $\varepsilon \downarrow 0$. Existence, uniqueness and regularity of this solution v are given by Proposition 3.1 just as for v_ε, and the proof of convergence is obtained as in Step 2 (with $\lambda = 0$).

Step 4: regime (GL_2'). Using the choice of the scalings for $\lambda_\varepsilon, h, F$ in the regime (GL_2'), equation (3.2) takes the following form, with $a_\varepsilon := \hat a^{\lambda_\varepsilon}$,

$$\partial_t \mathrm{v}_\varepsilon = \nabla \mathrm{p}_\varepsilon + (\alpha - \mathbb{J}\beta)\Big(\nabla^\perp \hat h - \hat F^\perp - \frac{2\lambda_\varepsilon^{-1} N_\varepsilon}{|\log\varepsilon|}\mathrm{v}_\varepsilon\Big)\operatorname{curl} \mathrm{v}_\varepsilon,$$
$$\mathrm{p}_\varepsilon := (\lambda_\varepsilon \alpha a_\varepsilon)^{-1}\operatorname{div}(a_\varepsilon \mathrm{v}_\varepsilon),$$

with initial data $\mathrm{v}_\varepsilon|_{t=0} = \mathrm{v}_\varepsilon^\circ \to \mathrm{v}^\circ$ in $\mathrm{L}^2_{\mathrm{uloc}}(\mathbb{R}^2)^2$. As by assumption $\lambda_\varepsilon^{-1} N_\varepsilon/|\log\varepsilon| \to 0$, it is formally clear that v_ε should converge to the solution v of equation (3.16) as $\varepsilon \downarrow 0$. Existence, uniqueness and regularity of this solution v are given by [37] as in Step 1, and the proof of convergence then similarly follows. ☐

3.2 Gross-Pitaevskii case

3.2.1 Properties of solutions to (3.3)

Let us examine the vorticity formulation of equation (3.3) for v_ε. Setting $\mathrm{m}_\varepsilon := \operatorname{curl} \mathrm{v}_\varepsilon$, equation (3.3) may be rewritten as a nonlinear nonlocal transport equation for the vorticity m_ε,

$$\begin{cases} \partial_t \mathrm{m}_\varepsilon = -\operatorname{div}(\Gamma_\varepsilon^\perp \mathrm{m}_\varepsilon), \quad \mathrm{m}_\varepsilon|_{t=0} = \operatorname{curl} \mathrm{v}_\varepsilon^\circ, \\ \operatorname{curl} \mathrm{v}_\varepsilon = \mathrm{m}_\varepsilon, \quad \operatorname{div}(a\mathrm{v}_\varepsilon) = 0. \end{cases} \tag{3.21}$$

Given the form of Γ_ε in (3.3), this equation can be seen as an "inhomogeneous" 2D Euler equation with "forcing". A detailed study of this kind of equations is given in the companion paper [37]. The following proposition states in particular that a solution v_ε always exists globally and satisfies the various properties of Assumption B(b), under suitable regularity assumptions on the initial data $\mathrm{v}_\varepsilon^\circ$.

Proposition 3.3. *Let $h : \mathbb{R}^2 \to \mathbb{R}$, $a := e^h$, $F : \mathbb{R}^2 \to \mathbb{R}^2$, and let $\mathrm{v}_\varepsilon^\circ : \mathbb{R}^2 \to \mathbb{R}^2$ be bounded in $W^{1,q}(\mathbb{R}^2)^2$ for all $q > 2$, satisfy $\operatorname{curl} \mathrm{v}_\varepsilon^\circ \in \mathcal{P}(\mathbb{R}^2)$. Assume that $h \in \mathrm{L}^\infty(\mathbb{R}^2)$, $\nabla h, F \in \mathrm{L}^4 \cap W^{2,\infty}(\mathbb{R}^2)^2$, that $a(x) \to 1$ uniformly as $|x| \uparrow \infty$, that $\mathrm{v}_\varepsilon^\circ$ is bounded in $W^{2,\infty}(\mathbb{R}^2)^2$ with $\operatorname{div}(a\mathrm{v}_\varepsilon^\circ) = 0$, and that $\operatorname{curl} \mathrm{v}_\varepsilon^\circ$ is bounded in $H^1(\mathbb{R}^2)$. Let the regime* (GP) *hold.*

Then, there exists a unique (global) solution $\mathrm{v}_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{v}_\varepsilon^\circ + H^2 \cap W^{1,\infty}(\mathbb{R}^2)^2)$ *of* (3.3) *on* $\mathbb{R}^+ \times \mathbb{R}^2$. *Moreover, all the properties of Assumption B(b) are satisfied, that is, for all* $t \geq 0$, *and all* $2 < q < \infty$,

$$\|(\mathrm{v}_\varepsilon^t, \nabla \mathrm{v}_\varepsilon^t)\|_{(\mathrm{L}^2+\mathrm{L}^q)\cap \mathrm{L}^\infty} \lesssim_{t,q} 1, \quad \|\mathrm{curl}\, \mathrm{v}_\varepsilon^t\|_{\mathrm{L}^1\cap\mathrm{L}^\infty} \lesssim_t 1,$$
$$\|\mathrm{p}_\varepsilon^t\|_{\mathrm{L}^q\cap\mathrm{L}^\infty} \lesssim_{t,q} 1, \quad \|\nabla \mathrm{p}_\varepsilon^t\|_{\mathrm{L}^2\cap\mathrm{L}^\infty} \lesssim_t 1, \quad \|\partial_t \mathrm{v}_\varepsilon^t\|_{\mathrm{L}^2} \lesssim_t 1, \quad \|\partial_t \mathrm{p}_\varepsilon^t\|_{\mathrm{L}^q} \lesssim_{t,q} 1.$$

Further, for all $\theta > 0$ *and* $\varrho \geq 1$, *we have for all* $t \geq 0$,

$$\|\nabla(\mathrm{p}_{\varepsilon,\varrho}^t - \mathrm{p}_\varepsilon^t)\|_{\mathrm{L}^2} \lesssim_{\theta,t} \varrho^{\theta-2} + \int_{|x|>\varrho} |\mathrm{curl}\, \mathrm{v}_\varepsilon^\circ|^2. \tag{3.22}$$

Proof. We split the proof into three steps.

Step 1: preliminary. In this step, we prove the following Meyers type elliptic regularity estimate: if $b \in \mathrm{L}^\infty(\mathbb{R}^2)$ satisfies $1/2 \leq b \leq 1$ pointwise, and $b(x) \to 1$ uniformly as $|x| \uparrow \infty$, then for all $g \in \mathrm{L}^1 \cap \mathrm{L}^2(\mathbb{R}^2)^2$ the decaying solution v of equation $-\mathrm{div}\,(b\nabla v) = \mathrm{div}\, g$ satisfies, for all $2 < q < \infty$,

$$\|v\|_{\mathrm{L}^q} \lesssim_q \|g\|_{\mathrm{L}^{2q/(q+2)}\cap\mathrm{L}^2} \lesssim \|g\|_{\mathrm{L}^1\cap\mathrm{L}^2}.$$

Let b be fixed as above. Set $b_r := \chi_r + b(1-\chi_r)$, and decompose the equation for v as follows,

$$-\mathrm{div}\,(b_r\nabla v) = \mathrm{div}\,\big(g + (b-b_r)\nabla v\big).$$

Given $1 < p < 2$, the Meyers perturbative argument [62] gives a value $\kappa_p > 0$ such that, if $\tilde b \in \mathrm{L}^\infty(\mathbb{R}^2)$ satisfies $\kappa_p \leq \tilde b \leq 1$, then for all $k \in \mathrm{L}^1 \cap \mathrm{L}^2(\mathbb{R}^2)^2$ the decaying solution w of equation $-\mathrm{div}\,(\tilde b \nabla w) = \mathrm{div}\, k$ satisfies $\|\nabla w\|_{\mathrm{L}^p} \lesssim_p \|k\|_{\mathrm{L}^p}$. By definition, for r large enough, the truncated coefficient b_r satisfies $\kappa_p \leq b_r \leq 1$, hence

$$\|\nabla v\|_{\mathrm{L}^p} \lesssim_p \|g + (b-b_r)\nabla v\|_{\mathrm{L}^p}.$$

Using the elementary energy estimate $\|\nabla v\|_{\mathrm{L}^2} \lesssim \|g\|_{\mathrm{L}^2}$, and noting that $b_r = b$ on $\mathbb{R}^2 \setminus B_{2r}$, we find by the Hölder inequality,

$$\|\nabla v\|_{\mathrm{L}^p} \lesssim_p \|g\|_{\mathrm{L}^p} + \|\nabla v\|_{\mathrm{L}^p(B_{2r})} \lesssim \|g\|_{\mathrm{L}^p} + r^{2(\frac1p-\frac12)}\|\nabla v\|_{\mathrm{L}^2} \lesssim \|g\|_{\mathrm{L}^p} + r^{2(\frac1p-\frac12)}\|g\|_{\mathrm{L}^2}.$$

On the other hand, rather decomposing the equation for v as follows,

$$-\triangle v = \mathrm{div}\,(g + (b-1)\nabla v),$$

we deduce from Riesz potential theory, with $2 < q := 2p/(2-p) < \infty$,

$$\|v\|_{\mathrm{L}^q} \lesssim_q \|g\|_{\mathrm{L}^p} + \|\nabla v\|_{\mathrm{L}^p}.$$

Combining this with the above, the conclusion follows.

Step 2: proof of Assumption B(b). The assumptions $\|\hat h\|_{W^{3,\infty}}, \|(\nabla \hat h, \hat F)\|_{\mathrm{L}^4\cap W^{2,\infty}} \lesssim 1$ yield $\|\lambda_\varepsilon^{-1}(\nabla^\perp h - F^\perp)\|_{\mathrm{L}^4\cap W^{2,\infty}} \lesssim 1$ in the considered regime, and also $\lambda_\varepsilon^{-1} N_\varepsilon / |\log \varepsilon| = 1$ and $\lambda_\varepsilon^{-1} \lesssim 1$. Further using the assumptions on the initial data v°, the results in [37] imply that there exists a unique (global) solution $\mathrm{v}_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{v}_\varepsilon^\circ + H^2 \cap W^{1,\infty}(\mathbb{R}^2)^2)$ of (3.3) on $\mathbb{R}^+ \times \mathbb{R}^2$ with initial data $\mathrm{v}_\varepsilon^\circ$. Moreover, it is shown in [37] that this solution satisfies in particular, for all $t \geq 0$,

$$\|\mathrm{v}_\varepsilon^t - \mathrm{v}_\varepsilon^\circ\|_{H^1\cap W^{1,\infty}} \lesssim_t 1, \qquad \|\mathrm{m}_\varepsilon^t\|_{H^1\cap\mathrm{L}^\infty} \lesssim_t 1, \qquad \int \mathrm{m}_\varepsilon^t = 1, \qquad \mathrm{m}_\varepsilon^t \geq 0. \tag{3.23}$$

(As such, in order to ensure $\mathrm{v}_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{v}_\varepsilon^\circ + H^2(\mathbb{R}^2)^2)$, the result in [37] would actually further require $h \in W^{s+3,\infty}(\mathbb{R}^2)$ and $\nabla h, F, \mathrm{v}^\circ \in W^{s+2,\infty}(\mathbb{R}^2)^2$ for some $s \in (0,1)$, due to the use of the Sobolev embedding for $H^{s+1}(\mathbb{R}^2)$ into $W^{s,\infty}(\mathbb{R}^2)$ in the proof of [37, Lemma 4.6]. However, this use of the Sobolev embedding is easily replaced by an a priori estimate for v_ε in $W^{s+1,\infty}(\mathbb{R}^2)^2$, for which it is already enough to assume e.g. $h \in W^{3,\infty}(\mathbb{R}^2)$ and $\nabla h, F, \mathrm{v}^\circ \in W^{2,\infty}(\mathbb{R}^2)^2$, as we do here.)

We claim that all the desired properties of v_ε follow from the bounds (3.23). Combining (3.23) with the assumption that $\mathrm{v}_\varepsilon^\circ$ is bounded in $W^{1,q}(\mathbb{R}^2)^2$ for all $q > 2$, we obtain

$$\|(\mathrm{v}_\varepsilon^t, \nabla \mathrm{v}_\varepsilon^t)\|_{(\mathrm{L}^2 + \mathrm{L}^q) \cap \mathrm{L}^\infty} \lesssim_{t,q} 1.$$

Applying the operator $\operatorname{div}(a\cdot)$ to both sides of equation (3.3), we find the following equation for the pressure, in the considered regime (GP),

$$-\operatorname{div}(\hat{a}\nabla \mathrm{p}_\varepsilon^t) = \operatorname{div}(\hat{a}\Gamma_\varepsilon^t \mathrm{m}_\varepsilon^t) = -\operatorname{div}\big(\hat{a}\mathrm{m}_\varepsilon^t(\lambda_\varepsilon^{-1}\nabla^\perp \hat{h} - \hat{F}^\perp - 2\,\mathrm{v}_\varepsilon^t)^\perp\big). \tag{3.24}$$

An energy estimate directly yields

$$\|\nabla \mathrm{p}_\varepsilon^t\|_{\mathrm{L}^2} \lesssim \|\hat{a}\mathrm{m}_\varepsilon^t(\lambda_\varepsilon^{-1}\nabla^\perp \hat{h} - \hat{F}^\perp - 2\,\mathrm{v}_\varepsilon^t)^\perp\|_{\mathrm{L}^2} \lesssim_t 1, \tag{3.25}$$

and similarly, first differentiating both sides of equation (3.24),

$$\|\nabla^2 \mathrm{p}_\varepsilon^t\|_{\mathrm{L}^2} \lesssim \|\nabla \mathrm{p}_\varepsilon^t\|_{\mathrm{L}^2} + \big\|\nabla\big(\hat{a}\mathrm{m}_\varepsilon^t(\lambda_\varepsilon^{-1}\nabla^\perp \hat{h} - \hat{F}^\perp - 2\,\mathrm{v}_\varepsilon^t)^\perp\big)\big\|_{\mathrm{L}^2} \lesssim_t 1. \tag{3.26}$$

Inserting (3.25) into equation (3.3) yields

$$\|\partial_t \mathrm{v}_\varepsilon^t\|_{\mathrm{L}^2} \le \|\nabla \mathrm{p}_\varepsilon^t\|_{\mathrm{L}^2} + \|\Gamma_\varepsilon^t \mathrm{m}_\varepsilon^t\|_{\mathrm{L}^2} \lesssim_t 1.$$

Applying to equation (3.24) the Meyers type result of Step 1, we find for all $2 < q < \infty$,

$$\|\mathrm{p}_\varepsilon^t\|_{\mathrm{L}^q} \lesssim_q \|\hat{a}\mathrm{m}_\varepsilon^t(\lambda_\varepsilon^{-1}\nabla^\perp \hat{h} - \hat{F}^\perp - 2\,\mathrm{v}_\varepsilon^t)^\perp\|_{\mathrm{L}^1 \cap \mathrm{L}^2} \lesssim_t 1.$$

Combining this with (3.26), we deduce from the Sobolev embedding $\|\mathrm{p}_\varepsilon^t\|_{\mathrm{L}^q \cap \mathrm{L}^\infty} \lesssim_{q,t} 1$ for all $q > 2$. First differentiating both sides of equation (3.24) with respect to the time variable, the Meyers type result of Step 1 further yields, for all $2 < q < \infty$,

$$\begin{aligned}
\|\partial_t \mathrm{p}_\varepsilon^t\|_{\mathrm{L}^q} &\lesssim_q \big\|\hat{a}\partial_t\big(\mathrm{m}_\varepsilon^t(\lambda_\varepsilon^{-1}\nabla^\perp \hat{h} - \hat{F}^\perp - 2\,\mathrm{v}_\varepsilon^t)^\perp\big)\big\|_{\mathrm{L}^1 \cap \mathrm{L}^2} \\
&\lesssim \|\mathrm{m}_\varepsilon^t\|_{\mathrm{L}^2 \cap \mathrm{L}^\infty}\|\partial_t \mathrm{v}_\varepsilon^t\|_{\mathrm{L}^2} + \|\Gamma_\varepsilon^t \partial_t \mathrm{m}_\varepsilon^t\|_{\mathrm{L}^1 \cap \mathrm{L}^2} \\
&\lesssim_t 1 + \|\Gamma_\varepsilon^t \partial_t \mathrm{m}_\varepsilon^t\|_{\mathrm{L}^1 \cap \mathrm{L}^2}.
\end{aligned}$$

Using equation (3.21) to express the time-derivative of the vorticity, and using the assumption $\|\lambda_\varepsilon^{-1}\nabla \hat{h} - \hat{F}\|_{\mathrm{L}^4 \cap W^{1,\infty}} \lesssim 1$, we find

$$\begin{aligned}
\|\Gamma_\varepsilon^t \partial_t \mathrm{m}_\varepsilon^t\|_{\mathrm{L}^1 \cap \mathrm{L}^2} &\lesssim \|\Gamma_\varepsilon^t\|^2_{\mathrm{L}^4 \cap \mathrm{L}^\infty}\|\nabla \mathrm{m}_\varepsilon^t\|_{\mathrm{L}^2} + \|\Gamma_\varepsilon^t\|^2_{W^{1,\infty}}\|\mathrm{m}_\varepsilon^t\|_{\mathrm{L}^1 \cap \mathrm{L}^2} \\
&\lesssim_t \|\Gamma_\varepsilon^t\|^2_{\mathrm{L}^4 \cap W^{1,\infty}} \lesssim 1 + \|\mathrm{v}_\varepsilon^t\|^2_{\mathrm{L}^4 \cap W^{1,\infty}} \lesssim_t 1,
\end{aligned}$$

and hence $\|\partial_t \mathrm{p}_\varepsilon^t\|_{\mathrm{L}^q} \lesssim_t 1$. All the stated estimates follow.

Step 3: proof of (3.22). For all $t \ge 0$, testing equation (3.24) against $(1-\chi_\varrho)\,\mathrm{p}_\varepsilon^t$, and using $|\nabla \chi_\varrho| \lesssim \varrho^{-1}(1-\chi_\varrho)^{1/2}$ and the inequality $2xy \le x^2 + y^2$, we find

$$\begin{aligned}
\int \hat{a}(1-\chi_\varrho)|\nabla \mathrm{p}_\varepsilon^t|^2 &= \int \hat{a}\,\mathrm{p}_\varepsilon^t\,\nabla\chi_\varrho \cdot \nabla \mathrm{p}_\varepsilon^t - \int \hat{a}(1-\chi_\varrho)\nabla \mathrm{p}_\varepsilon^t \cdot \Gamma_\varepsilon^t\,\mathrm{m}_\varepsilon^t + \int \hat{a}\mathrm{p}_\varepsilon^t\,\nabla\chi_\varrho \cdot \Gamma_\varepsilon^t\,\mathrm{m}_\varepsilon^t \\
&\le \frac{1}{2}\int \hat{a}(1-\chi_\varrho)|\nabla \mathrm{p}_\varepsilon^t|^2 + C\varrho^{-2}\int_{\varrho \le |x| \le 2\varrho} |\mathrm{p}_\varepsilon^t|^2 + C\int (1-\chi_\varrho)|\Gamma_\varepsilon^t|^2|\mathrm{m}_\varepsilon^t|^2.
\end{aligned}$$

Absorbing the first right-hand side term, and recalling that Step 2 gives $\|\Gamma_\varepsilon^t\|_{\mathrm{L}^\infty}$, $\|\mathrm{m}_\varepsilon^t\|_{\mathrm{L}^2}\lesssim_t 1$, and $\|\mathrm{p}_\varepsilon^t\|_{\mathrm{L}^p}\lesssim_{p,t} 1$ for all $2<p<\infty$, we obtain with the Hölder inequality,

$$\int(1-\chi_\varrho)|\nabla \mathrm{p}_\varepsilon^t|^2 \lesssim_t \varrho^{-2}\int_{\varrho\le|x|\le 2\varrho}|\mathrm{p}_\varepsilon^t|^2+\int(1-\chi_\varrho)|\mathrm{m}_\varepsilon^t|^2 \lesssim_{p,t} \varrho^{-4/p}+\int(1-\chi_\varrho)|\mathrm{m}_\varepsilon^t|^2,$$

and thus

$$\|\nabla(\mathrm{p}_{\varepsilon,\varrho}^t-\mathrm{p}_\varepsilon^t)\|_{\mathrm{L}^2}^2\lesssim\int(1-\chi_\varrho)|\nabla\mathrm{p}_\varepsilon^t|^2+\varrho^{-2}\int_{\varrho\le|x|\le2\varrho}|\mathrm{p}_\varepsilon^t|^2\lesssim_{p,t}\varrho^{-4/p}+\int(1-\chi_\varrho)|\mathrm{m}_\varepsilon^t|^2.$$

It remains to estimate the last right-hand side term. For all $t\ge0$, using again the bounds of Step 2 and the estimate $|\nabla\chi_\varrho|\lesssim\varrho^{-1}(1-\chi_\varrho)^{1/2}$, we deduce from equation (3.21),

$$\begin{aligned}
\partial_t\int(1-\chi_\varrho)|\mathrm{m}_\varepsilon^t|^2&=2\int(1-\chi_\varrho)\,\mathrm{m}_\varepsilon^t\mathrm{curl}\,(\Gamma_\varepsilon^t\mathrm{m}_\varepsilon^t)\\
&=2\int|\mathrm{m}_\varepsilon^t|^2\Gamma_\varepsilon^t\cdot\nabla^\perp\chi_\varrho-\int(1-\chi_\varrho)\Gamma_\varepsilon^t\cdot\nabla^\perp|\mathrm{m}_\varepsilon^t|^2\\
&=2\int|\mathrm{m}_\varepsilon^t|^2\Gamma_\varepsilon^t\cdot\nabla^\perp\chi_\varrho+\int|\mathrm{m}_\varepsilon^t|^2\mathrm{curl}\,\big((1-\chi_\varrho)\Gamma_\varepsilon^t\big)\\
&\lesssim_t\varrho^{-1}\int(1-\chi_\varrho)^{1/2}|\mathrm{m}_\varepsilon^t|^2+\int(1-\chi_\varrho)|\mathrm{m}_\varepsilon^t|^2\lesssim_t\varrho^{-2}+\int(1-\chi_\varrho)|\mathrm{m}_\varepsilon^t|^2,
\end{aligned}$$

hence by the Grönwall inequality,

$$\int(1-\chi_\varrho)|\mathrm{m}_\varepsilon^t|^2\lesssim_t\varrho^{-2}+\int(1-\chi_\varrho)|\mathrm{curl}\ \mathrm{v}_\varepsilon^\circ|^2,$$

and the result (3.22) follows. □

3.2.2 Passing to the limit in (3.3)

We now show how to pass to the limit in (3.3) as $\varepsilon\downarrow0$, which is easily achieved by a Grönwall type argument for the L^2-distance between v_ε and its limit. Note that in the limit, pinning effects are in this case only present through the constraint.

Lemma 3.4. *Let $h:\mathbb{R}^2\to\mathbb{R}$, $a:=e^h$, $F:\mathbb{R}^2\to\mathbb{R}^2$, and let $\mathrm{v}_\varepsilon:[0,T)\times\mathbb{R}^2\to\mathbb{R}^2$ be a solution of (3.3) as in Proposition 3.3, for some $T>0$. Then, in the regime* (GP), *with $\mathrm{v}_\varepsilon^\circ=\mathrm{v}^\circ$, we have $\mathrm{v}_\varepsilon\to\mathrm{v}$ in $\mathrm{L}^\infty_{\mathrm{loc}}([0,T);\mathrm{L}^2(\mathbb{R}^2)^2)$ as $\varepsilon\downarrow0$, where v is the unique solution of*

$$\partial_t\mathrm{v}=\nabla\mathrm{p}+(-\hat F+2\,\mathrm{v}^\perp)\mathrm{curl}\ \mathrm{v},\qquad\mathrm{div}\,(\hat a\mathrm{v})=0,\qquad\mathrm{v}|_{t=0}=\mathrm{v}^\circ\,.\tag{3.27}$$

Proof. Using the choice of the scalings for λ_ε,h,F in the regime (GP), equation (3.3) takes the following form,

$$\partial_t\mathrm{v}_\varepsilon=\nabla\mathrm{p}_\varepsilon+\big(\lambda_\varepsilon^{-1}\nabla\hat h-\hat F+2\,\mathrm{v}_\varepsilon^\perp\big)\mathrm{curl}\ \mathrm{v}_\varepsilon,\qquad\mathrm{div}\,(\hat a\mathrm{v}_\varepsilon)=0,\qquad\mathrm{v}_\varepsilon|_{t=0}=\mathrm{v}^\circ\,.$$

As $\lambda_\varepsilon^{-1}\to0$, it is formally clear that v_ε should converge to the solution v of equation (3.27) as $\varepsilon\downarrow0$. Note that the existence, uniqueness and regularity of this solution v are given by Proposition 3.3 just as for v_ε, and yields in particular the following bounds for all $t\in[0,T)$,

$$\|(\mathrm{v}^t,\mathrm{v}_\varepsilon^t)\|_{W^{1,\infty}}\lesssim_t1,\quad\|\mathrm{curl}\ \mathrm{v}_\varepsilon^t\|_{\mathrm{L}^1}\lesssim_t1,\quad\|(\mathrm{p}^t,\mathrm{p}_\varepsilon^t)\|_{\mathrm{L}^\infty}\lesssim_t1,\tag{3.28}$$

and for all $\theta > 0$,

$$\|(\mathrm{v}^t, \mathrm{v}_\varepsilon^t)\|_{\mathrm{L}^2(B_R)} \lesssim_{t,\theta} R^\theta, \quad \|(\mathrm{p}^t, \mathrm{p}_\varepsilon^t)\|_{\mathrm{L}^2(B_R)} \lesssim_{t,\theta} R^\theta. \tag{3.29}$$

For $R \ge 1$, we denote by $\xi_R^z(x) := e^{-|x-z|/R}$ the exponential cut-off at the scale R centered at $z \in R\mathbb{Z}^2$. Using the equations for $\mathrm{v}_\varepsilon, \mathrm{v}$, we find

$$\partial_t \int \hat{a}\xi_R^z|\mathrm{v}_\varepsilon - \mathrm{v}|^2 = 2\int \hat{a}\xi_R^z(\mathrm{v}_\varepsilon - \mathrm{v})\cdot\nabla(\mathrm{p}_\varepsilon - \mathrm{p}) + 2\int \hat{a}\xi_R^z(-\hat{F} + 2\,\mathrm{v}^\perp)\cdot(\mathrm{v}_\varepsilon - \mathrm{v})(\mathrm{curl}\,\mathrm{v}_\varepsilon - \mathrm{curl}\,\mathrm{v}) \\ + 2\lambda_\varepsilon^{-1}\int \hat{a}\xi_R^z\nabla\hat{h}\cdot(\mathrm{v}_\varepsilon - \mathrm{v})\mathrm{curl}\,\mathrm{v}_\varepsilon\,.$$

Integrating by parts in the first term with $\mathrm{div}\,(\hat{a}\xi_R^z(\mathrm{v}_\varepsilon - \mathrm{v})) = \hat{a}\nabla\xi_R^z\cdot(\mathrm{v}_\varepsilon - \mathrm{v})$, and using the weighted Delort-type identity (cf. (3.19)) in the form

$$(\mathrm{v}_\varepsilon - \mathrm{v})\mathrm{curl}\,(\mathrm{v}_\varepsilon - \mathrm{v}) = -\frac{1}{2}|\mathrm{v}_\varepsilon - \mathrm{v}|^2\nabla^\perp\hat{h} - \hat{a}^{-1}(\mathrm{div}\,(\hat{a}S_{\mathrm{v}_\varepsilon - \mathrm{v}}))^\perp,$$

we deduce

$$\partial_t \int \hat{a}\xi_R^z|\mathrm{v}_\varepsilon - \mathrm{v}|^2 = -2\int \hat{a}\nabla\xi_R^z\cdot(\mathrm{v}_\varepsilon - \mathrm{v})(\mathrm{p}_\varepsilon - \mathrm{p}) - \int \hat{a}\xi_R^z\nabla^\perp\hat{h}\cdot(-\hat{F} + 2\,\mathrm{v}^\perp)|\mathrm{v}_\varepsilon - \mathrm{v}|^2 \\ + 2\int \hat{a}S_{\mathrm{v}_\varepsilon - \mathrm{v}} : \nabla(\xi_R^z(\hat{F}^\perp + 2\,\mathrm{v})) + 2\lambda_\varepsilon^{-1}\int \hat{a}\xi_R^z\nabla\hat{h}\cdot(\mathrm{v}_\varepsilon - \mathrm{v})\mathrm{curl}\,\mathrm{v}_\varepsilon,$$

and hence, using (3.28)–(3.29), the assumption $\|(\nabla\hat{h}, \hat{F})\|_{W^{1,\infty}} \lesssim 1$, the property $|\nabla\xi_R^z| \lesssim R^{-1}\xi_R^z$ of the exponential cut-off, and the pointwise estimate $|S_w| \lesssim |w|^2$,

$$\partial_t \int \hat{a}\xi_R^z|\mathrm{v}_\varepsilon - \mathrm{v}|^2 \lesssim_{t,\theta} R^{-2(1-\theta)} + \lambda_\varepsilon^{-2} + \int \hat{a}\xi_R^z|\mathrm{v}_\varepsilon - \mathrm{v}|^2.$$

Choosing $\theta = 1/2$, the Grönwall inequality yields $\sup_z \int a_\varepsilon\xi_R^z|\mathrm{v}_\varepsilon - \mathrm{v}|^2 \lesssim_t R^{-1} + \lambda_\varepsilon^{-2}$, and the conclusion follows, letting $R \uparrow \infty$. ☐

4 Computations on the modulated energy

In this section, we adapt to the weighted case with pinning and forcing the computations of [82] involving the modulated energy excess. Their point is to compute the time-derivative of the modulated energy excess (1.13) and express it with only quadratic terms in the error instead of terms which initially appear as linear, thus making a Grönwall argument impossible. These computations are based on purely algebraic manipulations using all the equations and appropriate quantities that we will now describe.

For simplicity, in the estimates in this section, we focus on the non-oscillating case $\eta_\varepsilon = 1$, and we consider the regimes (GL_1), (GL_2), (GP), (GL_1'), and (GL_2').

4.1 Modulated energy

We first recall the definitions of modulated energy and energy excess in (1.10)–(1.13). In order to prove that $N_\varepsilon^{-1}j_\varepsilon$ is close to v_ε, we follow the strategy of [82], and consider the following *modulated energy*, which is modeled on the weighted energy density e_ε, plays the role of an adapted measure of the distance between

$N_\varepsilon^{-1} j_\varepsilon$ and v_ε, and is localized by means of the cut-off function χ_R at some scale $R \gg 1$ (to be later optimized as a function of ε),

$$\mathcal{E}_{\varepsilon,R} := \int \frac{a\chi_R}{2}\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big).$$

As usual, this modulated energy $\mathcal{E}_{\varepsilon,R}$ further needs to be renormalized by subtracting the expected self-interaction energy of the vortices (compare with Lemma 5.1), which then yields the following *modulated energy excess*,

$$\mathcal{D}_{\varepsilon,R} := \mathcal{E}_{\varepsilon,R} - \frac{|\log\varepsilon|}{2}\int a\chi_R\mu_\varepsilon = \int \frac{a\chi_R}{2}\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 - |\log\varepsilon|\mu_\varepsilon\Big).$$

As seen in the introduction, the cut-off χ_R is not needed in the Gross-Pitaevskii case, where we only treat the case when h and F decay at infinity. We write $\mathcal{E}_\varepsilon := \mathcal{E}_{\varepsilon,\infty}$ for the corresponding quantity without the cut-off χ_R in the definition (formally $R=\infty$), and also $\mathcal{D}_\varepsilon := \sup_{R\ge 1}\mathcal{D}_{\varepsilon,R}$.

On the one hand, rather than the L^2-norm restricted to the ball B_R centered at the origin, our methods further allow to consider the uniform $\mathrm{L}^2_{\mathrm{loc}}$-norm at the scale R: setting $\chi_R^z := \chi_R(\cdot - z)$, we define

$$\mathcal{E}^*_{\varepsilon,R} := \sup_z \mathcal{E}^z_{\varepsilon,R}, \qquad \mathcal{E}^z_{\varepsilon,R} := \int \frac{a\chi_R^z}{2}\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big),$$

where henceforth the supremum always implicitly runs over all lattice points $z \in R\mathbb{Z}^2$, and similarly

$$\mathcal{D}^*_{\varepsilon,R} := \sup_z \mathcal{D}^z_{\varepsilon,R}, \qquad \mathcal{D}^z_{\varepsilon,R} := \mathcal{E}^z_{\varepsilon,R} - \frac{|\log\varepsilon|}{2}\int a\chi_R^z\mu_\varepsilon.$$

Note that by definition we have for all $x \in \mathbb{R}^2$ and all $L > 0$,

$$\|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon\|_{\mathrm{L}^2(B_L(x))} + \varepsilon^{-1}\|1-|u_\varepsilon|^2\|_{\mathrm{L}^2(B_L(x))} \lesssim \Big(1+\frac{L}{R}\Big)^d \mathcal{E}^*_{\varepsilon,R}. \tag{4.1}$$

On the other hand, in order to simplify computations, we need as in [82] to add some suitable lower-order term, and rather consider, for some scale $\varrho \gg 1$ (to be later optimized as a function of ε),

$$\hat{\mathcal{E}}_{\varepsilon,\varrho,R} := \int \frac{a}{2}\Big(\chi_R|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a\chi_R}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 + (1-|u_\varepsilon|^2)(N_\varepsilon^2\psi_{\varepsilon,\varrho,R} + f\chi_R)\Big),$$

and similarly for the modulated energy excess,

$$\begin{aligned}\hat{\mathcal{D}}_{\varepsilon,\varrho,R} &:= \hat{\mathcal{E}}_{\varepsilon,\varrho,R} - \frac{|\log\varepsilon|}{2}\int a\chi_R\mu_\varepsilon \\ &= \int \frac{a}{2}\Big(\chi_R|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a\chi_R}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 + (1-|u_\varepsilon|^2)(N^2\psi_{\varepsilon,\varrho,R} + f\chi_R) - |\log\varepsilon|\chi_R\mu_\varepsilon\Big),\end{aligned} \tag{4.2}$$

where the field $\psi_{\varepsilon,\varrho,R}$ is chosen as follows,

$$\psi_{\varepsilon,\varrho,R} := 3\chi_R|\mathrm{v}_\varepsilon|^2 - \frac{|\log\varepsilon|}{N_\varepsilon}\chi_R\,\mathrm{v}_\varepsilon\cdot(\nabla^\perp h - F^\perp) + \frac{\lambda_\varepsilon\beta|\log\varepsilon|}{N_\varepsilon}\chi_R(2\Gamma_\varepsilon^\perp\cdot\mathrm{v}_\varepsilon + \mathrm{p}_{\varepsilon,\varrho}) + \frac{|\log\varepsilon|}{N_\varepsilon}\nabla\chi_R\cdot\mathrm{v}_\varepsilon^\perp, \tag{4.3}$$

with $\mathrm{p}_{\varepsilon,\varrho} := \chi_\varrho\,\mathrm{p}_\varepsilon$. This choice is motivated by the fact that it yields some crucial cancellations in the proof of Lemma 4.4. Again, replacing χ_R and $\mathrm{p}_{\varepsilon,\varrho}$ by χ_R^z and $\mathrm{p}^z_{\varepsilon,\varrho} = \chi^z_\varrho\,\mathrm{p}_\varepsilon$, we further define $\hat{\mathcal{E}}^z_{\varepsilon,\varrho,R}$ and $\hat{\mathcal{D}}^z_{\varepsilon,\varrho,R}$ for $z \in \mathbb{R}^2$, and we then set $\hat{\mathcal{E}}^*_{\varepsilon,\varrho,R} := \sup_z \hat{\mathcal{E}}^z_{\varepsilon,\varrho,R}$ and $\hat{\mathcal{D}}^*_{\varepsilon,\varrho,R} := \sup_z \hat{\mathcal{D}}^z_{\varepsilon,\varrho,R}$ (where again the supremum implicitly runs over all lattice points $z \in R\mathbb{Z}^2$). The truncation scale $\rho \gg 1$ is introduced here to cure the lack of integrability of the pressure p_ε in the Gross-Pitaevskii case: since the pressure p_ε does in general not belong to $\mathrm{L}^2(\mathbb{R}^2)$ (cf. Assumption B(b) above, which is indeed optimal in that respect), we define a truncated pressure $\mathrm{p}_{\varepsilon,\varrho} := \chi_\varrho \mathrm{p}_\varepsilon$. (In contrast, note that in the case without pinning and forcing the pressure always belongs to $\mathrm{L}^2(\mathbb{R}^2)$, see [82].) To unify notation, we set in the dissipative case $\mathrm{p}_{\varepsilon,\infty} := \mathrm{p}_\varepsilon$ with $\varrho := \infty$, and we then drop for simplicity the subscript ϱ, writing $\psi_{\varepsilon,R} := \psi_{\varepsilon,\infty,R}$, $\hat{\mathcal{E}}_{\varepsilon,R} := \hat{\mathcal{E}}_{\varepsilon,\infty,R}$, and so on.

In the dissipative case, as a consequence of (2.1) and of Assumption B(a), $\psi_{\varepsilon,R}$ is bounded uniformly with respect to R in $\mathrm{L}^p(\mathbb{R}^2)$ for all $2 < p \le \infty$ (but not in $\mathrm{L}^2(\mathbb{R}^2)$), and using the bound (2.1) we have in the considered regimes, for all $t \in [0,T)$ and $\theta > 0$,

$$\|\psi^t_{\varepsilon,R}\|_{\mathrm{L}^2} \lesssim_{t,\theta} 1 + \frac{|\log\varepsilon|}{N_\varepsilon}(\lambda_\varepsilon R^\theta + \lambda_\varepsilon^{1/2} + R^{-1+\theta}), \qquad \|\partial_t \psi_{\varepsilon,R}\|_{\mathrm{L}^2_t\mathrm{L}^2} \lesssim_{t,\theta} \frac{|\log\varepsilon|}{N_\varepsilon}. \tag{4.4}$$

In the Gross-Pitaevskii case, in the considered regime (GP), the bound (2.2) and Assumption B(b) rather yield, for all $t \in [0,T)$ and $\theta > 0$,

$$\|\psi^t_{\varepsilon,\varrho,R}\|_{\mathrm{L}^2} + \|\partial_t \psi^t_{\varepsilon,\varrho,R}\|_{\mathrm{L}^2} \lesssim_{t,\theta} 1 + \frac{|\log\varepsilon|}{N_\varepsilon}\lambda_\varepsilon \varrho^\theta \lesssim \varrho^\theta. \tag{4.5}$$

Based on these estimates, the following lemma states that the additional term in $\hat{\mathcal{E}}_{\varepsilon,\varrho,R}$ is indeed of lower order, so that the modulated energy $\hat{\mathcal{E}}_{\varepsilon,\varrho,R}$ itself controls the various quantities that we are interested in.

Lemma 4.1 (Neglecting lower-order terms)**.** *Let $h : \mathbb{R}^2 \to \mathbb{R}$, $a := e^h$, $F : \mathbb{R}^2 \to \mathbb{R}^2$ satisfy (2.1)–(2.2), let $u_\varepsilon : [0,T) \times \mathbb{R}^2 \to \mathbb{C}$, and let $\mathrm{v}_\varepsilon : [0,T) \times \mathbb{R}^2 \to \mathbb{R}^2$ be as in Assumption B. Further assume that $0 < \varepsilon \ll 1$ and $\varrho, R \gg 1$ satisfy for some $\theta > 0$, in the dissipative case (with $N_\varepsilon \lesssim |\log\varepsilon|$),*

$$\varepsilon\big(N_\varepsilon^2 + N_\varepsilon|\log\varepsilon|(\lambda_\varepsilon R^\theta + \lambda_\varepsilon^{1/2} + R^{-1+\theta}) + R + R\lambda_\varepsilon^2|\log\varepsilon|^2\big) \ll N_\varepsilon\Big(1 \wedge \frac{N_\varepsilon}{|\log\varepsilon|}\Big)^{1/2}. \tag{4.6}$$

or, in the Gross-Pitaevskii case (with $N_\varepsilon \ll \varepsilon^{-1}$),

$$\varepsilon N_\varepsilon^2(\varrho^\theta + R) \ll N_\varepsilon\Big(1 \wedge \frac{N_\varepsilon}{|\log\varepsilon|}\Big)^{1/2}. \tag{4.7}$$

Then for all $z \in \mathbb{R}^2$ we have

$$|\hat{\mathcal{E}}^{z,t}_{\varepsilon,\varrho,R} - \mathcal{E}^{z,t}_{\varepsilon,R}| = |\hat{\mathcal{D}}^{z,t}_{\varepsilon,\varrho,R} - \mathcal{D}^{z,t}_{\varepsilon,R}| \lesssim_t o(N_\varepsilon)\Big(1 \wedge \frac{N_\varepsilon}{|\log\varepsilon|}\Big)^{1/2}(\mathcal{E}^{z,t}_{\varepsilon,R})^{1/2}.$$

Proof. We focus on the dissipative case, the other case is similar. The Cauchy-Schwarz inequality yields

$$\begin{aligned}
|\hat{\mathcal{E}}^z_{\varepsilon,R} - \mathcal{E}^z_{\varepsilon,R}| &\lesssim \int |1-|u_\varepsilon|^2|(N_\varepsilon^2|\psi^z_{\varepsilon,R}| + |f|\chi^z_R) \\
&\le \Big(\int \chi^z_R(1-|u_\varepsilon|^2)^2\Big)^{1/2}\big(N_\varepsilon^2\|\psi^z_{\varepsilon,R}/(\chi^z_R)^{1/2}\|_{\mathrm{L}^2} + \|f\|_{\mathrm{L}^2(B_{2R}(z))}\big) \\
&\lesssim \varepsilon(\mathcal{E}^z_{\varepsilon,R})^{1/2}\big(N_\varepsilon^2\|\psi^z_{\varepsilon,R}/(\chi^z_R)^{1/2}\|_{\mathrm{L}^2} + R\|f\|_{\mathrm{L}^\infty}\big).
\end{aligned}$$

Arguing just as in (4.4), using (2.1), Assumption B(a), and the fact that $|\nabla\chi_R(x)/\chi_R^{1/2}(x)| \lesssim R^{-1}\mathbb{1}_{|x|\le 2R}$, the choice (4.3) of $\psi_{\varepsilon,R}$ yields, for all $\theta > 0$,

$$\|\psi_{\varepsilon,R}/\chi_R^{1/2}\|_{\mathrm{L}^2} \lesssim_{t,\theta} 1 + \frac{|\log\varepsilon|}{N_\varepsilon}(\lambda_\varepsilon R^\theta + \lambda_\varepsilon^{1/2} + R^{-1+\theta}).$$

Combined with (2.1) and with assumption (4.6), this proves the result. $\square$

4.2 Physical quantities and identities

Next to the *supercurrent density* $j_\varepsilon := \langle \nabla u_\varepsilon, iu_\varepsilon\rangle$ and the *vorticity* $\mu_\varepsilon := \operatorname{curl} j_\varepsilon$, we define the *vortex velocity* $V_\varepsilon := 2\langle \nabla u_\varepsilon, i\partial_t u_\varepsilon\rangle$. The following identities are easily checked from these definitions:

$$\partial_t j_\varepsilon = V_\varepsilon + \nabla\langle \partial_t u_\varepsilon, iu_\varepsilon\rangle, \qquad \partial_t \mu_\varepsilon = \operatorname{curl} V_\varepsilon, \tag{4.8}$$

and also, using equation (1.5) for u_ε,

$$\begin{aligned} \operatorname{div} j_\varepsilon = \langle \triangle u_\varepsilon, iu_\varepsilon\rangle &= \lambda_\varepsilon\alpha\langle \partial_t u_\varepsilon, iu_\varepsilon\rangle - j_\varepsilon\cdot\nabla h \\ &\quad - \frac{\lambda_\varepsilon\beta|\log\varepsilon|}{2}\partial_t(1-|u_\varepsilon|^2) + \frac{|\log\varepsilon|}{2}F^\perp\cdot\nabla(1-|u_\varepsilon|^2) + |\log\varepsilon| g(1-|u_\varepsilon|^2). \end{aligned}\tag{4.9}$$

We then consider the *weighted energy density*

$$e_\varepsilon := \frac{a}{2}\Big(|\nabla u_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 + (1-|u_\varepsilon|^2)f\Big).$$

In the same vein as when introducing the modulated energy and energy excess, we define the following *modulated vorticity* and *modulated velocity*,

$$\tilde\mu_\varepsilon := \operatorname{curl}(N_\varepsilon \mathrm{v}_\varepsilon + \langle \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon, iu_\varepsilon\rangle) = \mu_\varepsilon + \operatorname{curl}(N_\varepsilon\mathrm{v}_\varepsilon(1-|u_\varepsilon|^2)), \tag{4.10}$$

$$\tilde V_{\varepsilon,\varrho} := 2\langle \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon, i(\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_{\varepsilon,\varrho})\rangle = V_\varepsilon + N_\varepsilon\mathrm{v}_\varepsilon\partial_t|u_\varepsilon|^2 - N_\varepsilon\mathrm{p}_{\varepsilon,\varrho}\nabla|u_\varepsilon|^2. \tag{4.11}$$

For the computations, we will also need the 2×2 *stress-energy tensor* S_ε,

$$S_\varepsilon^{kl} := a\langle \partial_k u_\varepsilon, \partial_l u_\varepsilon\rangle - \frac{a}{2}\operatorname{Id}\Big(|\nabla u_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 + (1-|u_\varepsilon|^2)f\Big), \tag{4.12}$$

and its modulated version $\tilde S_\varepsilon$,

$$\begin{aligned} \tilde S_\varepsilon^{kl} := a\Big(&\langle \partial_k u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_{\varepsilon,k}, \partial_l u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_{\varepsilon,l}\rangle + N_\varepsilon^2(1-|u_\varepsilon|^2)\,\mathrm{v}_{\varepsilon,k}\mathrm{v}_{\varepsilon,l}\Big) \\ &- \frac{a}{2}\operatorname{Id}\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 + (1-|u_\varepsilon|^2)(N_\varepsilon^2|\mathrm{v}_\varepsilon|^2+f)\Big). \end{aligned}\tag{4.13}$$

We close this section with the following pointwise estimates.

Lemma 4.2. *We have*

$$\begin{gathered} |j_\varepsilon - N_\varepsilon v_\varepsilon| \le |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon| + |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon||1-|u_\varepsilon|^2| + N_\varepsilon|\mathrm{v}_\varepsilon||1-|u_\varepsilon|^2|, \\ |\mu_\varepsilon| \le 2|\nabla u_\varepsilon|^2 \le 4|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|^2 + 4N_\varepsilon^2|\mathrm{v}_\varepsilon|^2 + 4N_\varepsilon^2|1-|u_\varepsilon|^2||\mathrm{v}_\varepsilon|^2, \\ |V_\varepsilon| \le 2\big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon||\partial_t u_\varepsilon| + N_\varepsilon|\mathrm{v}_\varepsilon||\partial_t u_\varepsilon| + N_\varepsilon|1-|u_\varepsilon|^2||\mathrm{v}_\varepsilon||\partial_t u_\varepsilon|\big), \\ |\tilde V_{\varepsilon,\varrho}| \le 2|\partial_t u_\varepsilon||\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon| + 2N_\varepsilon|\mathrm{p}_{\varepsilon,\varrho}||\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon| + 2N_\varepsilon|\mathrm{p}_{\varepsilon,\varrho}||1-|u_\varepsilon|^2||\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|, \\ |\partial_t|u_\varepsilon|| \le |\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{p}_\varepsilon|, \\ |\nabla|u_\varepsilon|| \le |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|. \end{gathered}$$

Proof. The first estimate is obtained as follows,

$$\begin{aligned} |j_\varepsilon - N_\varepsilon\mathrm{v}_\varepsilon| &\le |\langle \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon, iu_\varepsilon\rangle| + N_\varepsilon|1-|u_\varepsilon|^2||\mathrm{v}_\varepsilon| \\ &\le |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon| + |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon||1-|u_\varepsilon|^2| + N_\varepsilon|\mathrm{v}_\varepsilon||1-|u_\varepsilon|^2|, \end{aligned}$$

while the estimates on V_ε and $\tilde V_{\varepsilon,\varrho}$ similarly follow the definitions. The estimate on μ_ε is a direct consequence of the representation $\mu_\varepsilon = \operatorname{curl}\langle \nabla u_\varepsilon, iu_\varepsilon\rangle = 2\langle \nabla_2 u_\varepsilon, i\nabla_1 u_\varepsilon\rangle$. Finally noting that

$$|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{p}_\varepsilon|^2 = |\partial_t|u_\varepsilon||^2 + |u_\varepsilon|^2\Big|\partial_t\frac{u_\varepsilon}{|u_\varepsilon|} - i\frac{u_\varepsilon}{|u_\varepsilon|}N_\varepsilon\mathrm{p}_\varepsilon\Big|^2,$$

the result on $\partial_t|u_\varepsilon|$ follows, and the result on $\nabla|u_\varepsilon|$ is obtained similarly. □

4.3 Divergence of the modulated stress-energy tensor

In the following lemma we explicitly compute the divergence of the modulated stress-energy tensor: as already mentioned, it will be crucial in the sequel in order to replace some linear terms in the error by quadratic ones (cf. Step 3 of the proof of Lemma 4.4 below).

Lemma 4.3. *Let $u_\varepsilon : [0,T)\times\mathbb{R}^2 \to \mathbb{C}$ be a solution of (1.5) as in Proposition 2.2, and let $\mathrm{v}_\varepsilon : [0,T)\times\mathbb{R}^2 \to \mathbb{R}^2$ be as in Assumption B. Then, defining by $(\mathrm{div}\ \tilde{S}_\varepsilon)_k := \sum_l \partial_l(\tilde{S}_\varepsilon)_{kl}$ the divergence of the 2-tensor $\tilde{S}_\varepsilon$, where $(\tilde{S}_\varepsilon)_{kl}$ denotes the (k,l)-component of $\tilde{S}_\varepsilon$, we have*

$$\begin{aligned}\mathrm{div}\ \tilde{S}_\varepsilon &= a\lambda_\varepsilon\alpha \left\langle \partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_{\varepsilon,\varrho}, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon \right\rangle - a\mu_\varepsilon(N_\varepsilon \mathrm{v}_\varepsilon^\perp - |\log\varepsilon| F/2) + aN_\varepsilon(N_\varepsilon \mathrm{v}_\varepsilon - j_\varepsilon)^\perp \mathrm{curl}\ \mathrm{v}_\varepsilon \\ &\quad + \frac{a\lambda_\varepsilon\beta}{2}|\log\varepsilon|\tilde{V}_{\varepsilon,\varrho} + aN_\varepsilon(N_\varepsilon \mathrm{v}_\varepsilon - j_\varepsilon)(\mathrm{div}\ \mathrm{v}_\varepsilon + \nabla h\cdot \mathrm{v}_\varepsilon - \lambda_\varepsilon\alpha\, \mathrm{p}_{\varepsilon,\varrho}) - \frac{a}{2}(1-|u_\varepsilon|^2)\nabla f \\ &\quad - \frac{a}{2}\nabla h\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{\varepsilon^2}(1-|u_\varepsilon|^2)^2 + (1-|u_\varepsilon|^2)(N_\varepsilon^2|\mathrm{v}_\varepsilon|^2 + f)\Big) + a\lambda_\varepsilon\beta N_\varepsilon|\log\varepsilon| \mathrm{v}_\varepsilon \partial_t(1-|u_\varepsilon|^2) \\ &\quad + a\lambda_\varepsilon\alpha N_\varepsilon^2 \mathrm{v}_\varepsilon \mathrm{p}_{\varepsilon,\varrho}(1-|u_\varepsilon|^2) + \frac{a\lambda_\varepsilon\beta}{2}N_\varepsilon|\log\varepsilon|\, \mathrm{p}_{\varepsilon,\varrho}\nabla|u_\varepsilon|^2 + \frac{a}{2}N_\varepsilon|\log\varepsilon|(F^\perp\cdot\nabla|u_\varepsilon|^2)\, \mathrm{v}_\varepsilon\,.\end{aligned}$$

Proof. A direct computation yields, for the stress-energy tensor,

$$\begin{aligned}\mathrm{div}\ S_\varepsilon &= a\left\langle \nabla u_\varepsilon, \triangle u_\varepsilon + \frac{au_\varepsilon}{\varepsilon^2}(1-|u_\varepsilon|^2) + \nabla h\cdot\nabla u_\varepsilon + fu_\varepsilon \right\rangle \\ &\quad - \frac{a}{2}\nabla h\Big(|\nabla u_\varepsilon|^2 + \frac{a}{\varepsilon^2}(1-|u_\varepsilon|^2)^2 + (1-|u_\varepsilon|^2)f\Big) - \frac{a}{2}(1-|u_\varepsilon|^2)\nabla f. \qquad (4.14)\end{aligned}$$

On the other hand, the modulated stress-energy tensor may be decomposed as

$$\tilde{S}_\varepsilon = S_\varepsilon - aN_\varepsilon \mathrm{v}_\varepsilon \otimes j_\varepsilon - aN_\varepsilon j_\varepsilon\otimes \mathrm{v}_\varepsilon + aN_\varepsilon^2 \mathrm{v}_\varepsilon\otimes \mathrm{v}_\varepsilon - \frac{aN_\varepsilon}{2}\,\mathrm{Id}\,\big(N_\varepsilon|\mathrm{v}_\varepsilon|^2 - 2\,\mathrm{v}_\varepsilon\cdot j_\varepsilon\big),$$

which, combined with (4.14), yields

$$\begin{aligned}\mathrm{div}\ \tilde{S}_\varepsilon &= a\left\langle \nabla u_\varepsilon, \triangle u_\varepsilon + \frac{au_\varepsilon}{\varepsilon^2}(1-|u_\varepsilon|^2) + \nabla h\cdot\nabla u_\varepsilon + fu_\varepsilon \right\rangle \\ &\quad - \frac{a}{2}\nabla h\Big(|\nabla u_\varepsilon|^2 + \frac{a}{\varepsilon^2}(1-|u_\varepsilon|^2)^2 + (1-|u_\varepsilon|^2)f\Big) - \frac{a}{2}(1-|u_\varepsilon|^2)\nabla f \\ &\quad - aN_\varepsilon\Big(j_\varepsilon\nabla h\cdot \mathrm{v}_\varepsilon + \mathrm{v}_\varepsilon\nabla h\cdot j_\varepsilon - N_\varepsilon \mathrm{v}_\varepsilon\nabla h\cdot \mathrm{v}_\varepsilon + \frac{1}{2}N_\varepsilon|\mathrm{v}_\varepsilon|^2\nabla h - \mathrm{v}_\varepsilon\cdot j_\varepsilon\nabla h\Big) \\ &\quad - aN_\varepsilon j_\varepsilon\,\mathrm{div}\ \mathrm{v}_\varepsilon - aN_\varepsilon(\mathrm{v}_\varepsilon\cdot\nabla)j_\varepsilon - aN_\varepsilon \mathrm{v}_\varepsilon \mathrm{div}\ j_\varepsilon - aN_\varepsilon(j_\varepsilon\cdot\nabla)\, \mathrm{v}_\varepsilon + aN_\varepsilon^2 \mathrm{v}_\varepsilon \mathrm{div}\ \mathrm{v}_\varepsilon + aN_\varepsilon^2(\mathrm{v}_\varepsilon\cdot\nabla)\, \mathrm{v}_\varepsilon \\ &\quad - aN_\varepsilon^2\sum_l \mathrm{v}_{\varepsilon,l}\,\nabla \mathrm{v}_{\varepsilon,l} + aN_\varepsilon\sum_l \mathrm{v}_{\varepsilon,l}\,\nabla j_{\varepsilon,l} + aN_\varepsilon\sum_l j_{\varepsilon,l}\nabla \mathrm{v}_{\varepsilon,l},\end{aligned}$$

where we denote by $\mathrm{v}_{\varepsilon,l}$ and $j_{\varepsilon,l}$ the l-th component of the vector fields v_ε and j_ε, respectively. Noting that $(F\cdot\nabla)G - \sum_l F_l\nabla G_l = F^\perp \mathrm{curl}\, G$, and using equation (1.5) for u_ε, this becomes

$$\begin{aligned}\mathrm{div}\ \tilde{S}_\varepsilon &= a\lambda_\varepsilon\left\langle (\alpha + i\beta|\log\varepsilon|)\partial_t u_\varepsilon, \nabla u_\varepsilon\right\rangle - a|\log\varepsilon|\langle\nabla u_\varepsilon, iF^\perp\cdot\nabla u_\varepsilon\rangle - a|\log\varepsilon| g j_\varepsilon \\ &\quad - \frac{a}{2}\nabla h\Big(|\nabla u_\varepsilon|^2 + N_\varepsilon^2|\mathrm{v}_\varepsilon|^2 - 2N_\varepsilon \mathrm{v}_\varepsilon\cdot j_\varepsilon + \frac{a}{\varepsilon^2}(1-|u_\varepsilon|^2)^2 + (1-|u_\varepsilon|^2)f\Big) \\ &\quad - \frac{a}{2}(1-|u_\varepsilon|^2)\nabla f - aN_\varepsilon\big(j_\varepsilon\nabla h\cdot \mathrm{v}_\varepsilon + \mathrm{v}_\varepsilon\nabla h\cdot j_\varepsilon - N_\varepsilon \mathrm{v}_\varepsilon\nabla h\cdot \mathrm{v}_\varepsilon\big) \\ &\quad + aN_\varepsilon\big(-\mathrm{v}_\varepsilon^\perp\mu_\varepsilon + (N_\varepsilon \mathrm{v}_\varepsilon - j_\varepsilon)^\perp \mathrm{curl}\ \mathrm{v}_\varepsilon - \mathrm{v}_\varepsilon \mathrm{div}\ j_\varepsilon + (N_\varepsilon \mathrm{v}_\varepsilon - j_\varepsilon)\,\mathrm{div}\ \mathrm{v}_\varepsilon\big). \qquad (4.15)\end{aligned}$$

Using identity (4.9), the first right-hand side term above may be rewritten as

$$
\begin{aligned}
&\lambda_\varepsilon \left\langle(\alpha + i\beta|\log\varepsilon|)\partial_t u_\varepsilon, \nabla u_\varepsilon\right\rangle \\
&= \lambda_\varepsilon\alpha \left\langle \partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_{\varepsilon,\varrho}, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon\right\rangle + N_\varepsilon\lambda_\varepsilon\alpha \mathrm{v}_\varepsilon\langle\partial_t u_\varepsilon, iu_\varepsilon\rangle \\
&\qquad + N_\varepsilon\lambda_\varepsilon\alpha\, \mathrm{p}_{\varepsilon,\varrho}\, j_\varepsilon - N_\varepsilon^2\lambda_\varepsilon\alpha|u_\varepsilon|^2\, \mathrm{p}_{\varepsilon,\varrho}\mathrm{v}_\varepsilon + \frac{\lambda_\varepsilon\beta}{2}|\log\varepsilon|V_\varepsilon \\
&= \lambda_\varepsilon\alpha \left\langle \partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_{\varepsilon,\varrho}, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon\right\rangle + N_\varepsilon \mathrm{v}_\varepsilon(\mathrm{div}\ j_\varepsilon + j_\varepsilon\cdot\nabla h) + \frac{1}{2}N_\varepsilon|\log\varepsilon|(F^\perp\cdot\nabla|u_\varepsilon|^2)\,\mathrm{v}_\varepsilon \\
&\qquad + \frac{\lambda_\varepsilon\beta}{2}N_\varepsilon|\log\varepsilon|\mathrm{v}_\varepsilon\partial_t(1-|u_\varepsilon|^2) + N_\varepsilon\lambda_\varepsilon\alpha\, \mathrm{p}_{\varepsilon,\varrho}\, j_\varepsilon - N_\varepsilon^2\lambda_\varepsilon\alpha|u_\varepsilon|^2\mathrm{p}_{\varepsilon,\varrho}\mathrm{v}_\varepsilon + \frac{\lambda_\varepsilon\beta}{2}|\log\varepsilon|V_\varepsilon.
\end{aligned}
$$

Inserting this into (4.15), recombining $|\nabla u_\varepsilon|^2 + N_\varepsilon^2|\mathrm{v}_\varepsilon|^2 - 2N_\varepsilon\mathrm{v}_\varepsilon\cdot j_\varepsilon = |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|^2 + N_\varepsilon^2(1-|u_\varepsilon|^2)|\mathrm{v}_\varepsilon|^2$, noting that $\langle\nabla u_\varepsilon, iF^\perp\cdot\nabla u_\varepsilon\rangle = -F\mu_\varepsilon/2$, and using (4.11) to transform V_ε into $\tilde V_{\varepsilon,\varrho}$, we obtain

$$
\begin{aligned}
\mathrm{div}\ \tilde S_\varepsilon &= a\lambda_\varepsilon\alpha \left\langle \partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_{\varepsilon,\varrho}, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon\right\rangle + aN_\varepsilon\mathrm{v}_\varepsilon(\mathrm{div}\ j_\varepsilon + j_\varepsilon\cdot\nabla h) \\
&\quad + \frac{a}{2}N_\varepsilon|\log\varepsilon|(F^\perp\cdot\nabla|u_\varepsilon|^2)\,\mathrm{v}_\varepsilon + a\lambda_\varepsilon\beta N_\varepsilon|\log\varepsilon|\mathrm{v}_\varepsilon\partial_t(1-|u_\varepsilon|^2) + \lambda_\varepsilon\alpha a N_\varepsilon\, \mathrm{p}_{\varepsilon,\varrho}\, j_\varepsilon \\
&\quad - aN_\varepsilon^2\lambda_\varepsilon\alpha|u_\varepsilon|^2\,\mathrm{p}_{\varepsilon,\varrho}\mathrm{v}_\varepsilon + \frac{a\lambda_\varepsilon\beta}{2}|\log\varepsilon|\tilde V_{\varepsilon,\varrho} + \frac{a\lambda_\varepsilon\beta}{2}N_\varepsilon|\log\varepsilon|\,\mathrm{p}_{\varepsilon,\varrho}\nabla|u_\varepsilon|^2 - a\mu_\varepsilon(N_\varepsilon\mathrm{v}_\varepsilon^\perp - |\log\varepsilon|F/2) \\
&\quad - \frac{a}{2}\nabla h\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|^2 + \frac{a}{\varepsilon^2}(1-|u_\varepsilon|^2)^2 + (1-|u_\varepsilon|^2)(N_\varepsilon^2|\mathrm{v}_\varepsilon|^2 + f)\Big) \\
&\quad - \frac{a}{2}(1-|u_\varepsilon|^2)\nabla f - aN_\varepsilon\big(j_\varepsilon\nabla h\cdot\mathrm{v}_\varepsilon + \mathrm{v}_\varepsilon\nabla h\cdot j_\varepsilon - N_\varepsilon\mathrm{v}_\varepsilon\nabla h\cdot\mathrm{v}_\varepsilon\big) \\
&\quad + aN_\varepsilon\big((N_\varepsilon\mathrm{v}_\varepsilon - j_\varepsilon)^\perp\mathrm{curl}\ \mathrm{v}_\varepsilon - \mathrm{v}_\varepsilon\mathrm{div}\ j_\varepsilon + (N_\varepsilon\mathrm{v}_\varepsilon - j_\varepsilon)\ \mathrm{div}\ \mathrm{v}_\varepsilon\,\big),
\end{aligned}
$$

and the result follows after straightforward simplifications. □

4.4 Time-derivative of the modulated energy excess

In the present section, we prove the following decomposition of the time-derivative of the modulated energy excess $\hat{\mathcal{D}}_{\varepsilon,\varrho,R}$. As will be seen in Sections 6–7, mean-field limit results are then essentially reduced to the estimation of the different terms in this decomposition. To simplify notation, it is stated here using truncations centered at $z=0$, but the translated result of course also holds for all $z\in\mathbb{R}^2$.

Lemma 4.4. *Let $\alpha\ge 0$, $\beta\in\mathbb{R}$, and let $h:\mathbb{R}^2\to\mathbb{R}$, $a:=e^h$, $F:\mathbb{R}^2\to\mathbb{R}^2$, $f:\mathbb{R}^2\to\mathbb{R}$ satisfy (2.1)–(2.2). Let $u_\varepsilon:[0,T)\times\mathbb{R}^2\to\mathbb{C}$ and $\mathrm{v}_\varepsilon:[0,T)\times\mathbb{R}^2\to\mathbb{R}^2$ be solutions of (1.5) and (3.1) as in Proposition 2.2 and as in Assumption B, respectively. Let $0<\varepsilon\ll 1$, $\varrho, R\gg 1$, and let $\bar\Gamma_\varepsilon:[0,T)\times\mathbb{R}^2\to\mathbb{R}^2$ be a given field with $\|\bar\Gamma_\varepsilon^t\|_{W^{1,\infty}}\lesssim_t 1$. Then, we have*

$$
\partial_t\hat{\mathcal{D}}_{\varepsilon,\varrho,R} = I^S_{\varepsilon,\varrho,R} + I^V_{\varepsilon,\varrho,R} + I^E_{\varepsilon,\varrho,R} + I^D_{\varepsilon,\varrho,R} + I^H_{\varepsilon,\varrho,R} + I^d_{\varepsilon,\varrho,R} + I^g_{\varepsilon,\varrho,R} + I^n_{\varepsilon,\varrho,R} + I'_{\varepsilon,\varrho,R},
$$

where we have set

$$
\begin{aligned}
I^S_{\varepsilon,\varrho,R} &:= -\int \chi_R \nabla \bar\Gamma_\varepsilon^\perp : \tilde S_\varepsilon,\\
I^V_{\varepsilon,\varrho,R} &:= \int \frac{a\chi_R}{2}\tilde V_{\varepsilon,\varrho}\cdot(-\lambda_\varepsilon\beta|\log\varepsilon|\Gamma_\varepsilon^\perp + |\log\varepsilon|(\nabla^\perp h - F^\perp) - 2N_\varepsilon \mathrm{v}_\varepsilon),\\
I^E_{\varepsilon,\varrho,R} &:= -\int \frac{a\chi_R}{2}\Gamma_\varepsilon\cdot(|\log\varepsilon|(\nabla^\perp h - F^\perp) - 2N_\varepsilon \mathrm{v}_\varepsilon)\mu_\varepsilon,\\
I^D_{\varepsilon,\varrho,R} &:= -\int \lambda_\varepsilon\alpha a\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_{\varepsilon,\varrho}|^2 - \int \lambda_\varepsilon \alpha a\chi_R\Gamma_\varepsilon^\perp\cdot\left\langle \partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{p}_{\varepsilon,\varrho}, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon\right\rangle,\\
I^H_{\varepsilon,\varrho,R} &:= \int \frac{a\chi_R}{2}\Gamma_\varepsilon^\perp\cdot\nabla h\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{\varepsilon^2}(1-|u_\varepsilon|^2)^2 - |\log\varepsilon|\mu_\varepsilon\Big),
\end{aligned}
$$

and also

$$
\begin{aligned}
I^d_{\varepsilon,\varrho,R} &:= \int a\chi_R N_\varepsilon\big(\bar\Gamma_\varepsilon^\perp\cdot(j_\varepsilon - N_\varepsilon\mathrm{v}_\varepsilon) + \langle \partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{p}_{\varepsilon,\varrho}, iu_\varepsilon\rangle\big)(\mathrm{div}\ \mathrm{v}_\varepsilon + \mathrm{v}_\varepsilon\cdot\nabla h - \lambda_\varepsilon\alpha\,\mathrm{p}_{\varepsilon,\varrho}),\\
I^g_{\varepsilon,\varrho,R} &:= \int a\chi_R N_\varepsilon(N_\varepsilon \mathrm{v}_\varepsilon - j_\varepsilon)\cdot(\Gamma_\varepsilon - \bar\Gamma_\varepsilon)\mathrm{curl}\ \mathrm{v}_\varepsilon + \int \lambda_\varepsilon\alpha a\chi_R(\Gamma_\varepsilon - \bar\Gamma_\varepsilon)^\perp\cdot\left\langle \partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{p}_{\varepsilon,\varrho}, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon\right\rangle\\
&\quad + \int \frac{a\chi_R}{2}(\bar\Gamma_\varepsilon - \Gamma_\varepsilon)^\perp\cdot\nabla h\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|^2 + \frac{a}{\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) + \int \frac{a\chi_R}{2}\lambda_\varepsilon\beta|\log\varepsilon|\tilde V_{\varepsilon,\varrho}\cdot(\Gamma_\varepsilon - \bar\Gamma_\varepsilon)^\perp\\
&\quad + \int a\chi_R(\bar\Gamma_\varepsilon - \Gamma_\varepsilon)\cdot(N_\varepsilon\mathrm{v}_\varepsilon + |\log\varepsilon|F^\perp/2)\mu_\varepsilon + \int a\chi_R\lambda_\varepsilon\beta N_\varepsilon|\log\varepsilon|(\bar\Gamma_\varepsilon - \Gamma_\varepsilon)^\perp\cdot\mathrm{v}_\varepsilon\,\partial_t|u_\varepsilon|^2,\\
I^n_{\varepsilon,\varrho,R} &:= -\int \nabla\chi_R\cdot\tilde S_\varepsilon\cdot\bar\Gamma_\varepsilon^\perp - \int a\nabla\chi_R\cdot\Big(\langle \partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{p}_{\varepsilon,\varrho}, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon\rangle + \frac{|\log\varepsilon|}{2}\tilde V_{\varepsilon,\varrho}^\perp\Big),
\end{aligned}
$$

and where the error $I'_{\varepsilon,\varrho,R}$ is estimated as follows, for all $\theta > 0$, in the dissipative case, in the considered regimes,

$$
\int_0^t |I'_{\varepsilon,\varrho,R}| \lesssim_{t,\theta} \varepsilon\big(\lambda_\varepsilon^{-1/2}N_\varepsilon^2 + R(1+\lambda_\varepsilon^2|\log\varepsilon|^2) + N_\varepsilon|\log\varepsilon|(1+\lambda_\varepsilon R^\theta)\big)(\mathcal{E}^*_{\varepsilon,R})^{1/2} \lesssim \varepsilon R|\log\varepsilon|^2(\mathcal{E}^*_{\varepsilon,R})^{1/2}, \tag{4.16}
$$

or in the Gross-Pitaevskii case (GP)*,*

$$
|I'_{\varepsilon,\varrho,R}| \lesssim_{t,\theta} \varepsilon N_\varepsilon\mathcal{E}^*_{\varepsilon,R} + N_\varepsilon(\mathcal{E}^*_{\varepsilon,R})^{1/2}\|\nabla(\mathrm{p}_\varepsilon - \mathrm{p}_{\varepsilon,\varrho})\|_{\mathrm{L}^2} + \varepsilon N_\varepsilon^2\varrho^\theta(\mathcal{E}^*_{\varepsilon,R})^{1/2}. \tag{4.17}
$$

Proof. We focus on the non-decaying setting, as the other case is similar. We split the proof into three steps, first computing the time-derivative $\partial_t\hat{\mathcal{E}}_{\varepsilon,\varrho,R}$, then deducing an expression for $\partial_t\hat{\mathcal{D}}_{\varepsilon,\varrho,R}$, and finally introducing the modulated stress-energy tensor to replace the linear terms by quadratic ones, which are better suited for a Grönwall argument.

Step 1: time-derivative of the modulated energy. In this step, we prove the following identity :

$$
\begin{aligned}
\partial_t\hat{\mathcal{E}}_{\varepsilon,\varrho,R} = &-\int a\nabla\chi_R\cdot\langle\partial_t u_\varepsilon, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon\rangle + \int \frac{aN_\varepsilon^2}{2}\partial_t\big((1-|u_\varepsilon|^2)(\psi_{\varepsilon,\varrho,R} - \chi_R|\mathrm{v}_\varepsilon|^2)\big)\\
&+ \int N_\varepsilon a\chi_R\langle\partial_t u_\varepsilon, iu_\varepsilon\rangle(\mathrm{div}\ \mathrm{v}_\varepsilon + \mathrm{v}_\varepsilon\cdot\nabla h)\\
&+ \int a\chi_R\Big(N_\varepsilon(N_\varepsilon\mathrm{v}_\varepsilon - j_\varepsilon)\cdot\partial_t\mathrm{v}_\varepsilon - \lambda_\varepsilon\alpha|\partial_t u_\varepsilon|^2 - N_\varepsilon\mathrm{v}_\varepsilon\cdot V_\varepsilon - \frac{|\log\varepsilon|}{2}F^\perp\cdot V_\varepsilon\Big).
\end{aligned} \tag{4.18}
$$

For that purpose, let us first compute the time-derivative of the modulated energy density

$$\begin{aligned}\frac{1}{2}\partial_t\Big(\chi_R|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a\chi_R}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 + (1-|u_\varepsilon|^2)(N_\varepsilon^2\psi_{\varepsilon,\varrho,R} + f\chi_R)\Big)\\ = \chi_R\langle \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon, \nabla\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\partial_t\mathrm{v}_\varepsilon - i\partial_t u_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon\rangle - \chi_R\langle\partial_t u_\varepsilon, \frac{au_\varepsilon}{\varepsilon^2}(1-|u_\varepsilon|^2)\rangle\\ + \frac{1}{2}\partial_t\big((1-|u_\varepsilon|^2)(N_\varepsilon^2\psi_{\varepsilon,\varrho,R} + f\chi_R)\big). \quad (4.19)\end{aligned}$$

Note that the first term in the right-hand side may be rewritten as

$$\begin{aligned}&\langle \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon, \nabla\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\partial_t\mathrm{v}_\varepsilon - i\partial_t u_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon\rangle\\ &= \langle\nabla u_\varepsilon, \nabla\partial_t u_\varepsilon\rangle - N_\varepsilon\partial_t\mathrm{v}_\varepsilon\cdot j_\varepsilon - N_\varepsilon\mathrm{v}_\varepsilon\cdot\langle\nabla u_\varepsilon, i\partial_t u_\varepsilon\rangle - N_\varepsilon\mathrm{v}_\varepsilon\cdot\langle iu_\varepsilon, \nabla\partial_t u_\varepsilon\rangle + \frac{N_\varepsilon^2}{2}|u_\varepsilon|^2\partial_t|\mathrm{v}_\varepsilon|^2 + \frac{N_\varepsilon^2}{2}|\mathrm{v}_\varepsilon|^2\partial_t|u_\varepsilon|^2\\ &= \operatorname{div}\langle\nabla u_\varepsilon, \partial_t u_\varepsilon\rangle - \langle\partial_t u_\varepsilon, \triangle u_\varepsilon\rangle - N_\varepsilon\partial_t\mathrm{v}_\varepsilon\cdot j_\varepsilon - N_\varepsilon\mathrm{v}_\varepsilon\cdot\langle\nabla u_\varepsilon, i\partial_t u_\varepsilon\rangle\\ &\qquad - N_\varepsilon\mathrm{v}_\varepsilon\cdot(\partial_t j_\varepsilon - \langle i\partial_t u_\varepsilon, \nabla u_\varepsilon\rangle) + \frac{N_\varepsilon^2}{2}\partial_t(|u_\varepsilon|^2|\mathrm{v}_\varepsilon|^2)\\ &= \operatorname{div}\langle\nabla u_\varepsilon, \partial_t u_\varepsilon\rangle - \langle\partial_t u_\varepsilon, \triangle u_\varepsilon\rangle - N_\varepsilon\mathrm{v}_\varepsilon\cdot\partial_t j_\varepsilon - N_\varepsilon j_\varepsilon\cdot\partial_t\mathrm{v}_\varepsilon + \frac{N_\varepsilon^2}{2}\partial_t(|u_\varepsilon|^2|\mathrm{v}_\varepsilon|^2), \quad (4.20)\end{aligned}$$

where

$$\begin{aligned}\operatorname{div}\langle\nabla u_\varepsilon, \partial_t u_\varepsilon\rangle &= \operatorname{div}\langle\partial_t u_\varepsilon, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon\rangle + \operatorname{div}(N_\varepsilon\mathrm{v}_\varepsilon\langle\partial_t u_\varepsilon, iu_\varepsilon\rangle)\\ &= \operatorname{div}\langle\partial_t u_\varepsilon, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon\rangle + N_\varepsilon\langle\partial_t u_\varepsilon, iu_\varepsilon\rangle\operatorname{div}\mathrm{v}_\varepsilon + N_\varepsilon\mathrm{v}_\varepsilon\cdot(\partial_t j_\varepsilon - V_\varepsilon). \quad (4.21)\end{aligned}$$

Combining (4.19), (4.20) and (4.21), the time-derivative of the energy density takes on the following guise, after straightforward simplifications,

$$\begin{aligned}\frac{1}{2}\partial_t\Big(\chi_R|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a\chi_R}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 + (1-|u_\varepsilon|^2)(N_\varepsilon^2\psi_{\varepsilon,\varrho,R} + f\chi_R)\Big)\\ = \chi_R\operatorname{div}\langle\partial_t u_\varepsilon, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon\rangle + N_\varepsilon\chi_R\langle\partial_t u_\varepsilon, iu_\varepsilon\rangle\operatorname{div}\mathrm{v}_\varepsilon - N_\varepsilon\chi_R\mathrm{v}_\varepsilon\cdot V_\varepsilon + N_\varepsilon\chi_R(N_\varepsilon\mathrm{v}_\varepsilon - j_\varepsilon)\cdot\partial_t\mathrm{v}_\varepsilon\\ - \chi_R\Big\langle\partial_t u_\varepsilon, \triangle u_\varepsilon + \frac{au_\varepsilon}{\varepsilon^2}(1-|u_\varepsilon|^2)\Big\rangle + \frac{1}{2}\partial_t\big((1-|u_\varepsilon|^2)(N_\varepsilon^2\psi_{\varepsilon,\varrho,R} - N_\varepsilon^2\chi_R|\mathrm{v}_\varepsilon|^2 + f\chi_R)\big).\end{aligned}$$

Integrating this identity in space yields

$$\begin{aligned}\partial_t\int\frac{a}{2}\Big(\chi_R|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a\chi_R}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 + (1-|u_\varepsilon|^2)(N_\varepsilon^2\psi_{\varepsilon,\varrho,R} + f\chi_R)\Big)\\ = \int a\chi_R\Big(N_\varepsilon\langle\partial_t u_\varepsilon, iu_\varepsilon\rangle\operatorname{div}\mathrm{v}_\varepsilon - N_\varepsilon\mathrm{v}_\varepsilon\cdot V_\varepsilon + N_\varepsilon(N_\varepsilon\mathrm{v}_\varepsilon - j_\varepsilon)\cdot\partial_t\mathrm{v}_\varepsilon - \Big\langle\partial_t u_\varepsilon, \triangle u_\varepsilon + \frac{au_\varepsilon}{\varepsilon^2}(1-|u_\varepsilon|^2)\Big\rangle\Big)\\ + \int\frac{a}{2}\partial_t\big((1-|u_\varepsilon|^2)(N_\varepsilon^2\psi_{\varepsilon,\varrho,R} - N_\varepsilon^2\chi_R|\mathrm{v}_\varepsilon|^2 + f\chi_R)\big) - \int\nabla(a\chi_R)\cdot\langle\partial_t u_\varepsilon, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon\rangle.\end{aligned}$$

Decomposing $\nabla(a\chi_R) = a\chi_R\nabla h + a\nabla\chi_R$, and using the equation (1.5) satisfied by u_ε in the form

$$\begin{aligned}\Big\langle\partial_t u_\varepsilon, \triangle u_\varepsilon + \frac{au_\varepsilon}{\varepsilon^2}(1-|u_\varepsilon|^2) + \nabla h\cdot\nabla u_\varepsilon\Big\rangle &= \big\langle\partial_t u_\varepsilon, \lambda_\varepsilon(\alpha + i\beta|\log\varepsilon|)\partial_t u_\varepsilon - i|\log\varepsilon|F^\perp\cdot\nabla u_\varepsilon - fu_\varepsilon\big\rangle\\ &= \lambda_\varepsilon\alpha|\partial_t u_\varepsilon|^2 + \frac{|\log\varepsilon|}{2}F^\perp\cdot V_\varepsilon - \frac{1}{2}f\partial_t|u_\varepsilon|^2,\end{aligned}$$

the result (4.18) follows after straightforward simplifications.

Step 2: time-derivative of the modulated energy excess. In this step, we prove the following identity:

$$
\begin{aligned}
\partial_t \hat{\mathcal{D}}_{\varepsilon,\varrho,R} &= \int \frac{a\chi_R}{2}\tilde{V}_{\varepsilon,\varrho}\cdot(|\log\varepsilon|(\nabla^\perp h - F^\perp) - 2N_\varepsilon \mathrm{v}_\varepsilon) + \int a\chi_R N_\varepsilon (N_\varepsilon \mathrm{v}_\varepsilon - j_\varepsilon)\cdot \Gamma_\varepsilon \mathrm{curl}\ \mathrm{v}_\varepsilon \\
&\quad - \int \lambda_\varepsilon \alpha a\chi_R |\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_{\varepsilon,\varrho}|^2 + \int a\chi_R N_\varepsilon \langle \partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_\varepsilon, iu_\varepsilon\rangle(\mathrm{div}\ \mathrm{v}_\varepsilon + \mathrm{v}_\varepsilon\cdot\nabla h - \lambda_\varepsilon\alpha\,\mathrm{p}_\varepsilon) \\
&\quad + \int a\chi_R N_\varepsilon (N_\varepsilon \mathrm{v}_\varepsilon - j_\varepsilon)\cdot\nabla(\mathrm{p}_\varepsilon - \mathrm{p}_{\varepsilon,\varrho}) - \int a\nabla\chi_R\cdot\Big(\langle\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_{\varepsilon,\varrho}, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon\rangle + \frac{|\log\varepsilon|}{2}\tilde{V}_{\varepsilon,\varrho}^\perp\Big) \\
&\quad - \int aN_\varepsilon^2 \mathrm{p}_\varepsilon (1-|u_\varepsilon|^2)\big(\mathrm{v}_\varepsilon\cdot\nabla\chi_R + \chi_R(\mathrm{div}\ \mathrm{v}_\varepsilon + \mathrm{v}_\varepsilon\cdot\nabla h)\big) + \int \frac{aN_\varepsilon^2}{2}\partial_t\big((1-|u_\varepsilon|^2)(\psi_{\varepsilon,\varrho,R} - \chi_R|\mathrm{v}_\varepsilon|^2)\big) \\
&\quad - \int \frac{aN_\varepsilon|\log\varepsilon|}{2}\partial_t(1-|u_\varepsilon|^2)\Big(\mathrm{v}_\varepsilon^\perp\cdot\nabla\chi_R + \lambda_\varepsilon\beta\chi_R \mathrm{p}_{\varepsilon,\varrho} - \chi_R \mathrm{v}_\varepsilon\cdot\Big(\nabla^\perp h - F^\perp - 2\frac{N_\varepsilon}{|\log\varepsilon|}\mathrm{v}_\varepsilon\Big)\Big) \\
&\quad - \int \frac{aN_\varepsilon|\log\varepsilon|}{2}\mathrm{p}_{\varepsilon,\varrho}\nabla(1-|u_\varepsilon|^2)\cdot\Big(\nabla^\perp\chi_R + \chi_R\Big(\nabla^\perp h - 2F^\perp - 2\frac{N_\varepsilon}{|\log\varepsilon|}\mathrm{v}_\varepsilon\Big)\Big). \qquad (4.22)
\end{aligned}
$$

Noting that by identity (4.8) we have

$$
|\log\varepsilon|\int a\chi_R\partial_t\mu_\varepsilon = |\log\varepsilon|\int a\chi_R \mathrm{curl}\, V_\varepsilon = -|\log\varepsilon|\int a\chi_R V_\varepsilon\cdot\nabla^\perp h - |\log\varepsilon|\int aV_\varepsilon\cdot\nabla^\perp\chi_R,
$$

it is immediate to deduce from (4.18) the following identity for the time-derivative of the modulated energy excess,

$$
\begin{aligned}
\partial_t \hat{\mathcal{D}}_{\varepsilon,\varrho,R} &= \int \frac{a\chi_R}{2} V_\varepsilon\cdot(|\log\varepsilon|(\nabla^\perp h - F^\perp) - 2N_\varepsilon\mathrm{v}_\varepsilon) + \int aN_\varepsilon\chi_R\langle\partial_t u_\varepsilon, iu_\varepsilon\rangle(\mathrm{div}\ \mathrm{v}_\varepsilon + \mathrm{v}_\varepsilon\cdot\nabla h) \\
&\quad + \int a\chi_R N_\varepsilon(N_\varepsilon\mathrm{v}_\varepsilon - j_\varepsilon)\cdot\partial_t\mathrm{v}_\varepsilon - \int\lambda_\varepsilon\alpha a\chi_R|\partial_t u_\varepsilon|^2 + \int\frac{aN_\varepsilon^2}{2}\partial_t\big((1-|u_\varepsilon|^2)(\psi_{\varepsilon,\varrho,R} - \chi_R|\mathrm{v}_\varepsilon|^2)\big) \\
&\quad - \int a\nabla\chi_R\cdot\Big(\langle\partial_t u_\varepsilon, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon\rangle + \frac{|\log\varepsilon|}{2}V_\varepsilon^\perp\Big). \qquad (4.23)
\end{aligned}
$$

Now using equation (3.1) for the time evolution of v_ε and an integration by parts, we find

$$
\begin{aligned}
&\int a\chi_R N_\varepsilon(N_\varepsilon\mathrm{v}_\varepsilon - j_\varepsilon)\cdot\partial_t\mathrm{v}_\varepsilon \\
&= \int a\chi_R N_\varepsilon(N_\varepsilon\mathrm{v}_\varepsilon - j_\varepsilon)\cdot\Gamma_\varepsilon\mathrm{curl}\ \mathrm{v}_\varepsilon + \int a\chi_R N_\varepsilon(N_\varepsilon\mathrm{v}_\varepsilon - j_\varepsilon)\cdot\nabla\mathrm{p}_\varepsilon \\
&= \int a\chi_R N_\varepsilon(N_\varepsilon\mathrm{v}_\varepsilon - j_\varepsilon)\cdot\Gamma_\varepsilon\mathrm{curl}\ \mathrm{v}_\varepsilon + \int a\chi_R N_\varepsilon(N_\varepsilon\mathrm{v}_\varepsilon - j_\varepsilon)\cdot\nabla(\mathrm{p}_\varepsilon - \mathrm{p}_{\varepsilon,\varrho}) \\
&\quad - \int a\chi_R N_\varepsilon\mathrm{p}_{\varepsilon,\varrho}(N_\varepsilon\,\mathrm{div}\ \mathrm{v}_\varepsilon - \mathrm{div}\ j_\varepsilon) - \int a\chi_R N_\varepsilon\mathrm{p}_{\varepsilon,\varrho}\,\nabla h\cdot(N_\varepsilon\mathrm{v}_\varepsilon - j_\varepsilon) - \int aN_\varepsilon\mathrm{p}_{\varepsilon,\varrho}\,\nabla\chi_R\cdot(N_\varepsilon\mathrm{v}_\varepsilon - j_\varepsilon).
\end{aligned}
$$

Combining this with identity (4.9) yields

$$\begin{aligned}
&\int a\chi_R N_\varepsilon(N_\varepsilon \mathrm{v}_\varepsilon - j_\varepsilon)\cdot\partial_t \mathrm{v}_\varepsilon \\
&= \int a\chi_R N_\varepsilon(N_\varepsilon \mathrm{v}_\varepsilon - j_\varepsilon)\cdot\Gamma_\varepsilon \mathrm{curl}\ \mathrm{v}_\varepsilon + \int a\chi_R N_\varepsilon(N_\varepsilon \mathrm{v}_\varepsilon - j_\varepsilon)\cdot\nabla(\mathrm{p}_\varepsilon - \mathrm{p}_{\varepsilon,\varrho}) \\
&\quad - \int a\chi_R N_\varepsilon \mathrm{p}_{\varepsilon,\varrho}\nabla h\cdot(N_\varepsilon \mathrm{v}_\varepsilon - j_\varepsilon) - \int aN_\varepsilon \mathrm{p}_{\varepsilon,\varrho}\nabla\chi_R\cdot(N_\varepsilon \mathrm{v}_\varepsilon - j_\varepsilon) \\
&\quad - \int a\chi_R N_\varepsilon \mathrm{p}_{\varepsilon,\varrho}\Big(N_\varepsilon \operatorname{div}\ \mathrm{v}_\varepsilon + j_\varepsilon\cdot\nabla h - \lambda_\varepsilon\alpha\langle\partial_t u_\varepsilon, iu_\varepsilon\rangle + \frac{|\log\varepsilon|}{2}F^\perp\cdot\nabla|u_\varepsilon|^2 - \frac{\lambda_\varepsilon\beta|\log\varepsilon|}{2}\partial_t|u_\varepsilon|^2\Big) \\
&= \int a\chi_R N_\varepsilon(N_\varepsilon \mathrm{v}_\varepsilon - j_\varepsilon)\cdot\Gamma_\varepsilon \mathrm{curl}\ \mathrm{v}_\varepsilon + \int a\chi_R N_\varepsilon(N_\varepsilon \mathrm{v}_\varepsilon - j_\varepsilon)\cdot\nabla(\mathrm{p}_\varepsilon - \mathrm{p}_{\varepsilon,\varrho}) - \int a\chi_R N_\varepsilon^2\, \mathrm{p}_{\varepsilon,\varrho}(\operatorname{div}\ \mathrm{v}_\varepsilon + \mathrm{v}_\varepsilon\cdot\nabla h) \\
&\quad - \int aN_\varepsilon \mathrm{p}_{\varepsilon,\varrho}\nabla\chi_R\cdot(N_\varepsilon \mathrm{v}_\varepsilon - j_\varepsilon) + \int a\chi_R N_\varepsilon \mathrm{p}_{\varepsilon,\varrho}\Big(\lambda_\varepsilon\alpha\langle\partial_t u_\varepsilon, iu_\varepsilon\rangle - \frac{|\log\varepsilon|}{2}F^\perp\cdot\nabla|u_\varepsilon|^2 + \frac{\lambda_\varepsilon\beta|\log\varepsilon|}{2}\partial_t|u_\varepsilon|^2\Big).
\end{aligned}$$

Inserting this into (4.23), we then find

$$\begin{aligned}
\partial_t\hat{\mathcal{D}}_{\varepsilon,\varrho,R} &= \int \frac{a\chi_R}{2}V_\varepsilon\cdot\big(|\log\varepsilon|(\nabla^\perp h - F^\perp) - 2N_\varepsilon \mathrm{v}_\varepsilon\big) + \int a\chi_R N_\varepsilon\langle\partial_t u_\varepsilon, iu_\varepsilon\rangle(\operatorname{div}\ \mathrm{v}_\varepsilon + \mathrm{v}_\varepsilon\cdot\nabla h + \lambda_\varepsilon\alpha\,\mathrm{p}_{\varepsilon,\varrho}) \\
&- \int a\chi_R N_\varepsilon^2\,\mathrm{p}_{\varepsilon,\varrho}(\operatorname{div}\ \mathrm{v}_\varepsilon + \mathrm{v}_\varepsilon\cdot\nabla h) + \int a\chi_R N_\varepsilon(N_\varepsilon \mathrm{v}_\varepsilon - j_\varepsilon)\cdot\Gamma_\varepsilon \mathrm{curl}\ \mathrm{v}_\varepsilon + \int a\chi_R N_\varepsilon(N_\varepsilon \mathrm{v}_\varepsilon - j_\varepsilon)\cdot\nabla(\mathrm{p}_\varepsilon - \mathrm{p}_{\varepsilon,\varrho}) \\
&\quad + \int \frac{aN_\varepsilon^2}{2}\partial_t\big((1-|u_\varepsilon|^2)(\psi_{\varepsilon,\varrho,R} - \chi_R|\mathrm{v}_\varepsilon|^2)\big) + \int \frac{a\chi_R}{2}N_\varepsilon|\log\varepsilon|\mathrm{p}_{\varepsilon,\varrho}\big(\lambda_\varepsilon\beta\partial_t|u_\varepsilon|^2 - F^\perp\cdot\nabla|u_\varepsilon|^2\big) \\
&\quad - \int \lambda_\varepsilon\alpha a\chi_R|\partial_t u_\varepsilon|^2 - \int a\nabla\chi_R\cdot\Big(\langle\partial_t u_\varepsilon, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon\rangle + \frac{|\log\varepsilon|}{2}V_\varepsilon^\perp + N_\varepsilon \mathrm{p}_{\varepsilon,\varrho}(N_\varepsilon \mathrm{v}_\varepsilon - j_\varepsilon)\Big). \qquad (4.24)
\end{aligned}$$

Using identity (4.11) to transform V_ε into $\tilde V_{\varepsilon,\varrho}$, the first right-hand side term may be rewritten as

$$\begin{aligned}
&\int \frac{a\chi_R}{2}V_\varepsilon\cdot(|\log\varepsilon|(\nabla^\perp h - F^\perp) - 2N_\varepsilon \mathrm{v}_\varepsilon) \\
&\qquad = \int \frac{a\chi_R}{2}\big(\tilde V_{\varepsilon,\varrho} + N_\varepsilon \mathrm{v}_\varepsilon\partial_t(1-|u_\varepsilon|^2) + N_\varepsilon \mathrm{p}_{\varepsilon,\varrho}\nabla|u_\varepsilon|^2\big)\cdot(|\log\varepsilon|(\nabla^\perp h - F^\perp) - 2N_\varepsilon \mathrm{v}_\varepsilon),
\end{aligned}$$

while the last right-hand side term of (4.24) becomes

$$\begin{aligned}
&\int a\nabla\chi_R\cdot\Big(\langle\partial_t u_\varepsilon, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon\rangle + \frac{|\log\varepsilon|}{2}V_\varepsilon^\perp + N_\varepsilon \mathrm{p}_{\varepsilon,\varrho}(N_\varepsilon \mathrm{v}_\varepsilon - j_\varepsilon)\Big) \\
&\qquad = \int a\nabla\chi_R\cdot\Big(\langle\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_{\varepsilon,\varrho}, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon\rangle + N_\varepsilon^2\,\mathrm{p}_{\varepsilon,\varrho}\mathrm{v}_\varepsilon(1-|u_\varepsilon|^2) \\
&\qquad\qquad + \frac{|\log\varepsilon|}{2}\tilde V_{\varepsilon,\varrho}^\perp + \frac{N_\varepsilon|\log\varepsilon|}{2}\mathrm{v}_\varepsilon^\perp\partial_t(1-|u_\varepsilon|^2) + \frac{N_\varepsilon|\log\varepsilon|}{2}\,\mathrm{p}_{\varepsilon,\varrho}\nabla^\perp|u_\varepsilon|^2\Big).
\end{aligned}$$

Further decomposing

$$\begin{aligned}
|\partial_t u_\varepsilon|^2 &= |\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_{\varepsilon,\varrho}|^2 + 2N_\varepsilon \mathrm{p}_\varepsilon\langle\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_{\varepsilon,\varrho}, iu_\varepsilon\rangle + N_\varepsilon^2|\mathrm{p}_{\varepsilon,\varrho}|^2 - (1-|u_\varepsilon|^2)N_\varepsilon^2|\mathrm{p}_{\varepsilon,\varrho}|^2, \\
\langle\partial_t u_\varepsilon, iu_\varepsilon\rangle &= \langle\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_{\varepsilon,\varrho}, iu_\varepsilon\rangle + |u_\varepsilon|^2 N_\varepsilon \mathrm{p}_{\varepsilon,\varrho},
\end{aligned}$$

the result (4.22) easily follows after straightforward simplifications.

Step 3: conclusion. In the right-hand side of (4.22), the term $\int a\chi_R N_\varepsilon(N_\varepsilon \mathrm{v}_\varepsilon - j_\varepsilon)\cdot\Gamma_\varepsilon \mathrm{curl}\,\mathrm{v}_\varepsilon$ is linear in $N_\varepsilon\mathrm{v}_\varepsilon - j_\varepsilon$, preventing a direct Grönwall argument. As already explained, just as in [82], the idea is to replace this bad term by others involving the modulated stress-energy tensor $\tilde S_\varepsilon$, which is indeed a nicer *quadratic* quantity. For that purpose, let us integrate the result of Lemma 4.3 in space against $\chi_R\bar\Gamma_\varepsilon^\perp$, where $\bar\Gamma_\varepsilon : [0,T)\to W^{1,\infty}(\mathbb{R}^d)$ is a given field (we would like to simply choose $\bar\Gamma_\varepsilon = \Gamma_\varepsilon$, but as we will see a suitable perturbation of it is needed), and obtain

$$\begin{aligned}\int \chi_R\bar\Gamma_\varepsilon^\perp\cdot\mathrm{div}\,\tilde S_\varepsilon &= \int \lambda_\varepsilon\alpha a\chi_R\bar\Gamma_\varepsilon^\perp\cdot\langle\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_{\varepsilon,\varrho}, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon\rangle - \int a\chi_R\bar\Gamma_\varepsilon\cdot(N_\varepsilon\mathrm{v}_\varepsilon + |\log\varepsilon|F^\perp/2)\mu_\varepsilon\\
&+\int a\chi_R N_\varepsilon(N_\varepsilon\mathrm{v}_\varepsilon - j_\varepsilon)\cdot\bar\Gamma_\varepsilon\mathrm{curl}\,\mathrm{v}_\varepsilon + \int\lambda_\varepsilon\beta\frac{a\chi_R}{2}|\log\varepsilon|\bar\Gamma_\varepsilon^\perp\cdot\tilde V_{\varepsilon,\varrho} + \int\lambda_\varepsilon\beta\frac{a\chi_R}{2}N_\varepsilon|\log\varepsilon|\,\mathrm{p}_{\varepsilon,\varrho}\,\bar\Gamma_\varepsilon^\perp\cdot\nabla|u_\varepsilon|^2\\
&+\int a\chi_R N_\varepsilon\bar\Gamma_\varepsilon^\perp\cdot(N_\varepsilon\mathrm{v}_\varepsilon - j_\varepsilon)(\mathrm{div}\,\mathrm{v}_\varepsilon + \nabla h\cdot\mathrm{v}_\varepsilon - \lambda_\varepsilon\alpha\,\mathrm{p}_{\varepsilon,\varrho}) + \int a\chi_R\lambda_\varepsilon\beta N_\varepsilon|\log\varepsilon|\partial_t(1-|u_\varepsilon|^2)(\bar\Gamma_\varepsilon^\perp\cdot\mathrm{v}_\varepsilon)\\
&-\int\frac{a\chi_R}{2}\bar\Gamma_\varepsilon^\perp\cdot\nabla h\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|^2 + \frac{a}{\varepsilon^2}(1-|u_\varepsilon|^2)^2 + (1-|u_\varepsilon|^2)(N_\varepsilon^2|\mathrm{v}_\varepsilon|^2+f)\Big) - \int\frac{a\chi_R}{2}(1-|u_\varepsilon|^2)\bar\Gamma_\varepsilon^\perp\cdot\nabla f\\
&+\int\lambda_\varepsilon\alpha a\chi_R N_\varepsilon^2\mathrm{p}_{\varepsilon,\varrho}(1-|u_\varepsilon|^2)(\bar\Gamma_\varepsilon^\perp\cdot\mathrm{v}_\varepsilon) + \int\frac{a\chi_R}{2}N_\varepsilon|\log\varepsilon|(F^\perp\cdot\nabla|u_\varepsilon|^2)(\bar\Gamma_\varepsilon^\perp\cdot\mathrm{v}_\varepsilon).\end{aligned}$$

In this last right-hand side, the term $\int a\chi_R N_\varepsilon(N_\varepsilon\mathrm{v}_\varepsilon - j_\varepsilon)\cdot\bar\Gamma_\varepsilon\mathrm{curl}\,\mathrm{v}_\varepsilon$ exactly corresponds to the bad term in the right-hand side of (4.22). Replacing it by this new expression involving the modulated stress-energy tensor, and treating as errors all the terms involving the difference $\bar\Gamma_\varepsilon - \Gamma_\varepsilon$, we find

$$\begin{aligned}\partial_t\hat{\mathcal D}_{\varepsilon,\varrho,R} &= \sum_{j=0}^3 T^j_{\varepsilon,R} + I^g_{\varepsilon,\varrho,R} + I^n_{\varepsilon,\varrho,R} - \int\chi_R\nabla\bar\Gamma_\varepsilon^\perp : \tilde S_\varepsilon - \int\lambda_\varepsilon\alpha a\chi_R\Gamma_\varepsilon^\perp\cdot\langle\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{p}_{\varepsilon,\varrho}, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon\rangle\\
&+\int\frac{a\chi_R}{2}\Gamma_\varepsilon^\perp\cdot\nabla h\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|^2 + \frac{a}{\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) - \int\lambda_\varepsilon\alpha a\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{p}_{\varepsilon,\varrho}|^2\\
&+\int a\chi_R\Gamma_\varepsilon\cdot(N_\varepsilon\mathrm{v}_\varepsilon + |\log\varepsilon|F^\perp/2)\mu_\varepsilon + \int\frac{a\chi_R}{2}\tilde V_{\varepsilon,\varrho}\cdot(-\lambda_\varepsilon\beta|\log\varepsilon|\Gamma_\varepsilon^\perp + |\log\varepsilon|(\nabla^\perp h - F^\perp) - 2N_\varepsilon\mathrm{v}_\varepsilon)\\
&+\int a\chi_R N_\varepsilon\big(\langle\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{p}_{\varepsilon,\varrho}, iu_\varepsilon\rangle + \bar\Gamma_\varepsilon^\perp\cdot(j_\varepsilon - N_\varepsilon\mathrm{v}_\varepsilon)\big)(\mathrm{div}\,\mathrm{v}_\varepsilon + \mathrm{v}_\varepsilon\cdot\nabla h - \lambda_\varepsilon\alpha\,\mathrm{p}_{\varepsilon,\varrho}),\end{aligned}$$

where $I^g_{\varepsilon,\varrho,R}$ and $I^n_{\varepsilon,\varrho,R}$ are given as in the statement, and where we have set

$$\begin{aligned}T^0_{\varepsilon,\varrho,R} &:= \int a\chi_R N_\varepsilon(N_\varepsilon\mathrm{v}_\varepsilon - j_\varepsilon)\cdot\nabla(\mathrm{p}_\varepsilon - \mathrm{p}_{\varepsilon,\varrho}),\\
T^1_{\varepsilon,\varrho,R} &:= \int\frac{a\chi_R}{2}(1-|u_\varepsilon|^2)(N_\varepsilon^2|\mathrm{v}_\varepsilon|^2+f)\bar\Gamma_\varepsilon^\perp\cdot\nabla h - \int aN_\varepsilon^2\mathrm{p}_{\varepsilon,\varrho}(1-|u_\varepsilon|^2)\big(\mathrm{v}_\varepsilon\cdot\nabla\chi_R + \chi_R(\mathrm{div}\,\mathrm{v}_\varepsilon + \mathrm{v}_\varepsilon\cdot\nabla h)\big)\\
&\quad+\int\frac{a\chi_R}{2}(1-|u_\varepsilon|^2)\bar\Gamma_\varepsilon^\perp\cdot\nabla f - \int\lambda_\varepsilon\alpha a\chi_R N_\varepsilon^2\,\mathrm{p}_{\varepsilon,\varrho}(1-|u_\varepsilon|^2)\bar\Gamma_\varepsilon^\perp\cdot\mathrm{v}_\varepsilon,\\
T^2_{\varepsilon,\varrho,R} &:= -\int\frac{aN_\varepsilon|\log\varepsilon|}{2}\,\mathrm{p}_{\varepsilon,\varrho}\nabla(1-|u_\varepsilon|^2)\cdot\Big(\nabla^\perp\chi_R + \chi_R\Big(\nabla^\perp h - 2F^\perp - \lambda_\varepsilon\beta\bar\Gamma_\varepsilon^\perp - 2\frac{N_\varepsilon}{|\log\varepsilon|}\mathrm{v}_\varepsilon\Big)\Big)\\
&\quad+\int\frac{a\chi_R}{2}N_\varepsilon|\log\varepsilon|\big(F^\perp\cdot\nabla(1-|u_\varepsilon|^2)\big)\bar\Gamma_\varepsilon^\perp\cdot\mathrm{v}_\varepsilon,\\
T^3_{\varepsilon,\varrho,R} &:= -\int\frac{aN_\varepsilon|\log\varepsilon|}{2}\partial_t(1-|u_\varepsilon|^2)\Big(\mathrm{v}_\varepsilon^\perp\cdot\nabla\chi_R + \lambda_\varepsilon\beta\chi_R\mathrm{p}_{\varepsilon,\varrho} - \chi_R\mathrm{v}_\varepsilon\cdot\Big(\nabla^\perp h - F^\perp - 2\lambda_\varepsilon\beta\Gamma_\varepsilon^\perp - 2\frac{N_\varepsilon}{|\log\varepsilon|}\mathrm{v}_\varepsilon\Big)\Big)\\
&\quad+\int\frac{aN_\varepsilon^2}{2}\partial_t\big((1-|u_\varepsilon|^2)(\psi_{\varepsilon,\varrho,R} - \chi_R|\mathrm{v}_\varepsilon|^2)\big).\end{aligned}$$

It remains to estimate these four error terms $T^i_{\varepsilon,\varrho,R}$, $0 \le i \le 3$. First consider the term $T^0_{\varepsilon,\varrho,R}$. In the dissipative case we take $\varrho = \infty$, and $T^0_{\varepsilon,\varrho,R} = 0$. In the Gross-Pitaevskii case, using the pointwise estimate of Lemma 4.2 for $j_\varepsilon - N_\varepsilon \mathrm{v}_\varepsilon$, and using Assumption B(b), with in particular

$$\|\nabla(\mathrm{p}^t_\varepsilon - \mathrm{p}^t_{\varepsilon,\varrho})\|_{\mathrm{L}^2\cap\mathrm{L}^\infty} \lesssim \|\nabla \mathrm{p}^t_\varepsilon\|_{\mathrm{L}^2\cap\mathrm{L}^\infty} + \varrho^{-1}\|\mathrm{p}^t_{\varepsilon,\varrho}\|_{\mathrm{L}^2\cap\mathrm{L}^\infty} \lesssim_t 1,$$

we find

$$\begin{aligned}|T^0_{\varepsilon,\varrho,R}| &\lesssim_t N_\varepsilon\|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon\|_{\mathrm{L}^2(B_{2R})}(\|\nabla(\mathrm{p}_\varepsilon - \mathrm{p}_{\varepsilon,\varrho})\|_{\mathrm{L}^2} + \|1-|u_\varepsilon|^2\|_{\mathrm{L}^2(B_{2R})}) \\ &\qquad + N_\varepsilon^2\|1-|u_\varepsilon|^2\|_{\mathrm{L}^2(B_{2R})}\|\nabla(\mathrm{p}_\varepsilon - \mathrm{p}_{\varepsilon,\varrho})\|_{\mathrm{L}^2} \\ &\lesssim_t \varepsilon N_\varepsilon \mathcal{E}^*_{\varepsilon,R} + (1+\varepsilon N_\varepsilon)N_\varepsilon(\mathcal{E}^*_{\varepsilon,R})^{1/2}\|\nabla(\mathrm{p}_\varepsilon - \mathrm{p}_{\varepsilon,\varrho})\|_{\mathrm{L}^2}.\end{aligned}$$

Second, using (2.1)–(2.2), Assumption B, and the assumption $\|\bar\Gamma_\varepsilon\|_{\mathrm{L}^\infty} \lesssim_t 1$, we obtain in the dissipative case

$$|T^1_{\varepsilon,\varrho,R}| \lesssim_t \varepsilon\big(\lambda_\varepsilon^{-1/2}N_\varepsilon^2 + R(1+\lambda_\varepsilon^2|\log\varepsilon|^2)\big)(\mathcal{E}^*_{\varepsilon,R})^{1/2},$$

or in the Gross-Pitaevskii case,

$$|T^1_{\varepsilon,\varrho,R}| \lesssim_t \varepsilon(N_\varepsilon^2 + \lambda_\varepsilon^2|\log\varepsilon|^2)(\mathcal{E}^*_{\varepsilon,R})^{1/2}.$$

Integrating by parts, $T^2_{\varepsilon,\varrho,R}$ takes the form

$$\begin{aligned}T^2_{\varepsilon,\varrho,R} = -\int &\frac{N_\varepsilon|\log\varepsilon|}{2}(1-|u_\varepsilon|^2) \\ &\times \operatorname{div}\bigg(a\mathrm{p}_{\varepsilon,\varrho}\nabla^\perp\chi_R + a\chi_R F^\perp(\bar\Gamma_\varepsilon^\perp\cdot\mathrm{v}_\varepsilon) + a\mathrm{p}_{\varepsilon,\varrho}\chi_R\Big(\nabla^\perp h - 2F^\perp - \lambda_\varepsilon\beta\bar\Gamma_\varepsilon^\perp - 2\frac{N_\varepsilon}{|\log\varepsilon|}\mathrm{v}_\varepsilon\Big)\bigg),\end{aligned}$$

and hence, again using (2.1)–(2.2), Assumption B, and the bound $\|\bar\Gamma_\varepsilon\|_{W^{1,\infty}} \lesssim 1$, we obtain, for all $\theta > 0$, in the dissipative case, in the considered regimes,

$$\|T^2_{\varepsilon,\varrho,R}\|_{\mathrm{L}^1_t} \lesssim_{t,\theta} \varepsilon N_\varepsilon|\log\varepsilon|(1+\lambda_\varepsilon R^\theta)(\mathcal{E}^*_{\varepsilon,R})^{1/2},$$

or in the Gross-Pitaevskii case,

$$|T^{2,t}_{\varepsilon,\varrho,R}| \lesssim_{t,\theta} \varepsilon N_\varepsilon|\log\varepsilon|(1+\lambda_\varepsilon\varrho^\theta)(\mathcal{E}^*_{\varepsilon,R})^{1/2} \lesssim \varepsilon N_\varepsilon^2\varrho^\theta(\mathcal{E}^*_{\varepsilon,R})^{1/2}.$$

Finally, we observe that the choice (4.3) of $\psi_{\varepsilon,\varrho,R}$ exactly yields

$$T^3_{\varepsilon,\varrho,R} = \int\frac{aN_\varepsilon^2}{2}(1-|u_\varepsilon|^2)\partial_t(\psi_{\varepsilon,\varrho,R} - \chi_R|\mathrm{v}_\varepsilon|^2) = \int\frac{aN_\varepsilon^2}{2}(1-|u_\varepsilon|^2)(\partial_t\psi_{\varepsilon,\varrho,R} - 2\chi_R\,\mathrm{v}_\varepsilon\cdot\partial_t\mathrm{v}_\varepsilon),$$

and hence, using (4.4)–(4.5) and Assumption B, in the dissipative case, we find

$$\|T^3_{\varepsilon,\varrho,R}\|_{\mathrm{L}^1_t} \lesssim \varepsilon N_\varepsilon^2\Big(1+\frac{|\log\varepsilon|}{N_\varepsilon}\Big)(\mathcal{E}^*_{\varepsilon,R})^{1/2} \lesssim \varepsilon N_\varepsilon|\log\varepsilon|(\mathcal{E}^*_{\varepsilon,R})^{1/2},$$

or in the Gross-Pitaevskii case,

$$|T^{3,t}_{\varepsilon,\varrho,R}| \lesssim \varepsilon N_\varepsilon^2\varrho^\theta(\mathcal{E}^*_{\varepsilon,R})^{1/2}.$$

The estimate (4.16) now follows from the above with $I'_{\varepsilon,\varrho,R} := T^0_{\varepsilon,\varrho,R} + T^1_{\varepsilon,\varrho,R} + T^2_{\varepsilon,\varrho,R} + T^3_{\varepsilon,\varrho,R}$. □

5 Vortex analysis

In this section, we first recall and revisit some standard tools for vortex analysis, which are needed in order to control the various terms appearing in the decomposition in Lemma 4.4. They will only be used in the dissipative case, so we may restrict to the situation when $N_\varepsilon \lesssim |\log \varepsilon|$.

5.1 Ball construction lower bounds

We need a version of the ball construction lower bounds à la Jerrard-Sandier [72, 52] which is *localizable* in order to be adapted both to the weighted case and to the setting of the infinite plane with no finite energy control (hence no a priori finiteness assumption on the number of vortices), and which further yields very small errors (we need an error of order $o(N_\varepsilon^2)$, which gets very small when N_ε diverges slowly). For that purpose we use the version developed in [75], which in particular allows to cover the plane with balls centered at the points of the lattice $R\mathbb{Z}^2$, make the standard ball construction in each ball of the covering, assemble all the constructed balls, and then discard some balls from the collection so as to make it disjoint again. The error in the lower bounds given by this ball construction is essentially $N_\varepsilon |\log r|$, where r is the total radius of the balls, so that we need to take r large enough (almost as large as $O(1)$ when N_ε diverges slowly), but here the pinning weight adds again a difficulty since it may vary significantly over the size of the balls of this construction, thus perturbing the lower bound itself.

The following preliminary result describes the precise contribution of the vortices to the energy, and in particular defines the vortex "locations".

Lemma 5.1 (Localized lower bound). *Let $h : \mathbb{R}^2 \to \mathbb{R}$, $a := e^h$, with $1 \lesssim a \leq 1$, let $u_\varepsilon : \mathbb{R}^2 \to \mathbb{C}$, $\mathrm{v}_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2$, with $\|\mathrm{curl}\, \mathrm{v}_\varepsilon\|_{\mathrm{L}^2 \cap \mathrm{L}^\infty} \lesssim 1$. Let $0 < \varepsilon \ll 1$, $1 \leq N_\varepsilon \lesssim |\log \varepsilon|$, $R \geq 1$, and assume that $\log \mathcal{E}^*_{\varepsilon,R} \ll |\log \varepsilon|$. Then, for some $\bar r \simeq 1$, for all $\varepsilon > 0$ small enough, and all $r \in (\varepsilon^{1/2}, \bar r)$, there exists a locally finite union of disjoint closed balls $\mathcal{B}^r_{\varepsilon,R} = \biguplus_j B^j$, $B^j := \bar B(y_j, r_j)$, monotone in r, covering the set $\{x : |u_\varepsilon(x)| < 1/2\}$, such that $\sum_{j : y_j \in B_R(z)} r_j \leq r$ for all $z \in R\mathbb{Z}^2$, and such that, letting $d_j := \deg(u_\varepsilon, \partial B^j)$ and $\nu^r_{\varepsilon,R} := 2\pi \sum_j d_j \delta_{y_j}$, the following hold,*

(i) Localized lower bound: *for all $\phi \in W^{1,\infty}(\mathbb{R}^2)$ with $\phi \geq 0$, we have for all j*

$$\frac{1}{2} \int_{B^j} \phi \Big(|\nabla u_\varepsilon - i u_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1 - |u_\varepsilon|^2)^2 \Big) \geq \pi \phi(y_j) |d_j| \log(r/\varepsilon) - O(r_j \mathcal{E}^*_{\varepsilon,R}) \|\nabla \phi\|_{\mathrm{L}^\infty} - O\bigg(r_j^2 N_\varepsilon^2 + |d_j| \log\Big(2 + \frac{\mathcal{E}^*_{\varepsilon,R}}{|\log \varepsilon|}\Big) \bigg) \|\phi\|_{\mathrm{L}^\infty}, \quad (5.1)$$

and similarly, for all $\phi \in W^{1,\infty}(\mathbb{R}^2)$ supported in a ball of radius R,

$$\frac{1}{2} \int_{\mathcal{B}^r_{\varepsilon,R}} \phi \Big(|\nabla u_\varepsilon - i u_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1 - |u_\varepsilon|^2)^2 \Big) \geq \frac{\log(r/\varepsilon)}{2} \int \phi |\nu^r_{\varepsilon,R}| - O(r \mathcal{E}^*_{\varepsilon,R}) \|\nabla \phi\|_{\mathrm{L}^\infty} - O\bigg(r^2 N_\varepsilon^2 + \Big(N_\varepsilon + \frac{\mathcal{E}^*_{\varepsilon,R}}{|\log \varepsilon|}\Big) \log\Big(2 + \frac{\mathcal{E}^*_{\varepsilon,R}}{|\log \varepsilon|}\Big) \bigg) \|\phi\|_{\mathrm{L}^\infty}; \quad (5.2)$$

(ii) Number of vortices:

$$\sup_z \int_{B_R(z)} |\nu^r_{\varepsilon,R}| \lesssim N_\varepsilon + \frac{\mathcal{E}^*_{\varepsilon,R}}{|\log \varepsilon|}; \quad (5.3)$$

(iii) Jacobian estimate: *for all* $\gamma \in [0,1]$,

$$\sup_z \|\nu^r_{\varepsilon,R} - \tilde\mu_\varepsilon\|_{(C^\gamma_c(B_R(z)))^*} \lesssim r^\gamma \Big(N_\varepsilon + \frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big) + \varepsilon^{\gamma/2}\mathcal{E}^*_{\varepsilon,R} + \varepsilon^2.$$

Proof. Step 1: proof of (i)–(ii). We use the notation $\tilde{\mathcal{E}}^*_{\varepsilon,R} := \sup_z \int_{B_R(z)} \tilde e_\varepsilon$, with

$$\tilde e_\varepsilon := \frac12|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a_{\min}}{4\varepsilon^2}(1-|u_\varepsilon|^2)^2, \qquad a_{\min} := \inf_x a(x) \ge \frac12.$$

Note that by assumption we have in particular $\tilde{\mathcal{E}}^*_{\varepsilon,R} \le \mathcal{E}^*_{\varepsilon,R} \lesssim \varepsilon^{-1/5}$. We may apply [75, Proposition 2.1] with $\Omega_\varepsilon = \mathbb{R}^2$, $A_\varepsilon = N_\varepsilon \mathrm{v}_\varepsilon$, with ε replaced by $\varepsilon/\sqrt{a_{\min}}$, and with open cover $(U_\alpha)_\alpha = (B_R(z))_{z\in R\mathbb{Z}^2}$ (note that the argument in [75] indeed works identically on the whole space, and that the energy bound is only needed uniformly on all elements of the open cover). For some $\varepsilon_0, C_0, \bar r \simeq 1$, for all $\varepsilon < \varepsilon_0$ and all $r \in (\varepsilon^{1/2}, \bar r)$, we obtain a locally finite collection $\mathcal{B}^r_{\varepsilon,R}$ of disjoint closed balls covering the set $\{x : |u_\varepsilon(x)| < 1/2\}$, such that for all $B \in \mathcal{B}^r_{\varepsilon,R}$ we have

$$\int_B \Big(\tilde e_\varepsilon + \frac{N_\varepsilon^2}{2}|\mathrm{curl}\ \mathrm{v}_\varepsilon|^2\Big) \ge \pi|d_B|\Big(\log\frac{r}{\varepsilon\bar C_B} - C_0\Big),$$

where we have set $d_B := \deg(u_\varepsilon, \partial B)$, and where $\bar C_B$ is defined as in [75, equation (2.4)]. Moreover, the construction in [75] ensures that the collection $\mathcal{B}^r_{\varepsilon,R}$ is monotone in r, and that $B_R(z)\cap\mathcal{B}^r_{\varepsilon,R}$ has total radius bounded by r for all $z \in R\mathbb{Z}^2$. By [75, Lemma 2.1], we have $\bar C_B \le 16|\log\varepsilon|^{-1}\tilde{\mathcal{E}}^*_{\varepsilon,R} \lesssim |\log\varepsilon|^{-1}\mathcal{E}^*_{\varepsilon,R}$, so that the above becomes, for all $B \in \mathcal{B}^r_{\varepsilon,R}$,

$$\int_B \Big(\tilde e_\varepsilon + \frac{N_\varepsilon^2}{2}|\mathrm{curl}\ \mathrm{v}_\varepsilon|^2\Big) \ge \pi|d_B|\log(r/\varepsilon) - |d_B| O\Big(\log\Big(2 + \frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)\Big). \tag{5.4}$$

Let $r \in (\varepsilon^{1/2}, \bar r)$ be fixed, and set $\mathcal{B}^r_{\varepsilon,R} = \biguplus_j B^j$, $B^j := \bar B(y_j, r_j)$, with corresponding degrees $d_j := d_{B^j}$. Noting that by assumption we have

$$\int_{B^j} |\mathrm{curl}\ \mathrm{v}_\varepsilon|^2 \lesssim |B^j| \lesssim r_j^2,$$

the result (5.4) takes the following form, for all j,

$$\int_{B^j} \tilde e_\varepsilon \ge \pi|d_j|\log(r/\varepsilon) - |d_j| O\Big(\log\Big(2 + \frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)\Big) - O(r_j^2 N_\varepsilon^2). \tag{5.5}$$

Using the assumption $\log\mathcal{E}^*_{\varepsilon,R} \ll |\log\varepsilon|$ and the choice $r > \varepsilon^{1/2}$, the above right-hand side is bounded from below by $\frac{\pi}{2}|d_j||\log\varepsilon|(1-o(1)) - O(r_j^2N_\varepsilon^2)$, and hence, summing over $B^j \in \mathcal{B}^r_{\varepsilon,R}$ with $y_j \in B_R(z)$, we find for all $\varepsilon > 0$ small enough,

$$\frac{\pi}{3}|\log\varepsilon| \sum_{j:y_j\in B_R(z)} |d_j| \ \le\ \int_{B_{R+1}(z)\cap\mathcal{B}^r_{\varepsilon,R}} \tilde e_\varepsilon + O(N_\varepsilon^2) \sum_{j:y_j\in B_R(z)} r_j^2 \ \lesssim\ \mathcal{E}^*_{\varepsilon,R} + r^2N_\varepsilon^2,$$

and hence, with the choice $N_\varepsilon \lesssim |\log\varepsilon|$ and $r \lesssim 1$,

$$\sum_{j:y_j\in B_R(z)} |d_j| \lesssim N_\varepsilon + \frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}, \tag{5.6}$$

that is, item (ii). Let us now prove item (i). Let $\phi \in W^{1,\infty}(\mathbb{R}^2)$, $\phi \geq 0$. For all $B^j \in \mathcal{B}^r_{\varepsilon,R}$, we have from (5.5)

$$\int_{B^j} \phi \tilde{e}_\varepsilon \geq \phi(y_j) \int_{B^j} \tilde{e}_\varepsilon - r_j \|\nabla\phi\|_{\mathrm{L}^\infty} \int_{B^j} \tilde{e}_\varepsilon$$
$$\geq \pi\phi(y_j)|d_j|\log(r/\varepsilon) - \phi(y_j)|d_j|\, O\bigg(\log\Big(2+\frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)\bigg) - \phi(y_j)O(r_j^2N_\varepsilon^2) - r_j\|\nabla\phi\|_{\mathrm{L}^\infty}\int_{B^j}\tilde{e}_\varepsilon,$$

hence

$$\int_{B^j} \phi\tilde{e}_\varepsilon \geq \pi\phi(y_j)|d_j|\log(r/\varepsilon) - O\bigg(r_j^2N_\varepsilon^2 + |d_j|\Big(2+\frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)\bigg)\|\phi\|_{\mathrm{L}^\infty} - O(r_j\mathcal{E}^*_{\varepsilon,R})\|\nabla\phi\|_{\mathrm{L}^\infty}.$$

Further assuming that ϕ is supported in $B_R(z)$ for some $z \in R\mathbb{Z}^2$, summing the above with respect to j with $y_j \in B_R$, setting $\nu^r_{\varepsilon,R} := 2\pi\sum_j d_j\delta_{y_j}$, and using (5.6), we find

$$\int_{\mathcal{B}^r_{\varepsilon,R}} \phi\tilde{e}_\varepsilon \geq \frac{\log(r/\varepsilon)}{2}\int\phi|\nu^r_{\varepsilon,R}| - O\bigg(r^2N_\varepsilon^2 + \Big(N_\varepsilon + \frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)\log\Big(2+\frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)\bigg)\|\phi\|_{\mathrm{L}^\infty} - O(r\mathcal{E}^*_{\varepsilon,R})\|\nabla\phi\|_{\mathrm{L}^\infty}.$$

Item (ii) then follows by definition of $\tilde{e}_\varepsilon$ with $a_{\min} \leq a$.

Step 2: proof of (iii). Using item (i) and arguing just as in [82, item (5) of Proposition 4.4], for $\gamma \in [0,1]$, we obtain for all $r \in (\varepsilon^{1/2}, \bar{r})$ and all $\phi \in C^\gamma_c(\mathbb{R}^2)$ supported in $B_R(z)$ for some $z \in R\mathbb{Z}^2$,

$$\begin{aligned}\bigg|\int\phi(\nu^r_{\varepsilon,R} - \tilde{\mu}_\varepsilon)\bigg| &\lesssim r^\gamma\|\phi\|_{C^\gamma}\sum_{j:y_j\in B_R(z)}|d_j| \\ &\qquad + \varepsilon^{\gamma/2}\|\phi\|_{C^\gamma}\int_{B_R}\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{(1-|u_\varepsilon|^2)^2}{2\varepsilon^2} + N_\varepsilon|1-|u_\varepsilon|^2||\mathrm{curl}\ \mathrm{v}_\varepsilon|\Big) \\ &\lesssim r^\gamma\Big(N_\varepsilon + \frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)|\phi|_{C^\gamma} + \Big(\varepsilon^{\gamma/2}\mathcal{E}^*_{\varepsilon,R} + \varepsilon^{2+\gamma/2}N_\varepsilon^2\int_{B_R}|\mathrm{curl}\ \mathrm{v}_\varepsilon|^2\Big)\|\phi\|_{C^\gamma},\end{aligned} \tag{5.7}$$

and the result follows from the assumption $\|\mathrm{curl}\ \mathrm{v}_\varepsilon\|_{\mathrm{L}^2} \lesssim 1$. ☐

In Section 6 below, strong estimates are proved on the modulated energy excess $\mathcal{D}^*_{\varepsilon,R}$, but these estimates involve the modulated energy $\mathcal{E}^*_{\varepsilon,R}$ itself. In order to buckle the argument, it is thus crucial to independently find an optimal control on $\mathcal{E}^*_{\varepsilon,R}$, or equivalently on the number of vortices. Note that in the case without pinning and forcing no cut-off is needed and this difficulty is absent (the excess is then indeed simply defined by $\mathcal{D}_\varepsilon = \mathcal{E}_\varepsilon - \pi N_\varepsilon|\log\varepsilon|$, cf. [82]). This control of $\mathcal{E}^*_{\varepsilon,R}$ is the main content of the following result, and allows to further refine the conclusions of Lemma 5.1 above. Particular attention is needed in the regime $N_\varepsilon \lesssim \log|\log\varepsilon|$ to ensure an error as small as $o(N_\varepsilon^2)$ in the lower bound. Various useful corollaries are further included. In particular, item (vi) gives an optimal control of the energy inside the balls, measured in L^p for any $p < 2$; and since this result in L^p is already enough for our purposes, it is not necessary here to adapt the more precise Lorentz estimates of [83, Corollary 1.2] to the present weighted context, and we instead use a more direct argument adapted from [86].

Proposition 5.2 (Refined lower bound)**.** *Let $h : \mathbb{R}^2 \to \mathbb{R}$, $a := e^h$, with $1 \lesssim a \leq 1$ and $\|\nabla h\|_{\mathrm{L}^\infty} \lesssim 1$, let $u_\varepsilon : \mathbb{R}^2 \to \mathbb{C}$, $\mathrm{v}_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2$, with $\|\mathrm{curl}\ \mathrm{v}_\varepsilon\|_{\mathrm{L}^1\cap\mathrm{L}^\infty}, \|\mathrm{v}_\varepsilon\|_{\mathrm{L}^\infty} \lesssim 1$. Let $0 < \varepsilon \ll 1$, $1 \ll N_\varepsilon \lesssim |\log\varepsilon|$, and $R \geq 1$ with $|\log\varepsilon| \lesssim R \lesssim |\log\varepsilon|^n$ for some $n \geq 1$, and assume that $\mathcal{D}^*_{\varepsilon,R} \lesssim N_\varepsilon^2$. Then $\mathcal{E}^*_{\varepsilon,R} \lesssim N_\varepsilon|\log\varepsilon|$ holds for all $\varepsilon > 0$ small enough.*

Moreover, for some $\bar{r} \simeq 1$, for all $\varepsilon > 0$ small enough and all $r \in (\varepsilon^{1/2}, \bar{r})$, there exists a locally finite union of disjoint closed balls $\mathcal{B}^r_{\varepsilon,R}$, monotone in r and covering the set $\{x : |u_\varepsilon(x)| < 1/2\}$, and for all

$r_0 \in (\varepsilon^{1/2}, \bar r)$ and $r \ge r_0$ there exists a locally finite union of disjoint closed balls $\tilde{\mathcal{B}}^{r_0,r}_{\varepsilon,R}$, monotone in r and covering the set $\{x : ||u_\varepsilon(x)| - 1| \ge |\log\varepsilon|^{-1}\}$, such that $\mathcal{B}^{r_0}_{\varepsilon,R} \subset \tilde{\mathcal{B}}^{r_0,r_0}_{\varepsilon,R}$, such that for all $z \in R\mathbb{Z}^2$ the sum of the radii of the balls of the collection $\mathcal{B}^r_{\varepsilon,R}$ centered at points of $B_R(z)$ is bounded by r and the sum of the radii of the balls of the collection $\tilde{\mathcal{B}}^{r_0,r}_{\varepsilon,R}$ centered at points of $B_R(z)$ is bounded by Cr, and such that, letting $\mathcal{B}^r_{\varepsilon,R} = \biguplus_j B^j$, $B^j := \bar B(y_j, s_j)$, $d_j := \deg(u_\varepsilon, \partial B^j)$, and defining the point-vortex measure $\nu^r_{\varepsilon,R} := 2\pi \sum_j d_j \delta_{y_j}$, the following properties hold,

(i) Lower bound: *in the regime $N_\varepsilon \gg \log|\log\varepsilon|$, for $e^{-o(N_\varepsilon)} \le r \ll N_\varepsilon |\log\varepsilon|^{-1}$, we have for all $z \in \mathbb{R}^2$,*

$$\frac{1}{2}\int_{\mathcal{B}^r_{\varepsilon,R}} a\chi^z_R\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) \ge \frac{|\log\varepsilon|}{2}\int a\chi^z_R |\nu^r_{\varepsilon,R}| - o(N_\varepsilon^2), \tag{5.8}$$

while in the regime $1 \ll N_\varepsilon \lesssim \log|\log\varepsilon|$ we have for all $e^{-o(N_\varepsilon)} \le r \ll 1$ and all $r_0 \le r$ with $\varepsilon^{1/2} < r_0 \ll N_\varepsilon|\log\varepsilon|^{-1}$, for all $z \in \mathbb{R}^2$,

$$\frac{1}{2}\int_{\tilde{\mathcal{B}}^{r_0,r}_{\varepsilon,R}} a\chi^z_R\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) \ge \frac{|\log\varepsilon|}{2}\int a\chi^z_R \nu^{r_0}_{\varepsilon,R} - o(N_\varepsilon^2); \tag{5.9}$$

(ii) Number of vortices: *for $\varepsilon^{1/2} < r \ll 1$,*

$$\sup_z \int_{B_R(z)} |\nu^r_{\varepsilon,R}| \lesssim N_\varepsilon, \tag{5.10}$$

and moreover in the regime $1 \ll N_\varepsilon \ll |\log\varepsilon|^{1/2}$ the measure $\nu^r_{\varepsilon,R}$ is nonnegative for all $e^{-o(1)N_\varepsilon^{-1}|\log\varepsilon|} \le r < \bar r$;

(iii) Jacobian estimate: *for $\varepsilon^{1/2} < r \ll 1$, for all $\gamma \in [0,1]$,*

$$\sup_z \|\nu^r_{\varepsilon,R} - \tilde\mu_\varepsilon\|_{(C^\gamma_c(B_R(z)))^*} \lesssim r^\gamma N_\varepsilon + \varepsilon^{\gamma/2} N_\varepsilon |\log\varepsilon|, \tag{5.11}$$

$$\sup_z \|\mu_\varepsilon - \tilde\mu_\varepsilon\|_{(C^\gamma_c(B_R(z)))^*} \lesssim \varepsilon^\gamma N_\varepsilon |\log\varepsilon|^{n+1}, \tag{5.12}$$

hence in particular, for all $\gamma \in (0,1]$,

$$\sup_z \|\tilde\mu_\varepsilon\|_{(C^\gamma_c(B_R(z)))^*} \simeq_\gamma N_\varepsilon, \qquad \sup_z \|\mu_\varepsilon\|_{(C^\gamma_c(B_R(z)))^*} \simeq_\gamma N_\varepsilon; \tag{5.13}$$

(iv) Excess energy estimate: *for all $\phi \in W^{1,\infty}(\mathbb{R}^2)$ supported in a ball of radius R,*

$$\int_{\mathbb{R}^2} \phi\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 - |\log\varepsilon|\mu_\varepsilon\Big) \lesssim (\mathcal{D}^*_{\varepsilon,R} + o(N_\varepsilon^2))\|\phi\|_{W^{1,\infty}}; \tag{5.14}$$

(v) Energy outside small balls: *in the regime $N_\varepsilon \gg \log|\log\varepsilon|$, we have for all $e^{-o(N_\varepsilon)} \le r < \bar r$,*

$$\sup_z \int_{\mathbb{R}^2\setminus\mathcal{B}^r_{\varepsilon,R}} \chi^z_R\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) \le \mathcal{D}^*_{\varepsilon,R} + o(N_\varepsilon^2), \tag{5.15}$$

while in the regime $1 \ll N_\varepsilon \lesssim \log|\log\varepsilon|$ we have for all $r \ge e^{-o(N_\varepsilon)}$ and all $r_0 \le r$ with $\varepsilon^{1/2} < r_0 \ll N_\varepsilon|\log\varepsilon|^{-1}$,

$$\sup_z \int_{\mathbb{R}^2\setminus\tilde{\mathcal{B}}^{r_0,r}_{\varepsilon,R}} \chi^z_R\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) \le \mathcal{D}^*_{\varepsilon,R} + o(N_\varepsilon^2); \tag{5.16}$$

(vi) L^p-estimate inside small balls: *in the regime* $N_\varepsilon \gg \log|\log\varepsilon|$*, we have for all* $\varepsilon^{1/2} < r < \bar r$*, for all* $1 \le p < 2$*,*

$$\sup_z \int_{\mathcal{B}^r_{\varepsilon,R}} \chi^z_R |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^p \lesssim_p (\mathcal{D}^*_{\varepsilon,R} + o(N_\varepsilon^2))^{p/2}, \tag{5.17}$$

while in the regime $1 \ll N_\varepsilon \lesssim \log|\log\varepsilon|$ *we have for all* $r > \varepsilon^{1/2}$ *and all* $r_0 \le r$ *with* $\varepsilon^{1/2} < r_0 \ll N_\varepsilon|\log\varepsilon|^{-1}$*, for all* $1 \le p < 2$*,*

$$\sup_z \int_{\tilde{\mathcal{B}}^{r_0,r}_{\varepsilon,R}} \chi^z_R |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^p \lesssim_p (\mathcal{D}^*_{\varepsilon,R} + o(N_\varepsilon^2))^{p/2}. \tag{5.18}$$

Proof. We split the proof into nine steps. The main work consists in checking that the assumptions imply the optimal bound on the energy $\mathcal{E}^*_{\varepsilon,R} \lesssim N_\varepsilon|\log\varepsilon|$. The conclusion is obtained in Step 5 for the regime $N_\varepsilon \gtrsim \log|\log\varepsilon|$, but only in Step 8 for the complementary regime. The various other conclusions are then deduced in Step 9.

Step 1: rough a priori estimate on the energy. In this step, we prove $\mathcal{E}^*_{\varepsilon,R} \lesssim R^2|\log\varepsilon|^2$, and hence by the choice of R we deduce $\mathcal{E}^*_{\varepsilon,R} \lesssim |\log\varepsilon|^m$ for some $m \ge 4$. Decomposing $\mu_\varepsilon = N_\varepsilon \mathrm{curl}\, \mathrm{v}_\varepsilon + \mathrm{curl}\,(j_\varepsilon - N_\varepsilon \mathrm{v}_\varepsilon)$, the assumption $\mathcal{D}^*_{\varepsilon,R} \lesssim N_\varepsilon^2$ yields for all $z \in \mathbb{R}^2$,

$$\mathcal{E}^z_{\varepsilon,R} \le \mathcal{D}^*_{\varepsilon,R} + \frac{|\log\varepsilon|}{2} \int a\chi^z_R \mu_\varepsilon \lesssim N_\varepsilon^2 + N_\varepsilon|\log\varepsilon| \int a\chi^z_R|\mathrm{curl}\, \mathrm{v}_\varepsilon| + |\log\varepsilon| \int |\nabla(a\chi^z_R)||j_\varepsilon - N_\varepsilon \mathrm{v}_\varepsilon|. \tag{5.19}$$

Using the pointwise estimate of Lemma 4.2 for $j_\varepsilon - N_\varepsilon \mathrm{v}_\varepsilon$, using $|\nabla(a\chi^z_R)| \lesssim \mathbb{1}_{B_{2R}(z)}$, $\|\mathrm{curl}\, \mathrm{v}_\varepsilon\|_{\mathrm{L}^1} \lesssim 1$, and $\|\mathrm{v}_\varepsilon\|_{\mathrm{L}^\infty} \lesssim 1$, we obtain

$$\begin{aligned}
\mathcal{E}^z_{\varepsilon,R} &\lesssim |\log\varepsilon|^2 + |\log\varepsilon|\Big(\int_{B_{2R}(z)} (1-|u_\varepsilon|^2)^2\Big)^{1/2}\Big(\int_{B_{2R}(z)} |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2\Big)^{1/2} \\
&\qquad + R|\log\varepsilon|\Big(\int_{B_{2R}(z)} |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2\Big)^{1/2} + RN_\varepsilon|\log\varepsilon|\Big(\int_{B_{2R}(z)} (1-|u_\varepsilon|^2)^2\Big)^{1/2} \\
&\lesssim |\log\varepsilon|^2 + \varepsilon|\log\varepsilon|\mathcal{E}^*_{\varepsilon,R} + R|\log\varepsilon|(\mathcal{E}^*_{\varepsilon,R})^{1/2}.
\end{aligned}$$

Taking the supremum over z, and absorbing $\mathcal{E}^*_{\varepsilon,R}$ into the left-hand side, the result follows.

Step 2: applying Lemma 5.1. The result of Step 1 yields in particular $\log\mathcal{E}^*_{\varepsilon,R} \ll |\log\varepsilon|$, which allows to apply Lemma 5.1. For fixed $r \in (\varepsilon^{1/2}, \bar r)$, let $\mathcal{B}^r_{\varepsilon,R} = \biguplus_j B^j$ denote the union of disjoint closed balls given by Lemma 5.1, and let $\nu^r_{\varepsilon,R}$ denote the associated point-vortex measure. Using Lemma 5.1(ii) in the form

$$\int_{B_R(z)} |\nu^r_{\varepsilon,R}| = \sum_{j: y_j \in B_R(z)} |d_j| \lesssim N_\varepsilon + \frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}, \tag{5.20}$$

Lemma 5.1(i) gives, for all $\phi \in W^{1,\infty}(\mathbb{R}^2)$ with $\phi \ge 0$, if ϕ is supported in a ball of radius R,

$$\begin{aligned}
\frac12 \int_{\mathcal{B}^r_{\varepsilon,R}} \phi\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) &\ge \frac{|\log\varepsilon|}{2} \int \phi|\nu^r_{\varepsilon,R}| - O(r\mathcal{E}^*_{\varepsilon,R})\|\nabla\phi\|_{\mathrm{L}^\infty} \\
&\quad - O\bigg(r^2N_\varepsilon^2 + |\log r|\Big(N_\varepsilon + \frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big) + \Big(N_\varepsilon + \frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)\log\Big(2 + \frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)\bigg)\|\phi\|_{\mathrm{L}^\infty}.
\end{aligned} \tag{5.21}$$

We now prove the following consequence of these bounds,

$$\sup_z \int_{\mathbb{R}^2\setminus\mathcal{B}^r_{\varepsilon,R}} a\chi_R^z\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)$$
$$\le \mathcal{D}^*_{\varepsilon,R} + O\bigg(r\mathcal{E}^*_{\varepsilon,R} + (|\log r| + r|\log\varepsilon|)\Big(N_\varepsilon + \frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big) + \Big(N_\varepsilon + \frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)\log\Big(2 + \frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)\bigg). \quad (5.22)$$

First, the lower bound (5.21) applied to $\phi = a\chi_R^z$ is rewritten as follows,

$$\frac12\int_{\mathbb{R}^2\setminus\mathcal{B}^r_{\varepsilon,R}} a\chi_R^z\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)$$
$$\le T^{r,z}_{\varepsilon,R} + O\bigg(r\mathcal{E}^*_{\varepsilon,R} + r^2N_\varepsilon^2 + |\log r|\Big(N_\varepsilon + \frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big) + \Big(N_\varepsilon + \frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)\log\Big(2 + \frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)\bigg),$$

where we have set

$$T^{r,z}_{\varepsilon,R} := \frac12\int a\chi_R^z\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 - |\log\varepsilon|\nu^r_{\varepsilon,R}\Big).$$

If $\nu^r_{\varepsilon,R}$ was replaced by μ_ε in this last expression, we would recognize the definition of the excess $\mathcal{D}^z_{\varepsilon,R}$, and the result (5.22) would follow. Hence, in order to deduce (5.22), it only remains to check that for all $\phi\in W^{1,\infty}(\mathbb{R}^2)$ supported in a ball of radius R,

$$\Big|\int\phi(\mu_\varepsilon - \nu^r_{\varepsilon,R})\Big| \lesssim r\Big(N_\varepsilon + \frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)\|\phi\|_{W^{1,\infty}} + o(|\log\varepsilon|^{-1})\|\phi\|_{W^{1,\infty}}. \quad (5.23)$$

Using the result of Step 1 in the form $\varepsilon^{1/2}|\log\varepsilon|\mathcal{E}^*_{\varepsilon,R}\ll 1$, Lemma 5.1(iii) with $\gamma=1$ yields

$$\Big|\int\phi(\tilde\mu_\varepsilon - \nu^r_{\varepsilon,R})\Big| \lesssim r\Big(N_\varepsilon + \frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)\|\phi\|_{W^{1,\infty}} + o(|\log\varepsilon|^{-1})\|\phi\|_{W^{1,\infty}}.$$

It remains to replace $\tilde\mu_\varepsilon$ by μ_ε in this estimate. By definition (4.10), with $\|\mathrm{v}_\varepsilon\|_{\mathrm{L}^\infty}\lesssim 1$ and $|\nabla\phi|\le \mathbb{1}_{B_R(z)}\|\phi\|_{W^{1,\infty}}$, and using the result of Step 1 in the form $\varepsilon RN_\varepsilon|\log\varepsilon|(\mathcal{E}^*_{\varepsilon,R})^{1/2}=o(1)$, we find

$$\begin{aligned}\Big|\int\phi(\tilde\mu_\varepsilon-\mu_\varepsilon)\Big| &\le N_\varepsilon\int_{B_{2R}(z)}|\nabla\phi||\mathrm{v}_\varepsilon||1-|u_\varepsilon|^2|\\ &\lesssim RN_\varepsilon\|\phi\|_{W^{1,\infty}}\Big(\int_{B_{2R}(z)}(1-|u_\varepsilon|^2)^2\Big)^{1/2}\\ &\lesssim \varepsilon RN_\varepsilon(\mathcal{E}^*_{\varepsilon,R})^{1/2}\|\phi\|_{W^{1,\infty}} = o(|\log\varepsilon|^{-1})\|\phi\|_{W^{1,\infty}},\end{aligned} \quad (5.24)$$

and the result (5.23) follows.

Step 3: energy and number of vortices. In this step, we show that (5.20) is essentially an equality, in the sense that for all $\varepsilon^{1/2}<r\ll 1$,

$$\sup_z\int\chi_R^z|\nu^r_{\varepsilon,R}| \lesssim N_\varepsilon + \frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|} \lesssim N_\varepsilon + \sup_z\int\chi_R^z|\nu^r_{\varepsilon,R}|. \quad (5.25)$$

The lower bound already follows from (5.20). We now turn to the upper bound. Since the energy excess satisfies $\mathcal{D}^z_{\varepsilon,R}\lesssim N_\varepsilon^2$, we deduce from (5.23)

$$\mathcal{E}^z_{\varepsilon,R} \le \mathcal{D}^z_{\varepsilon,R} + \frac{|\log\varepsilon|}{2}\int a\chi_R^z\mu_\varepsilon \le \frac{|\log\varepsilon|}{2}\int a\chi_R^z\nu^r_{\varepsilon,R} + O\Big(N_\varepsilon^2 + r|\log\varepsilon|\Big(N_\varepsilon + \frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)\Big). \quad (5.26)$$

Taking the supremum in z, and absorbing $\mathcal{E}^*_{\varepsilon,R}$ in the left-hand side with $r \ll 1$, the upper bound in (5.25) follows.

Step 4: estimate on the negative part of the vorticity. In this step, we prove that for all $\varepsilon^{1/2} < r \ll 1$,

$$\sup_z \int \chi_R^z |\nu_{\varepsilon,R}^r| \le (1+o(1)) \sup_z \int \chi_R^z \nu_{\varepsilon,R}^r + O(N_\varepsilon). \tag{5.27}$$

This result is used in Step 5 below in order to replace $\int a\chi_R^z \nu_{\varepsilon,R}^r$ (resp. $\int a\chi_R^z \mu_\varepsilon$) by $\int \chi_R^z \nu_{\varepsilon,R}^r$ (resp. $\int \chi_R^z \mu_\varepsilon$), which happens to be crucial if we want to avoid integrability assumptions on ∇h, as we do here. The lower bound (5.21) of Step 2 with $\phi = a\chi_R^y$ yields for all $y \in \mathbb{R}^2$, using the upper bound in (5.25) to replace the energy $\mathcal{E}^*_{\varepsilon,R}$ in the error terms,

$$\begin{aligned}\mathcal{E}^y_{\varepsilon,R} \ge \frac{1}{2}\int_{\mathcal{B}^r_{\varepsilon,R}} a\chi_R^y \Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) \ge \frac{|\log\varepsilon|}{2}\int a\chi_R^y |\nu_{\varepsilon,R}^r| \\ - O\bigg((|\log r| + r|\log\varepsilon|)\Big(N_\varepsilon + \sup_z \int \chi_R^z |\nu_{\varepsilon,R}^r|\Big) + \Big(N_\varepsilon + \sup_z \int \chi_R^z|\nu_{\varepsilon,R}^r|\Big)\log\Big(2 + \frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)\bigg).\end{aligned}$$

For $e^{-o(|\log\varepsilon|)} < r \ll 1$, using the result of Step 1 in the form $\log \mathcal{E}^*_{\varepsilon,R} \ll |\log\varepsilon|$, we obtain for all $y \in \mathbb{R}^2$,

$$\mathcal{E}^y_{\varepsilon,R} \ge \frac{|\log\varepsilon|}{2}\int a\chi_R^y|\nu_{\varepsilon,R}^r| - o(|\log\varepsilon|)\sup_z \int \chi_R^z |\nu_{\varepsilon,R}^r| - O(N_\varepsilon|\log\varepsilon|). \tag{5.28}$$

On the other hand, the upper bound (5.26) yields

$$\mathcal{E}^y_{\varepsilon,R} \le \frac{|\log\varepsilon|}{2}\int a\chi_R^y \nu_{\varepsilon,R}^r + O(N_\varepsilon|\log\varepsilon|) + o(1)\mathcal{E}^*_{\varepsilon,R}, \tag{5.29}$$

and thus, taking the supremum over y and absorbing $\mathcal{E}^*_{\varepsilon,R}$ in the left-hand side,

$$\mathcal{E}^*_{\varepsilon,R} \le \frac{1}{2}|\log\varepsilon|(1+o(1))\sup_z \int a\chi_R^z|\nu_{\varepsilon,R}^r| + O(N_\varepsilon|\log\varepsilon|),$$

so that (5.29) takes the form, for all $y \in \mathbb{R}^2$,

$$\mathcal{E}^y_{\varepsilon,R} \le \frac{|\log\varepsilon|}{2}\int a\chi_R^y\nu_{\varepsilon,R}^r + O(N_\varepsilon|\log\varepsilon|) + o(|\log\varepsilon|)\sup_z\int \chi_R^z|\nu_{\varepsilon,R}^r|.$$

Combining this with (5.28), dividing both sides by $\frac{1}{2}|\log\varepsilon|$, and taking the supremum over y, we find

$$\sup_z \int \chi_R^z (\nu_{\varepsilon,R}^r)^- \le \sup_z \int a\chi_R^z(|\nu_{\varepsilon,R}^r| - \nu_{\varepsilon,R}^r) \le O(N_\varepsilon) + o(1)\sup_z \int \chi_R^z|\nu_{\varepsilon,R}^r|.$$

This implies

$$\sup_z \int \chi_R^z|\nu_{\varepsilon,R}^r| = \sup_z \int \chi_R^z(\nu_{\varepsilon,R}^r + 2(\nu_{\varepsilon,R}^r)^-) \le \sup_z \int \chi_R^z\nu_{\varepsilon,R}^r + O(N_\varepsilon) + o(1)\sup_z\int a\chi_R^z|\nu_{\varepsilon,R}^r|,$$

and the result (5.27) follows after absorbing the last term in the left-hand side.

Step 5: refined bound on the energy. In this step, we prove $\mathcal{E}^*_{\varepsilon,R} \lesssim (N_\varepsilon + \log|\log\varepsilon|)|\log\varepsilon|$. By (5.20) this implies in particular $\sup_z \int \chi_R^z|\nu_{\varepsilon,R}^r| \lesssim N_\varepsilon + \log|\log\varepsilon|$. In the regime $N_\varepsilon \gtrsim \log|\log\varepsilon|$, these bounds are already the optimal ones. The regime with a "small" number of vortices $1 \ll N_\varepsilon \ll \log|\log\varepsilon|$ is treated in Steps 6–8 below. Let $\varepsilon^{1/2} < r \ll 1$ to be suitably chosen later. Using (5.23), the bound on the energy excess $\mathcal{D}^*_{\varepsilon,R} \lesssim N_\varepsilon^2$ yields for all $z \in R\mathbb{Z}^2$

$$\mathcal{E}^z_{\varepsilon,R} \le \mathcal{D}^z_{\varepsilon,R} + \frac{|\log\varepsilon|}{2}\int a\chi^z_R\mu_\varepsilon \lesssim N_\varepsilon^2 + |\log\varepsilon|\int \chi^z_R|\nu^r_{\varepsilon,R}| + r(N_\varepsilon|\log\varepsilon| + \mathcal{E}^*_{\varepsilon,R}),$$

and hence, using the result (5.27) of Step 4,

$$\mathcal{E}^*_{\varepsilon,R} \lesssim N_\varepsilon|\log\varepsilon| + |\log\varepsilon|\sup_z\int \chi^z_R\nu^r_{\varepsilon,R} + r\mathcal{E}^*_{\varepsilon,R}.$$

Using (5.23) again, and absorbing $\mathcal{E}^*_{\varepsilon,R}$ in the left-hand side with $r \ll 1$, this takes the form

$$\mathcal{E}^*_{\varepsilon,R} \lesssim N_\varepsilon|\log\varepsilon| + |\log\varepsilon|\sup_z\int \chi^z_R\mu_\varepsilon. \tag{5.30}$$

It remains to estimate $\int \chi^z_R\mu_\varepsilon$. Decomposing $\mu_\varepsilon = N_\varepsilon \mathrm{curl}\,\mathrm{v}_\varepsilon + \mathrm{curl}\,(j_\varepsilon - N_\varepsilon\mathrm{v}_\varepsilon)$, using the pointwise estimate of Lemma 4.2 for $j_\varepsilon - N_\varepsilon\mathrm{v}_\varepsilon$, using $|\nabla\chi^z_R| \lesssim R^{-1}\mathbb{1}_{B_{2R}(z)}$, $\|\nabla\chi^z_R\|_{\mathrm{L}^2} \lesssim 1$, $\|\mathrm{curl}\,\mathrm{v}_\varepsilon\|_{\mathrm{L}^1} \lesssim 1$, $\|\mathrm{v}_\varepsilon\|_{\mathrm{L}^\infty} \lesssim 1$, and using the result of Step 1 in the forms $\varepsilon R^{-1}\mathcal{E}^*_{\varepsilon,R} \lesssim 1$ and $\varepsilon(\mathcal{E}^*_{\varepsilon,R})^{1/2} \lesssim 1$, we find

$$\int \chi^z_R\mu_\varepsilon = N_\varepsilon\int \chi^z_R\mathrm{curl}\,\mathrm{v}_\varepsilon - \int \nabla^\perp\chi^z_R\cdot(j_\varepsilon - N_\varepsilon\mathrm{v}_\varepsilon) \lesssim N_\varepsilon + \int |\nabla\chi^z_R||\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|.$$

Regarding the last integral, we distinguish between the contributions inside and outside the balls $\mathcal{B}^r_{\varepsilon,R}$, with $|\nabla\chi^z_R| \lesssim R^{-1}\mathbb{1}_{B_{2R}(z)} \le R^{-1}\chi^z_{2R}$, $\|\nabla\chi^z_R\|_{\mathrm{L}^2} \lesssim 1$, and $|B_{2R}(z)\cap\mathcal{B}^r_{\varepsilon,R}| \lesssim r^2$,

$$\begin{aligned}\int \chi^z_R\mu_\varepsilon &\lesssim N_\varepsilon + \int_{\mathbb{R}^2\setminus\mathcal{B}^r_{\varepsilon,R}} |\nabla\chi^z_R||\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon| + R^{-1}\int_{B_{2R}(z)\cap\mathcal{B}^r_{\varepsilon,R}} |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|\\ &\lesssim N_\varepsilon + \Big(\int_{\mathbb{R}^2\setminus\mathcal{B}^r_{\varepsilon,R}} \chi^z_{2R}|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|^2\Big)^{1/2} + rR^{-1}\Big(\int_{B_{2R}(z)} |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|^2\Big)^{1/2}. \end{aligned}\tag{5.31}$$

Estimating the last right-hand side term by $rR^{-1}(\mathcal{E}^*_{\varepsilon,R})^{1/2}$, using (5.22) to estimate the first, using the bound on the energy excess $\mathcal{D}^*_{\varepsilon,R} \lesssim N_\varepsilon^2$, and noting that $k^{1/2}\log^{1/2}(2+k) \ll k$ holds for $k \gg 1$, we obtain

$$\begin{aligned}\int \chi^z_R\mu_\varepsilon &\lesssim N_\varepsilon + (\mathcal{D}^*_{\varepsilon,R})^{1/2} + rR^{-1}(\mathcal{E}^*_{\varepsilon,R})^{1/2} + r^{1/2}(N_\varepsilon|\log\varepsilon| + \mathcal{E}^*_{\varepsilon,R})^{1/2}\\ &\qquad + \Big(N_\varepsilon + \frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)^{1/2}\Big(|\log r| + \log\Big(2 + \frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)\Big)^{1/2}\\ &\lesssim N_\varepsilon + r^{1/2}(N_\varepsilon|\log\varepsilon|)^{1/2} + r^{1/2}(\mathcal{E}^*_{\varepsilon,R})^{1/2} + o(1)\frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|} + |\log r|^{1/2}\Big(N_\varepsilon + \frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)^{1/2}.\end{aligned}$$

Combining this with (5.30) yields

$$\frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|} \lesssim N_\varepsilon + r^{1/2}(N_\varepsilon|\log\varepsilon|)^{1/2} + r^{1/2}(\mathcal{E}^*_{\varepsilon,R})^{1/2} + o(1)\frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|} + |\log r|^{1/2}\Big(N_\varepsilon + \frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)^{1/2},$$

and hence,

$$\frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|} \lesssim N_\varepsilon + |\log r| + r^{1/2}|\log\varepsilon|.$$

The result then follows from the choice $r = |\log \varepsilon|^{-2}$.

Step 6: refined lower bound in the regime with a "small" number of vortices. In this step, we treat the regime $1 \ll N_\varepsilon \lesssim \log|\log \varepsilon|$, for which the result of Step 5 is not optimal. More precisely, we consider the whole regime $1 \ll N_\varepsilon \lesssim |\log \varepsilon|$ and we show the following: for all $r_0 \in (\varepsilon^{1/2}, \bar r)$ and $r \geq r_0$, there exists a locally finite union of disjoint closed balls $\tilde{\mathcal{B}}^{r_0,r}_{\varepsilon,R}$, monotone in r, covering the set $\{x : ||u_\varepsilon(x)|-1| \geq |\log \varepsilon|^{-1}\}$, such that for all z the sum of the radii of the balls intersecting $B_R(z)$ is bounded by Cr, and such that for all $\varepsilon > 0$ small enough, and all $r_0 \leq r$ satisfying

$$\varepsilon^{1/2} < r_0 \ll N_\varepsilon^2 |\log \varepsilon|^{-1} (N_\varepsilon + \log|\log \varepsilon|)^{-1}, \qquad e^{-o(N_\varepsilon)} \leq r \ll 1, \tag{5.32}$$

we have for all $z \in \mathbb{R}^2$,

$$\frac{1}{2}\int_{\tilde{\mathcal{B}}^{r_0,r}_{\varepsilon,R}} a\chi_R^z \Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) \geq \frac{|\log \varepsilon|}{2}\int a\chi_R^z \nu^{r_0}_{\varepsilon,R} - o(1)\Big(\frac{\mathcal{E}^*_{\varepsilon,R}}{|\log \varepsilon|}\Big)^2 - o(N_\varepsilon^2), \tag{5.33}$$

and similarly, for all $B = \bar B(y_B, r_B) \in \tilde{\mathcal{B}}^{r_0,r}_{\varepsilon,R}$ with degree d_B,

$$\begin{aligned}\frac{1}{2}\int_B \Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)&\\ \geq \frac{|\log \varepsilon|}{2}\int_B \nu^{r_0}_{\varepsilon,R} - E_B\Big(N_\varepsilon + \frac{\mathcal{E}^*_{\varepsilon,R}}{|\log \varepsilon|}\Big)^2 - C|d_B|\log\Big(2 + \frac{\mathcal{E}^*_{\varepsilon,R}}{|\log \varepsilon|}\Big),&\end{aligned} \tag{5.34}$$

where the E_B's satisfy $\sup_z \sum_{B\in\tilde{\mathcal{B}}^{r_0,r}_{\varepsilon,R}\cap B_R(z)} |E_B| \ll 1$. In the sequel, we focus on (5.33), while (5.34) is proved similarly, based on Lemma 5.1(i) in the localized form (5.1) rather than in the form (5.2). We split the proof into three further substeps.

Substep 6.1: enlarged balls. In this step, given some fixed $r_0 \in (\varepsilon^{1/2}, \bar r)$, we construct the enlarged collections of balls $\tilde{\mathcal{B}}^{r_0,r}_{\varepsilon,R}$ for $r \geq r_0$. According to [74, Proposition 4.8], and using the energy estimate of Step 5, we have

$$\mathcal{H}^1(\{x \in B_R(z), ||u_\varepsilon(x)|-1| \geq |\log \varepsilon|^{-1}\}) \leq C\varepsilon|\log \varepsilon|^2 \mathcal{E}^*_{\varepsilon,R} \leq C\varepsilon|\log \varepsilon|^4,$$

where $\mathcal{H}^1$ denotes the 1-dimensional Haussdorff measure. From [74, Section 4.4.1] and [75, Section 2.2], it follows that we may cover the set $\{x : ||u_\varepsilon(x)|-1| \geq |\log \varepsilon|^{-1}\}$ by a locally finite union of disjoint closed balls such that for all z the sum of the radii of the balls intersecting $B_R(z)$ is bounded by $C\varepsilon|\log \varepsilon|^4$. We then combine this collection of balls with the collection $\mathcal{B}^{r_0}_{\varepsilon,R}$. Inductively merging as in [74, Lemma 4.1] any two such balls that intersect into a ball with the same total radius, we obtain a new collection $\tilde{\mathcal{B}}^{r_0}_{\varepsilon,R}$ of disjoint closed balls that cover the set $\{x : ||u_\varepsilon(x)|-1| \geq |\log \varepsilon|^{-1}\}$, and such that for all z the sum of the radii of the balls intersecting $B_R(z)$ is bounded by $r_0 + C\varepsilon|\log \varepsilon|^6 \leq Cr_0$.

Let us now grow the balls of this new collection $\tilde{\mathcal{B}}^{r_0}_{\varepsilon,R}$ following Sandier's ball construction, as described e.g. in [74, Theorem 4.2]. This consists in growing simultaneously all the balls keeping their centers fixed and multiplying their radius by the same factor t. If some balls touch at some point during the growth, the corresponding balls are merged into one larger ball containing the previous ones and of same total radius. This construction ensures that the balls always remain disjoint. Stopping the growth process at some value of the factor t, and setting $r = tr_0$, we denote by $\tilde{\mathcal{B}}^{r_0,r}_{\varepsilon,R}$ the corresponding locally finite collection of disjoint closed balls. By construction, for all z, the sum of the radii of the balls that intersect $B_R(z)$ is bounded by $Ct(r_0 + C\varepsilon|\log \varepsilon|^6) \leq Cr$. Note that by construction $\mathcal{B}^{r_0}_{\varepsilon,R} \subset \tilde{\mathcal{B}}^{r_0}_{\varepsilon,R} = \tilde{\mathcal{B}}^{r_0,r_0}_{\varepsilon,R}$, but for $r > r_0$ the collection $\tilde{\mathcal{B}}^{r_0,r}_{\varepsilon,R}$ has a priori no clear relation with the collection $\mathcal{B}^{r}_{\varepsilon,R}$.

Substep 6.2: preliminary estimate. According to [83, Lemma 3.2] (applied with $c = d$ and $\lambda = 1$), we have, for any $\mathbb{S}^1$-valued map v with degree d on a generic ball B, and for any vector field $A : \partial B \to \mathbb{R}^2$,

$$\frac{1}{2}\int_{\partial B} |\nabla v - ivA|^2 + \frac{1}{2}\int_B |\mathrm{curl}\, A|^2 \geq \frac{\pi d^2}{r} - \frac{\pi d^2}{2} + \frac{1}{2}\int_{\partial B} \Big|\nabla v - ivA - ivd\frac{\tau}{r}\Big|^2,$$

where τ denotes the unit tangent to the circle ∂B. Applying it to $v = \frac{u_\varepsilon}{|u_\varepsilon|}$ and $A = N_\varepsilon \mathrm{v}_\varepsilon$, and noting that $|\nabla u_\varepsilon - iu_\varepsilon F|^2 = |u_\varepsilon|^2 |\nabla \frac{u_\varepsilon}{|u_\varepsilon|} - i\frac{u_\varepsilon}{|u_\varepsilon|}F|^2 + |\nabla |u_\varepsilon||^2$ holds for any real-valued vector field F, we obtain the following improved lower bound on annuli: if $||u_\varepsilon| - 1| \leq |\log \varepsilon|^{-1}$ holds on ∂B, then we have

$$\begin{aligned}\frac{1}{2}(1 + O(|\log \varepsilon|^{-1}))\int_{\partial B} |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{1}{2}N_\varepsilon^2 \int_B |\mathrm{curl}\ \mathrm{v}_\varepsilon|^2 \\ \geq \frac{\pi d^2}{r} - \frac{\pi d^2}{2} + \frac{1}{2}(1 - O(|\log \varepsilon|^{-1}))\int_{\partial B} \Big|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon - iu_\varepsilon d\frac{\tau}{r}\Big|^2. \end{aligned} \tag{5.35}$$

Substep 6.3: proof of (5.33)*.* Let $r_0 > 0$ be chosen as in (5.32). We start from Lemma 5.1(i) with $\phi = a\chi_R^z$, combined with the refined energy estimate of Step 5 and the choice of r_0, which yields

$$\begin{aligned}\frac{1}{2}\int_{\mathcal{B}_{\varepsilon,R}^{r_0}} a\chi_R^z \Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1 - |u_\varepsilon|^2)^2\Big) \\ \geq \frac{\log(r_0/\varepsilon)}{2}\int a\chi_R^z |\nu_{\varepsilon,R}^{r_0}| - o(N_\varepsilon^2) - C\Big(N_\varepsilon + \frac{\mathcal{E}_{\varepsilon,R}^*}{|\log \varepsilon|}\Big)\log\Big(2 + \frac{\mathcal{E}_{\varepsilon,R}^*}{|\log \varepsilon|}\Big).\end{aligned} \tag{5.36}$$

We next need to show that this lower bound for the energy is essentially maintained during the ball growth and merging process, hence holds as well for the collections $\tilde{\mathcal{B}}_{\varepsilon,R}^{r_0,r}$ with $r > r_0$.

Assume that some ball $B = \bar{B}(y, s)$ gets grown into $B' = \bar{B}(y, ts)$ without merging, for some $t \geq 1$, and assume that $B' \setminus B$ does not intersect $\tilde{\mathcal{B}}_{\varepsilon,R}^{r_0}$, so that $||u_\varepsilon| - 1| \leq |\log \varepsilon|^{-1}$ holds on $B' \setminus B$. Let d denote the degree of B (hence of B'). Since by assumption we have

$$|a(x)\chi_R^z(x) - a(y)\chi_R^z(y)| \leq \chi_R^z(y)|a(x) - a(y)| + a(x)|\chi_R^z(x) - \chi_R^z(y)| \leq C\big(R^{-1} + \chi_R^z(y)\big)|x - y|, \tag{5.37}$$

we may write

$$\begin{aligned}\frac{1}{2}\int_{B'\setminus B} a\chi_R^z \Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1 - |u_\varepsilon|^2)^2\Big) \geq \frac{a(y)\chi_R^z(y)}{2}\int_{B'\setminus B} |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 \\ - CR^{-1}\int_{B'\setminus B} |\cdot - y||\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 - C\chi_R^z(y)\int_{B'\setminus B} |\cdot - y||\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2.\end{aligned}$$

Using that $|u_\varepsilon| \leq 1 + |\log \varepsilon|^{-1}$ holds on $B' \setminus B$, the last right-hand side term above is estimated as follows,

$$\begin{aligned}&\int_{B'\setminus B} |\cdot - y||\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 \\ &\leq 2\int_{B'\setminus B} |\cdot - y|\, |u_\varepsilon|^2 \Big|\frac{d\tau}{|\cdot - y|}\Big|^2 + 2\int_{B'\setminus B} |\cdot - y|\, \Big|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon - iu_\varepsilon \frac{d\tau}{|\cdot - y|}\Big|^2 \\ &\leq Cd^2 ts + 2ts \int_{B'\setminus B} \Big|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon - iu_\varepsilon \frac{d\tau}{|\cdot - y|}\Big|^2,\end{aligned} \tag{5.38}$$

where $\tau(x) = (x-y)^\perp/|x-y|$ is the unit tangent to each circle centered at y, and we may then deduce

$$\frac{1}{2}\int_{B'\setminus B} a\chi_R^z\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) \ge \frac{a(y)\chi_R^z(y)}{2}\int_{B'\setminus B}|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2$$
$$-CtsR^{-1}\int_{B'\setminus B}|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 - Cd^2ts\chi_R^z(y) - Cts\chi_R^z(y)\int_{B'\setminus B}\Big|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon - iu_\varepsilon\frac{d\tau}{|\cdot - y|}\Big|^2. \quad (5.39)$$

Again using that $||u_\varepsilon|-1| \le |\log\varepsilon|^{-1}$ holds on $B'\setminus B$, the estimate (5.35) on the ball $B(y,\rho)$ for ρ integrated between s and ts takes the form

$$(1+O(|\log\varepsilon|^{-1}))\frac{1}{2}\int_{B'\setminus B}|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 \ge \pi d^2\log t - \frac{\pi}{2}d^2ts - \frac{1}{2}N_\varepsilon^2 ts\int_{B'}|\mathrm{curl}\ \mathrm{v}_\varepsilon|^2$$
$$+(1-O(|\log\varepsilon|^{-1}))\frac{1}{2}\int_{B'\setminus B}\Big|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon - iu_\varepsilon\frac{d\tau}{|\cdot - y|}\Big|^2. \quad (5.40)$$

Combining this with (5.39), we are then led to

$$\begin{aligned}(1+C|\log\varepsilon|^{-1})\frac{1}{2}\int_{B'\setminus B} a\chi_R^z\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)\\ \ge a(y)\chi_R^z(y)\pi d^2\log t - Cd^2ts - \frac{1}{2}N_\varepsilon^2 ts\int_{B'}|\mathrm{curl}\ \mathrm{v}_\varepsilon|^2 - CtsR^{-1}\int_{B'\setminus B}|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2\\ +\Big(\frac{a(y)}{2}(1-C|\log\varepsilon|^{-1}) - Cts\Big)\chi_R^z(y)\int_{B'\setminus B}\Big|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon - iu_\varepsilon\frac{d\tau}{|\cdot - y|}\Big|^2.\end{aligned}$$

For ε small enough and $ts \le \min\{1, \frac{1}{4C}\inf a\} =: \tilde r$ (note that by assumption $\tilde r \simeq 1$), the last right-hand side term is nonnegative, so that we conclude

$$\begin{aligned}(1+C|\log\varepsilon|^{-1})\frac{1}{2}\int_{B'\setminus B} a\chi_R^z\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)\\ \ge a(y)\chi_R^z(y)\pi d^2\log t - Cd^2ts - \frac{1}{2}N_\varepsilon^2 ts\int_{B'}|\mathrm{curl}\ \mathrm{v}_\varepsilon|^2 - CtsR^{-1}\int_{B'\setminus B}|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2\\ \ge a(y)\chi_R^z(y)\pi d^2\log t - Cts(d^2+N_\varepsilon^2) - CtsR^{-1}\mathcal{E}^*_{\varepsilon,R}. \quad (5.41)\end{aligned}$$

If the ball $B = \bar B(y,s)$ belongs to the collection $\tilde{\mathcal{B}}_{\varepsilon,R}^{r_0,r}$ for some $r \ge r_0$, only a finite number of balls of the collection $\mathcal{B}_{\varepsilon,R}^{r_0}$ are included in the ball B. Denote them by $B^j = \bar B(y_j, s_j)$, $j = 1,\ldots,k$. By definition, the degree d of B is then equal to $d = \sum_j d_j$, where d_j denotes the degree of B^j. We may then write

$$\begin{aligned}a(y)\chi_R^z(y)d^2 \ge a(y)\chi_R^z(y)\sum_j d_j &\ge \sum_j a(y_j)\chi_R^z(y_j)d_j - C\sum_j |d_j||y-y_j|\mathbb{1}_{B_{2R}(z)}(y_j)\\ &\ge \sum_j a(y_j)\chi_R^z(y_j)d_j - Cs\sum_j |d_j|\mathbb{1}_{B_{2R}(z)}(y_j),\end{aligned}$$

and hence, in terms of the point-vortex measure $\nu_{\varepsilon,R}^{r_0}$,

$$a(y)\chi_R^z(y)d^2 \ge \frac{1}{2\pi}\int_B a\chi_R^z\nu_{\varepsilon,R}^{r_0} - Cs\int_{B_{2R}(z)}|\nu_{\varepsilon,R}^{r_0}|. \quad (5.42)$$

Therefore, if the ball $B=\bar B(y,s)$ belongs to the collection $\tilde{\mathcal B}^{r_0,r}_{\varepsilon,R}$ for some $r\ge r_0$ and gets grown without merging into a ball $B'=\bar B(y,ts)$ for some $t\ge1$ with $ts\le\tilde r$, then combining (5.41) and (5.42) yields

$$\begin{aligned}(1+C|\log\varepsilon|^{-1})\frac12\int_{B'\setminus B}a\chi_R^z\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)\\ \ge\frac{\log t}{2}\int_B a\chi_R^z\nu_{\varepsilon,R}^{r_0}-Cs\log t\int_{B_{2R}(z)}|\nu_{\varepsilon,R}^{r_0}|-Cts\Big(N_\varepsilon+\int_{B_{2R}(z)}|\nu^{r_0}_{\varepsilon,R}|\Big)^2-CtsR^{-1}\mathcal E^*_{\varepsilon,R},\end{aligned}$$

and hence, using Lemma 5.1(ii), the inequality $|\log t|\le t$ for $t\ge1$, and the choice $R\gtrsim|\log\varepsilon|$,

$$\frac12\int_{B'\setminus B}a\chi_R^z\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)\ge\frac{\log t}{2}\int_B a\chi_R^z\nu_{\varepsilon,R}^{r_0}-Cts\Big(N_\varepsilon+\frac{\mathcal E^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)^2.$$

By construction of the ball growth and merging process, this easily implies the following: if a ball $B=\bar B(y_B,s_B)$ belongs to the collection $\tilde{\mathcal B}^{r_0,r}_{\varepsilon,R}$ for some $r_0\le r\le\tilde r$, then we have

$$\frac12\int_{B\setminus\tilde{\mathcal B}^{r_0}_{\varepsilon,R}}a\chi_R^z\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)\ge\frac{\log(r/r_0)}{2}\int_B a\chi_R^z\nu_{\varepsilon,R}^{r_0}-Cs_B\Big(N_\varepsilon+\frac{\mathcal E^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)^2.$$

Summing this estimate over all the balls B of the collection $\tilde{\mathcal B}^{r_0,r}_{\varepsilon,R}$ that intersect $B_{2R}(z)$, and recalling that the sum of the radii of these balls is by construction bounded by Cr, we deduce for any $r_0\le r\le\tilde r$,

$$\frac12\int_{\tilde{\mathcal B}^{r_0,r}_{\varepsilon,R}\setminus\tilde{\mathcal B}^{r_0}_{\varepsilon,R}}a\chi_R^z\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)\ge\frac{\log(r/r_0)}{2}\int a\chi_R^z\nu_{\varepsilon,R}^{r_0}-Cr\Big(N_\varepsilon+\frac{\mathcal E^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)^2.$$

Combining this with (5.36), and recalling that by definition $\mathcal B^{r_0}_{\varepsilon,R}\subset\tilde{\mathcal B}^{r_0}_{\varepsilon,R}$, we deduce

$$\begin{aligned}\frac12\int_{\tilde B^{r_0,r}_{\varepsilon,R}}a\chi_R^z\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)\\ \ge\frac{\log(r/\varepsilon)}{2}\int a\chi_R^z\nu_{\varepsilon,R}^{r_0}-Cr\Big(N_\varepsilon+\frac{\mathcal E^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)^2-o(N_\varepsilon^2)-C\Big(N_\varepsilon+\frac{\mathcal E^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)\log\Big(2+\frac{\mathcal E^*_{\varepsilon,R}}{|\log\varepsilon|}\Big),\end{aligned}\tag{5.43}$$

and hence, using Lemma 5.1(ii) and the choice (5.32) of r,

$$\begin{aligned}&\frac12\int_{\tilde B^{r_0,r}_{\varepsilon,R}}a\chi_R^z\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)\\ &\ge\frac{|\log\varepsilon|}{2}\int a\chi_R^z\nu_{\varepsilon,R}^{r_0}-C|\log r|\Big(N_\varepsilon+\frac{\mathcal E^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)-Cr\Big(\frac{\mathcal E^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)^2-o(N_\varepsilon^2)-C\Big(N_\varepsilon+\frac{\mathcal E^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)\log\Big(2+\frac{\mathcal E^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)\\ &\ge\frac{|\log\varepsilon|}{2}\int a\chi_R^z\nu_{\varepsilon,R}^{r_0}-o(1)\Big(\frac{\mathcal E^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)^2-o(N_\varepsilon^2),\end{aligned}$$

that is, (5.33).

Step 8: optimal bound on the energy. In this step, we prove $\mathcal E^*_{\varepsilon,R}\lesssim N_\varepsilon|\log\varepsilon|$, thus completing the result of Step 5 in all regimes. Note that by Step 3 this also implies $\sup_z\int\chi_R^z|\nu^r_{\varepsilon,R}|\lesssim N_\varepsilon$. By Step 5, it only remains to consider the regime with a "small" number of vortices $1\ll N_\varepsilon\lesssim\log|\log\varepsilon|$. Let $r_0\le r\ll1$ be fixed as in (5.32). On the one hand, using the estimate (5.23), we deduce from the result (5.33) of Step 7,

$$\frac{1}{2}\int_{\mathbb{R}^2\setminus\tilde{\mathcal{B}}^{r_0,r}_{\varepsilon,R}} a\chi_R^z\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) \\ \le \mathcal{D}^z_{\varepsilon,R}+O\bigg(r_0|\log\varepsilon|\Big(N_\varepsilon+\frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)\bigg)+o(1)\Big(\frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)^2+o(N_\varepsilon^2)$$

and hence, using the assumption $\mathcal{D}^*_{\varepsilon,R}\lesssim N_\varepsilon^2$, the suboptimal energy bound of Step 5, and the choice (5.32) of r_0,

$$\frac{1}{2}\int_{\mathbb{R}^2\setminus\tilde{\mathcal{B}}^{r_0,r}_{\varepsilon,R}} a\chi_R^z\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)\lesssim N_\varepsilon^2+o(1)\Big(\frac{\mathcal{E}^*_{\varepsilon,R}}{|\log\varepsilon|}\Big)^2. \tag{5.44}$$

On the other hand, combining the estimates (5.30) and (5.31) (with $\mathcal{B}^r_{\varepsilon,R}$ replaced by $\tilde{\mathcal{B}}^{r_0,r}_{\varepsilon,R}$) of Step 5, we find

$$\mathcal{E}^*_{\varepsilon,R}\lesssim N_\varepsilon|\log\varepsilon|+|\log\varepsilon|\Big(\sup_z\int_{\mathbb{R}^2\setminus\tilde{\mathcal{B}}^{r_0,r}_{\varepsilon,R}}\chi_R^z|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2\Big)^{1/2}+r|\log\varepsilon|R^{-1}(\mathcal{E}^*_{\varepsilon,R})^{1/2}.$$

Now inserting (5.44) yields

$$\mathcal{E}^*_{\varepsilon,R}\lesssim N_\varepsilon|\log\varepsilon|+o(1)\mathcal{E}^*_{\varepsilon,R}+|\log\varepsilon|R^{-1}(\mathcal{E}^*_{\varepsilon,R})^{1/2},$$

and thus, recalling the choice $R\gtrsim|\log\varepsilon|$, and absorbing $\mathcal{E}^*_{\varepsilon,R}$ in the left-hand side, the result $\mathcal{E}^*_{\varepsilon,R}\lesssim N_\varepsilon|\log\varepsilon|$ follows.

Step 9: conclusion. The optimal energy bound $\mathcal{E}^*_{\varepsilon,R}\lesssim N_\varepsilon|\log\varepsilon|$ is now proved. In the present step, we check that the rest of the statements follow from this bound. We split the proof into seven further substeps.

Substep 9.1: proof of (i). The result (5.8) follows from (5.21) in Step 2 with $\phi=a\chi_R^z$, combined with the optimal energy bound. Repeating the argument of Step 6 with the optimal energy bound rather than with the suboptimal bound of Step 5, the choice (5.32) can be replaced by $\varepsilon^{1/2}<r_0\ll N_\varepsilon|\log\varepsilon|^{-1}$. For such a choice of r_0, and for $r\ge r_0$ as in (5.32), the result (5.33) together with the optimal energy bound directly implies the result (5.9) for a "small" number of vortices $1\ll N_\varepsilon\lesssim\log|\log\varepsilon|$.

Substep 9.2: proof of (ii). The bound (5.10) on the number of vortices follow from the result (5.25) of Step 3 together with the optimal energy bound. It remains to prove that in the regime $1\ll N_\varepsilon\ll|\log\varepsilon|^{1/2}$ for $e^{-o(1)N_\varepsilon^{-1}|\log\varepsilon|}\le r<\bar r$ each ball of the collection $\mathcal{B}^r_{\varepsilon,R}$ has a nonnegative degree. This is a refinement of the result of Step 4. The lower bound (5.21) of Step 2 with $\phi=a\chi_R^z$ can be rewritten as follows, using the optimal energy bound, for all $z\in\mathbb{R}^2$,

$$|\log\varepsilon|\int a\chi_R^z(\nu^r_{\varepsilon,R})^-=\frac{|\log\varepsilon|}{2}\int a\chi_R^z(|\nu^r_{\varepsilon,R}|-\nu^r_{\varepsilon,R}) \\ \le\mathcal{E}^z_{\varepsilon,R}-\frac{|\log\varepsilon|}{2}\int a\chi_R^z\nu^r_{\varepsilon,R}+O\Big(rN_\varepsilon|\log\varepsilon|+r^2N_\varepsilon^2+N_\varepsilon|\log r|\Big)+o(N_\varepsilon^2),$$

and hence, using (5.23) to replace $\nu^r_{\varepsilon,R}$ by μ_ε in the right-hand side, and using the assumption $\mathcal{D}^z_{\varepsilon,R}\lesssim N_\varepsilon^2$, we find

$$|\log\varepsilon|\int\chi_R^z(\nu^r_{\varepsilon,R})^-\lesssim N_\varepsilon^2+rN_\varepsilon|\log\varepsilon|+N_\varepsilon|\log r|. \tag{5.45}$$

Dividing both sides by $|\log\varepsilon|$, we deduce in the regime $N_\varepsilon \ll |\log\varepsilon|^{1/2}$ with $e^{-o(1)N_\varepsilon^{-1}|\log\varepsilon|} \le r \ll N_\varepsilon^{-1}$,

$$\sup_z \int \chi_R^z (\nu_{\varepsilon,R}^r)^- \ll 1,$$

which means that for ε small enough there exists no single ball $B^j \in \mathcal{B}_{\varepsilon,R}^r$ with negative degree $d_j < 0$. This proves the result for $r \ll N_\varepsilon^{-1}$. Now for $N_\varepsilon^{-1} \lesssim r < \bar r$ the same property must hold, since, by monotonicity of the collection $\mathcal{B}_{\varepsilon,R}^r$ with respect to r, for any $r > r'$ the degree of a ball $B \in \mathcal{B}_{\varepsilon,R}^r$ equals the sum of the degrees of all the balls $B' \in \mathcal{B}_\varepsilon(r')$ with $B' \subset B$.

Substep 9.3: proof of (v). In the regime $N_\varepsilon \gg \log|\log\varepsilon|$, for $e^{-o(N_\varepsilon)} \le r \ll N_\varepsilon|\log\varepsilon|^{-1}$, the result (5.15) follows from (5.22) together with the optimal energy bound. Monotonicity of $\mathcal{B}_{\varepsilon,R}^r$ with respect to r then implies (5.15) for all $r \ge e^{-o(N_\varepsilon)}$ in the regime $N_\varepsilon \gg \log|\log\varepsilon|$. In the regime $1 \ll N_\varepsilon \lesssim \log|\log\varepsilon|$, it suffices to argue as for (5.22) in Step 2, but with the lower bound (5.21) replaced by its refined version (5.33): for $r_0 \le r$ with $\varepsilon^{1/2} < r_0 \ll N_\varepsilon|\log\varepsilon|^{-1}$ and $e^{-o(N_\varepsilon)} \le r \ll 1$, the estimate (5.33) together with (5.23) indeed yields

$$\begin{aligned}
&\frac12 \int_{\mathbb{R}^2\setminus\tilde{\mathcal{B}}_{\varepsilon,R}^{r_0,r}} a\chi_R^z\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)\\
&\qquad\le \frac12 \int_{\mathbb{R}^2} a\chi_R^z\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) - \frac{|\log\varepsilon|}{2}\int a\chi_R^z\nu_{\varepsilon,R}^{r_0} + o(N_\varepsilon^2)\\
&\qquad\qquad\le \mathcal{D}_{\varepsilon,R}^* + r_0 N_\varepsilon|\log\varepsilon| + o(N_\varepsilon^2) = \mathcal{D}_{\varepsilon,R}^* + o(N_\varepsilon^2),
\end{aligned}$$

and the result (5.16) follows by monotonicity of $\tilde{\mathcal{B}}_{\varepsilon,R}^{r_0,r}$ with respect to r.

Substep 9.4: proof of (iii). The Jacobian estimate (5.11) follows from Lemma 5.1(iii) together with the optimal energy bound, and the estimate (5.12) with $\gamma = 1$ similarly follows from (5.24). The result (5.12) for all $\gamma \in [0,1]$ then follows by interpolation (as e.g. in [53]) provided we also manage to prove, for all $\phi \in \mathrm{L}^\infty(\mathbb{R}^2)$ supported in a ball $B_R(z)$,

$$\Big|\int \phi(\tilde\mu_\varepsilon - \mu_\varepsilon)\Big| \lesssim RN_\varepsilon|\log\varepsilon|\|\phi\|_{\mathrm{L}^\infty}. \tag{5.46}$$

Let $\phi \in \mathrm{L}^\infty(\mathbb{R}^2)$ be supported in $B_R(z)$, for some $z \in R\mathbb{Z}^2$. By definition (4.10), we find

$$\begin{aligned}
\int \phi(\tilde\mu_\varepsilon - \mu_\varepsilon) &= N_\varepsilon \int \phi\left((1-|u_\varepsilon|^2)\mathrm{curl}\ \mathrm{v}_\varepsilon + 2(\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon, u_\varepsilon)\cdot \mathrm{v}_\varepsilon^\perp\right)\\
&\le N_\varepsilon\|\phi\|_{\mathrm{L}^\infty}\int_{B_R(z)} \big(|1-|u_\varepsilon|^2||\mathrm{curl}\ \mathrm{v}_\varepsilon| + 2|\mathrm{v}_\varepsilon||1-|u_\varepsilon|^2||\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon| + 2|\mathrm{v}_\varepsilon||\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|\big),
\end{aligned}$$

and hence we obtain with the optimal energy bound, with $\|\mathrm{v}_\varepsilon\|_{\mathrm{L}^\infty}, \|\mathrm{curl}\ \mathrm{v}_\varepsilon\|_{\mathrm{L}^2} \lesssim 1$,

$$\int \phi(\tilde\mu_\varepsilon - \mu_\varepsilon) \lesssim \big(\varepsilon N_\varepsilon^2|\log\varepsilon| + RN_\varepsilon|\log\varepsilon|\big)\|\phi\|_{\mathrm{L}^\infty},$$

that is, (5.46).

Substep 9.5: proof of (iv) in the regime $N_\varepsilon \gg \log|\log\varepsilon|$. We focus on the regime $N_\varepsilon \gg \log|\log\varepsilon|$. Let $\varepsilon^{1/2} < r \ll 1$ to be later optimized as a function of ε. We write as before $\mathcal{B}_{\varepsilon,R}^r = \biguplus_j B^j$, $B^j = \bar B(y_j, r_j)$, we denote by d_j the degree of B^j, and we set $\nu^r_{\varepsilon,R} = 2\pi \sum_j d_j \delta_{y_j}$. Given $\phi \in W^{1,\infty}(\mathbb{R}^2)$ supported in the ball $B_R(z)$, we decompose

$$\begin{aligned}
&\int_{\mathbb{R}^2} \phi\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 - |\log\varepsilon|\nu^r_{\varepsilon,R}\Big)\\
&\le \int_{\mathbb{R}^2\setminus\mathcal{B}^r_{\varepsilon,R}} \phi\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)\\
&\qquad + \sum_j \bigg| \int_{B^j} \phi\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) - 2\pi\phi(y_j)d_j|\log\varepsilon|\bigg|\\
&\le \|\phi\|_{\mathrm{L}^\infty} \int_{\mathbb{R}^2\setminus\mathcal{B}^r_{\varepsilon,R}} \chi^z_R\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)\\
&\qquad + \|\phi\|_{\mathrm{L}^\infty}\sum_j \chi^z_R(y_j)\bigg| \int_{B^j} \Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) - 2\pi d_j|\log\varepsilon|\bigg|\\
&\qquad + r\|\nabla\phi\|_{\mathrm{L}^\infty} \int_{B_{2R}(z)\cap\mathcal{B}^r_{\varepsilon,R}} \Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big).
\end{aligned}\tag{5.47}$$

Combined with the optimal energy bound, the localized lower bound (5.1) in Lemma 5.1(i) with $\phi = 1$ yields for all j,

$$\frac12 \int_{B^j} \Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) \ge \pi|d_j||\log\varepsilon| - O\big(r_j^2N_\varepsilon^2 + |d_j||\log r| + |d_j|\log N_\varepsilon\big),$$

and hence

$$\begin{aligned}
&\bigg| \int_{B^j} \Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) - 2\pi|d_j||\log\varepsilon|\bigg|\\
&\qquad \le \int_{B^j} \Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) - 2\pi|d_j||\log\varepsilon| + O\big(r_j^2N_\varepsilon^2 + |d_j||\log r| + |d_j|\log N_\varepsilon\big).
\end{aligned}$$

Noting that $\chi^z_R(y_j) \le \chi^z_R(y) + O(R^{-1}r_j)\chi^z_{2R}(y_j)$ holds for $y \in B_R(z)$, using the optimal energy bound and $R \gtrsim |\log\varepsilon|$, we obtain

$$\begin{aligned}
&\chi^z_R(y_j)\bigg| \int_{B^j} \Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) - 2\pi|d_j||\log\varepsilon|\bigg|\\
&\qquad \le \int_{B^j} \chi^z_R\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) - 2\pi\chi^z_R(y_j)|d_j||\log\varepsilon|\\
&\qquad\qquad + \chi^z_{2R}(y_j)O\big(r_jN_\varepsilon + r_j^2N_\varepsilon^2 + |d_j||\log r| + |d_j|\log N_\varepsilon\big).
\end{aligned}$$

Inserting this into (5.47), and using the bound of item (ii) on the number of vortices, we find

$$\begin{aligned}
&\int \phi\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 - |\log\varepsilon|\nu^r_{\varepsilon,R}\Big)\\
&\le \|\phi\|_{\mathrm{L}^\infty} \int \chi^z_R\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 - |\log\varepsilon|\nu^r_{\varepsilon,R}\Big) + O\big(1 + r^2N_\varepsilon^2 + N_\varepsilon|\log r| + N_\varepsilon\log N_\varepsilon\big)\|\phi\|_{\mathrm{L}^\infty}\\
&\qquad + r\|\nabla\phi\|_{\mathrm{L}^\infty} \int_{B_{2R}(z)\cap\mathcal{B}^r_{\varepsilon,R}} \Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big),
\end{aligned}$$

where the last term is estimated by $r\mathcal{E}^*_{\varepsilon,R}\|\nabla\phi\|_{\mathrm{L}^\infty} \lesssim rN_\varepsilon|\log\varepsilon|\|\nabla\phi\|_{\mathrm{L}^\infty}$, and where (5.23) can be used to replace $\nu^r_{\varepsilon,R}$ by μ_ε in both sides up to an error of order $(rN_\varepsilon|\log\varepsilon|+1)\|\phi\|_{W^{1,\infty}}$. In the present regime $N_\varepsilon \gg \log|\log\varepsilon|$, we may choose $e^{-o(N_\varepsilon)} \le r \ll N_\varepsilon|\log\varepsilon|^{-1}$, and the conclusion (5.14) follows for that choice.

Substep 9.6: proof of (iv) in the regime $1 \ll N_\varepsilon \lesssim \log|\log\varepsilon|$. We turn to the regime $1 \ll N_\varepsilon \lesssim \log|\log\varepsilon|$, in which case the proof needs to be adapted in the spirit of the computations in Step 7. Let $\phi \in W^{1,\infty}(\mathbb{R}^2)$ be supported in the ball $B_R(z)$, and let $e^{-o(1)|\log\varepsilon|/N_\varepsilon} \le r_0 \ll N_\varepsilon/|\log\varepsilon|$. First arguing as in Substep 9.5 with this choice of r_0, we obtain

$$\begin{aligned}\int_{\mathcal{B}^{r_0}_{\varepsilon,R}} \phi\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 - \log(r_0/\varepsilon)\nu^{r_0}_{\varepsilon,R}\Big)\\ \le \|\phi\|_{\mathrm{L}^\infty}\int_{\mathcal{B}^{r_0}_{\varepsilon,R}} \chi^z_R\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 - \log(r_0/\varepsilon)\nu^{r_0}_{\varepsilon,R}\Big) + o(N_\varepsilon^2)\|\phi\|_{W^{1,\infty}}. \quad (5.48)\end{aligned}$$

Now we consider the modified ball collection $\tilde{\mathcal{B}}^{r_0,r}_{\varepsilon,R}$ with $r \ge r_0$, as constructed in Step 7.1. Assume that some ball $B = \bar{B}(y,s)$ gets grown into $B' = \bar{B}(y,ts)$ without merging, for some $t \ge 1$, and assume that $B' \setminus B$ does not intersect $\tilde{\mathcal{B}}^{r_0}_{\varepsilon,R}$, so that by construction $||u_\varepsilon|-1| \le |\log\varepsilon|^{-1}$ holds on $B'\setminus B$. Let d denote the degree of B (hence of B'). We may then decompose

$$\begin{aligned}\Big|\frac12\int_{B'\setminus B}\phi\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) - \pi\phi(y)d\log t\Big|\\ \le \|\phi\|_{\mathrm{L}^\infty}\Big|\frac12\int_{B'\setminus B}\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) - \pi d\log t\Big|\\ + \|\nabla\phi\|_{\mathrm{L}^\infty}\frac12\int_{B'\setminus B}|\cdot-y|\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big),\end{aligned}$$

and hence, arguing as in (5.38),

$$\begin{aligned}\Big|\frac12\int_{B'\setminus B}\phi\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) - \pi\phi(y)d\log t\Big|\\ \le \|\phi\|_{\mathrm{L}^\infty}\Big|\frac12\int_{B'\setminus B}\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) - \pi d\log t\Big|\\ + ts\|\nabla\phi\|_{\mathrm{L}^\infty}\bigg(Cd^2 + \int_{B'\setminus B}\Big|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon - iu_\varepsilon\frac{d\tau}{|\cdot-y|}\Big|^2 + \int_{B'\setminus B}\frac{a}{4\varepsilon^2}(1-|u_\varepsilon|^2)^2\bigg). \quad (5.49)\end{aligned}$$

Let us estimate the last right-hand side term of (5.49). Applying the lower bound (5.33) with ε replaced by 2ε (with $\varepsilon < 1/2$), together with the optimal energy bound, we obtain, for $r \ge r_0$ with $e^{-o(N_\varepsilon)} \le r \ll 1$,

$$\begin{aligned}\frac{|\log\varepsilon|}{2}\int a\chi^z_R|\nu^{r_0}_{\varepsilon,R}| - \frac{\log 2}{2}\int a\chi^z_R|\nu^{r_0}_{\varepsilon,R}| - o(N_\varepsilon^2) = \frac{|\log(2\varepsilon)|}{2}\int a\chi^z_R|\nu^{r_0}_{\varepsilon,R}| - o(N_\varepsilon^2)\\ \le \frac12\int_{\tilde{\mathcal{B}}^{r_0,r}_{\varepsilon,R}} a\chi^z_R\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2(2\varepsilon)^2}(1-|u_\varepsilon|^2)^2\Big)\\ \le \mathcal{D}^*_{\varepsilon,R} + \frac{|\log\varepsilon|}{2}\int a\chi^z_R\mu_\varepsilon - \frac{3}{16\varepsilon^2}\int_{\tilde{\mathcal{B}}^{r_0,r}_{\varepsilon,R}} a^2\chi^z_R(1-|u_\varepsilon|^2)^2.\end{aligned}$$

Using (5.23), the bound of item (ii) on the number of vortices, and the choice of r_0, we then find

$$\frac{3}{16\varepsilon^2}\int_{\tilde{\mathcal{B}}^{r_0,r}_{\varepsilon,R}} a^2\chi_R^z(1-|u_\varepsilon|^2)^2 \le \mathcal{D}^*_{\varepsilon,R}+\frac{|\log\varepsilon|}{2}\int a\chi_R^z(\mu_\varepsilon-\nu^{r_0}_{\varepsilon,R})+\frac{\log 2}{2}\int a\chi_R^z|\nu^{r_0}_{\varepsilon,R}|+o(N_\varepsilon^2)$$
$$\le \mathcal{D}^*_{\varepsilon,R}+o(N_\varepsilon^2)\lesssim N_\varepsilon^2.$$

Combining this with the result (5.16) of item (v), we deduce the (suboptimal) estimate

$$\sup_z\int_{\mathbb{R}^d}\frac{\chi_R^z}{\varepsilon^2}(1-|u_\varepsilon|^2)^2\lesssim N_\varepsilon^2. \tag{5.50}$$

Injecting this result into (5.49), together with the bound of item (ii) on the number of vortices, we find

$$\left|\frac12\int_{B'\setminus B}\phi\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)-\pi\phi(y)d\log t\right|$$
$$\le CtsN_\varepsilon^2\|\nabla\phi\|_{\mathrm{L}^\infty}+\|\phi\|_{\mathrm{L}^\infty}\left|\frac12\int_{B'\setminus B}\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)-\pi d\log t\right|$$
$$+\,ts\|\nabla\phi\|_{\mathrm{L}^\infty}\int_{B'\setminus B}\left|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon-iu_\varepsilon\frac{d\tau}{|\cdot-y|}\right|^2. \tag{5.51}$$

Recalling the improved lower bound (5.40), and combining it with the bound of item (ii) on the number of vortices, and with the assumption $\|\mathrm{curl}\ \mathrm{v}_\varepsilon\|_{\mathrm{L}^\infty}\lesssim 1$, we find for $ts\le 1$,

$$(1+O(|\log\varepsilon|^{-1}))\frac12\int_{B'\setminus B}\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)\ge \pi d\log t-CtsN_\varepsilon^2$$
$$+\,(1-O(|\log\varepsilon|^{-1}))\frac12\int_{B'\setminus B}\left|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon-iu_\varepsilon\frac{d\tau}{|\cdot-y|}\right|^2.$$

Injecting this estimate into (5.51) yields

$$\left|\frac12\int_{B'\setminus B}\phi\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)-\pi\phi(y)d\log t\right|$$
$$\le C\|\phi\|_{W^{1,\infty}}\left(\frac12\int_{B'\setminus B}\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)-\pi d\log t\right)$$
$$+\,CtsN_\varepsilon^2\|\phi\|_{W^{1,\infty}}+C\|\phi\|_{W^{1,\infty}}|\log\varepsilon|^{-1}\int_{B'\setminus B}\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big).$$

Arguing as in (5.42), together with the bound of item (ii) on the number of vortices, we find

$$\left|2\pi\phi(y)d\log t-\log t\int_B\phi\nu^{r_0}_{\varepsilon,R}\right|\le\|\nabla\phi\|_{\mathrm{L}^\infty}s\log t\int_B|\nu^{r_0}_{\varepsilon,R}|\le C\|\nabla\phi\|_{\mathrm{L}^\infty}tsN_\varepsilon,$$

so that the above becomes

$$\left|\frac12\int_{B'\setminus B}\phi\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)-\frac{\log t}{2}\int_{B'}\phi\nu^{r_0}_{\varepsilon,R}\right|$$
$$\le C\|\phi\|_{W^{1,\infty}}\left(\frac12\int_{B'\setminus B}\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)-\pi d\log t\right)$$
$$+\,CtsN_\varepsilon^2\|\phi\|_{W^{1,\infty}}+C\|\phi\|_{W^{1,\infty}}|\log\varepsilon|^{-1}\int_{B'\setminus B}\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big).$$

By construction of the ball growth and merging process, this easily implies the following: if a ball $B=\bar B(y_B,s_B)$ belongs to the collection $\tilde{\mathcal B}_{\varepsilon,R}^{r_0,r}$ for some $r_0\le r\le 1$, then we have

$$\begin{aligned}\Big|\frac12\int_{B\setminus\tilde{\mathcal B}_{\varepsilon,R}^{r_0}}\phi\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)-\frac{\log(r/r_0)}{2}\int_B\phi\nu_{\varepsilon,R}^{r_0}\Big|\\
\le C\|\phi\|_{W^{1,\infty}}\bigg(\int_{B\setminus\tilde{\mathcal B}_{\varepsilon,R}^{r_0}}\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)-\log(r/r_0)\int_B\nu_{\varepsilon,R}^{r_0}\bigg)\\
+Cs_BN_\varepsilon^2\|\phi\|_{W^{1,\infty}}+C\|\phi\|_{W^{1,\infty}}|\log\varepsilon|^{-1}\int_B\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big).\end{aligned}$$

Since by assumption ϕ is supported in $B_R(z)$, we may write $\phi=\chi_R^z\phi$. Using that $|\chi_R^z(y)-\chi_R^z(y_B)|\lesssim s_BR^{-1}$ holds for all $y\in B$, and recalling the choice $R\gtrsim|\log\varepsilon|$, we then find

$$\begin{aligned}\Big|\frac12\int_{B\setminus\tilde{\mathcal B}_{\varepsilon,R}^{r_0}}\phi\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)-\frac{\log(r/r_0)}{2}\int_B\phi\nu_{\varepsilon,R}^{r_0}\Big|\\
\le\chi_R^z(y_B)\Big|\frac12\int_{B\setminus\tilde{\mathcal B}_{\varepsilon,R}^{r_0}}\phi\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)-\frac{\log(r/r_0)}{2}\int_B\phi\nu_{\varepsilon,R}^{r_0}\Big|\\
+C\|\phi\|_{\mathrm{L}^\infty}rR^{-1}\bigg(\int_B\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)+\log(r/r_0)\int_B|\nu_{\varepsilon,R}^{r_0}|\bigg)\\
\le C\|\phi\|_{W^{1,\infty}}\bigg(\int_{B\setminus\tilde{\mathcal B}_{\varepsilon,R}^{r_0}}\chi_R^z\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)-\log(r/r_0)\int_B\chi_R^z\nu_{\varepsilon,R}^{r_0}\bigg)\\
+Cs_BN_\varepsilon^2\|\phi\|_{W^{1,\infty}}+C\|\phi\|_{W^{1,\infty}}|\log\varepsilon|^{-1}\bigg(\int_B\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)+\log(r/r_0)\int_B|\nu_{\varepsilon,R}^{r_0}|\bigg).\end{aligned}$$

Summing this estimate over all balls B of the collection $\tilde{\mathcal B}_{\varepsilon,R}^{r_0,r}$ that intersect $B_R(z)$, recalling that the sum of the radii of these balls is by construction bounded by Cr, and using the optimal energy bound and the bound of item (ii) on the number of vortices, we deduce

$$\begin{aligned}\Big|\frac12\int_{\tilde{\mathcal B}_{\varepsilon,R}^{r_0,r}\setminus\tilde{\mathcal B}_{\varepsilon,R}^{r_0}}\phi\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)-\frac{\log(r/r_0)}{2}\int_{\mathbb R^2}\phi\nu_{\varepsilon,R}^{r_0}\Big|\\
\le C\|\phi\|_{W^{1,\infty}}\bigg(\int_{\tilde{\mathcal B}_{\varepsilon,R}^{r_0,r}\setminus\tilde{\mathcal B}_{\varepsilon,R}^{r_0}}\chi_R^z\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)-\log(r/r_0)\int_{\mathbb R^2}\chi_R^z\nu_{\varepsilon,R}^{r_0}\bigg)\\
+CrN_\varepsilon^2\|\phi\|_{W^{1,\infty}}+C\|\phi\|_{W^{1,\infty}}|\log\varepsilon|^{-1}\bigg(\mathcal E_{\varepsilon,2R}^z+|\log\varepsilon|\int_{B_{2R}(z)}|\nu_{\varepsilon,R}^{r_0}|\bigg)\\
\le C\|\phi\|_{W^{1,\infty}}\bigg(\int_{\tilde{\mathcal B}_{\varepsilon,R}^{r_0,r}\setminus\tilde{\mathcal B}_{\varepsilon,R}^{r_0}}\chi_R^z\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)-\log(r/r_0)\int_{\mathbb R^2}\chi_R^z\nu_{\varepsilon,R}^{r_0}+o(N_\varepsilon^2)\bigg).\end{aligned}$$

Combining this with (5.48), and recalling that by definition $\mathcal B_{\varepsilon,R}^{r_0}\subset\tilde{\mathcal B}_{\varepsilon,R}^{r_0}$, we deduce

$$\begin{aligned}\Big|\frac12\int_{\tilde{\mathcal B}_{\varepsilon,R}^{r_0,r}}\phi\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)-\frac{\log(r/\varepsilon)}{2}\int_{\mathbb R^2}\phi\nu_{\varepsilon,R}^{r_0}\Big|\\
\le C\|\phi\|_{W^{1,\infty}}\bigg(\int_{\tilde{\mathcal B}_{\varepsilon,R}^{r_0,r}}\chi_R^z\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)-\log(r/\varepsilon)\int_{\mathbb R^2}\chi_R^z\nu_{\varepsilon,R}^{r_0}+o(N_\varepsilon^2)\bigg).\end{aligned}$$

Using (5.23) to replace $\nu_{\varepsilon,R}^{r_0}$ by μ_ε in both sides up to an error of order $(r_0 N_\varepsilon|\log\varepsilon|+1)\|\phi\|_{W^{1,\infty}} \ll N_\varepsilon^2\|\phi\|_{W^{1,\infty}}$, the result (5.14) follows.

Substep 9.7: proof of (vi). We adapt an argument by Struwe [86] (see also [76, Proof of Lemma 4.7]). Recalling that $|B_{2R}(z)\cap\mathcal{B}_{\varepsilon,R}^r|\lesssim r^2$, a direct application of the Hölder inequality yields

$$\int_{\mathcal{B}_{\varepsilon,R}^r}\chi_R^z|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^p \lesssim r^{2-p}\Big(\int_{\mathcal{B}_{\varepsilon,R}^r}\chi_R^z|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2\Big)^{p/2}\lesssim r^{2-p}(N_\varepsilon|\log\varepsilon|)^{p/2},$$

which only implies the result if we are allowed to choose the total radius r small enough. Otherwise, it is useful to rather work on dyadic "annuli". For all integer $0\le k\le K_\varepsilon:=\lfloor\log_2(r/\varepsilon^{1/2})\rfloor$, define the "annulus" $E_k:=\mathcal{B}_{\varepsilon,R}^{r2^{-k}}\setminus\mathcal{B}_{\varepsilon,R}^{r2^{-k-1}}$. We set for simplicity $s_k:=r2^{-k}$. Applying the Hölder inequality separately on each annulus yields

$$\begin{aligned}\int_{\mathcal{B}_{\varepsilon,R}^r}\chi_R^z|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^p \le \Big(\int_{\mathcal{B}_{\varepsilon,R}^{\sqrt{\varepsilon}}}\chi_R^z|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2\Big)^{p/2}|B_{2R}(z)\cap\mathcal{B}_{\varepsilon,R}^{\sqrt{\varepsilon}}|^{1-p/2}&\\ +\sum_{k=0}^{K_\varepsilon}\Big(\int_{E_k}\chi_R^z|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2\Big)^{p/2}|B_{2R}(z)\cap E_k|^{1-p/2}&.\end{aligned}$$

Using that $|B_{2R}(z)\cap\mathcal{B}_{\varepsilon,R}^{\sqrt{\varepsilon}}|\lesssim\varepsilon$, that $|B_{2R}(z)\cap E_k|\lesssim s_k^2$, and that the integral over $\mathcal{B}_{\varepsilon,R}^{\sqrt{\varepsilon}}$ in the right-hand side is bounded by $\mathcal{E}_{\varepsilon,R}^z\lesssim N_\varepsilon|\log\varepsilon|$, we deduce

$$\int_{\mathcal{B}_{\varepsilon,R}^r}\chi_R^z|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^p\lesssim\varepsilon^{1-p/2}(N_\varepsilon|\log\varepsilon|)^{p/2}+\sum_{k=0}^{K_\varepsilon}s_k^{2-p}\Big(\int_{\mathbb{R}^2\setminus\mathcal{B}_{\varepsilon,R}^{s_{k+1}}}\chi_R^z|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2\Big)^{p/2}. \tag{5.52}$$

It remains to estimate the last integrals. Using Lemma 5.1(i)–(ii) in the forms (5.2) and (5.3), together with the optimal energy bound, we obtain

$$\begin{aligned}\frac12\int_{\mathcal{B}_{\varepsilon,R}^{s_{k+1}}}a\chi_R^z\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)&\\ \ge\frac{|\log\varepsilon|}{2}\int a\chi_R^z\nu_{\varepsilon,R}^{s_{k+1}}-O\big(N_\varepsilon|\log s_{k+1}|+s_{k+1}N_\varepsilon|\log\varepsilon|\big)-o(N_\varepsilon^2)&,\end{aligned}$$

and hence, using (5.23) to replace $\nu_{\varepsilon,R}^{s_{k+1}}$ by μ_ε,

$$\frac12\int_{\mathbb{R}^2\setminus\mathcal{B}_{\varepsilon,R}^{s_{k+1}}}a\chi_R^z|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2\le\mathcal{D}_{\varepsilon,R}^z+O(N_\varepsilon|\log s_{k+1}|+s_{k+1}N_\varepsilon|\log\varepsilon|)+o(N_\varepsilon^2).$$

If $r\ll N_\varepsilon|\log\varepsilon|^{-1}$, then $s_k\le r\ll N_\varepsilon|\log\varepsilon|^{-1}$ for all k, so that we find

$$\frac12\int_{\mathbb{R}^2\setminus\mathcal{B}_{\varepsilon,R}^{s_{k+1}}}\chi_R^z|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2\lesssim N_\varepsilon^2+N_\varepsilon(|\log r|+k). \tag{5.53}$$

Inserting this into (5.52) yields for all $p<2$, with $r\ll N_\varepsilon|\log\varepsilon|^{-1}$,

$$\begin{aligned}\int_{\mathcal{B}_{\varepsilon,R}^r}\chi_R^z|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^p&\lesssim\varepsilon^{1-p/2}(N_\varepsilon|\log\varepsilon|)^{p/2}+\sum_{k=0}^{K_\varepsilon}(r2^{-k})^{2-p}\Big(N_\varepsilon^p+N_\varepsilon^{p/2}|\log r|^{p/2}+N_\varepsilon^{p/2}k^{p/2}\Big)\\ &\lesssim_p\varepsilon^{1-p/2}(N_\varepsilon|\log\varepsilon|)^{p/2}+r^{2-p}N_\varepsilon^p+r^{2-p}N_\varepsilon^{p/2}|\log r|^{p/2}.\end{aligned}$$

In the regime $N_\varepsilon \gg \log|\log\varepsilon|$, we may choose $e^{-o(N_\varepsilon)} \le r \ll N_\varepsilon|\log\varepsilon|^{-1}$, and the above yields for that choice

$$\int_{\mathcal{B}^r_{\varepsilon,R}} \chi_R^z|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^p \ll_p N_\varepsilon^p. \tag{5.54}$$

Combining this with the result (5.15) of item (v) and with the Hölder inequality, the result (5.17) easily follows.

We now consider the regime $1 \ll N_\varepsilon \lesssim \log|\log\varepsilon|$. In that case, we need to prove (5.54) for larger values of the radius $r \ge e^{-o(N_\varepsilon)}$, and the above argument no longer holds. Given $\varepsilon^{1/2} < r_0 \ll N_\varepsilon|\log\varepsilon|^{-1}$, we replace the initial total radius $\varepsilon^{1/2}$ by r_0, and for $r_0 \le r \ll 1$ we consider the modified dyadic "annuli" $\tilde E_k := \tilde{\mathcal{B}}^{r_0,r2^{-k}}_{\varepsilon,R} \setminus \tilde{\mathcal{B}}^{r_0,r2^{-k-1}\vee r_0}_{\varepsilon,R}$, with $0 \le k \le K := \lfloor \log_2(r/r_0)\rfloor$. We set for simplicity $\tilde s_k := (r2^{-k}) \vee r_0$. The decomposition (5.52) is then replaced by

$$\int_{\tilde{\mathcal{B}}^{r_0,r}_{\varepsilon,R}} \chi_R^z|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^p \lesssim r_0^{2-p}(N_\varepsilon|\log\varepsilon|)^{p/2} + \sum_{k=0}^K s_k^{2-p}\Big(\int_{\mathbb{R}^2\setminus\tilde{\mathcal{B}}^{r_0,\tilde s_{k+1}}_{\varepsilon,R}} \chi_R^z|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2\Big)^{p/2}, \tag{5.55}$$

where it remains to adapt the estimate (5.53) for the last integrals. The lower bound (5.43) of Step 7 together with the optimal energy bound and with the bound of item (ii) on the number of vortices yields

$$\begin{aligned}\frac12\int_{\tilde{\mathcal{B}}^{r_0,\tilde s_{k+1}}_{\varepsilon,R}} a\chi_R^z\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) &\ge \frac{\log(\tilde s_{k+1}/\varepsilon)}{2}\int a\chi_R^z\nu^{r_0}_{\varepsilon,R} - o(N_\varepsilon^2)\\ &\ge \frac{|\log\varepsilon|}{2}\int a\chi_R^z\nu^{r_0}_{\varepsilon,R} - O(N_\varepsilon|\log s_{k+1}|) - o(N_\varepsilon^2),\end{aligned}$$

and hence, using (5.23) to replace $\nu^{r_0}_{\varepsilon,R}$ by μ_ε,

$$\frac12\int_{\mathbb{R}^2\setminus\tilde{\mathcal{B}}^{r_0,\tilde s_{k+1}}_{\varepsilon,R}} a\chi_R^z|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 \le \mathcal{D}^z_{\varepsilon,R} + O(N_\varepsilon|\log s_{k+1}| + r_0N_\varepsilon|\log\varepsilon|) + o(N_\varepsilon^2).$$

The choice $r_0 \ll N_\varepsilon|\log\varepsilon|^{-1}$ then yields

$$\frac12\int_{\mathbb{R}^2\setminus\tilde{\mathcal{B}}^{r_0,\tilde s_{k+1}}_{\varepsilon,R}} a\chi_R^z|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 \lesssim N_\varepsilon^2 + N_\varepsilon(|\log r| + k).$$

Inserting this into (5.55), the result (5.18) follows as before. □

Based on the above vortex-balls construction, we have the following approximation result, which is easily obtained just as in [74, Proposition 9.6].

Lemma 5.3. *Let $\varepsilon^{1/2} < r_0 \le r < \bar r$, and let $\mathcal{B}^r_{\varepsilon,R}$ and $\tilde{\mathcal{B}}^{r_0,r}_{\varepsilon,R}$ denote the collections of the balls constructed in Proposition 5.2. Then, given $\Gamma_\varepsilon \in W^{2,\infty}(\mathbb{R}^2)^2$, there exist approximate vector fields $\bar\Gamma_\varepsilon, \tilde\Gamma_\varepsilon \in W^{2,\infty}(\mathbb{R}^2)^2$ such that $\bar\Gamma_\varepsilon$ is constant in each ball of the collection $\mathcal{B}^r_{\varepsilon,R}$ and $\tilde\Gamma_\varepsilon$ is constant in each ball of the collection $\tilde{\mathcal{B}}^{r_0,r}_{\varepsilon,R}$, such that $\|\bar\Gamma_\varepsilon\|_{\mathrm{L}^\infty} \le \|\Gamma_\varepsilon\|_{\mathrm{L}^\infty}$ and $\|\tilde\Gamma_\varepsilon\|_{\mathrm{L}^\infty} \le \|\Gamma_\varepsilon\|_{\mathrm{L}^\infty}$, such that for all $0 \le \gamma \le 1$,*

$$\|\bar\Gamma_\varepsilon - \Gamma_\varepsilon\|_{C^\gamma} + \|\tilde\Gamma_\varepsilon - \Gamma_\varepsilon\|_{C^\gamma} \lesssim r^{1-\gamma}\|\nabla\Gamma_\varepsilon\|_{\mathrm{L}^\infty},$$

and such that for all $R \ge 1$,

$$\sup_z\|\nabla(\bar\Gamma_\varepsilon - \Gamma_\varepsilon)\|_{\mathrm{L}^1(B_R(z))} + \sup_z\|\nabla(\tilde\Gamma_\varepsilon - \Gamma_\varepsilon)\|_{\mathrm{L}^1(B_R(z))} \lesssim rR^2\|\nabla\Gamma_\varepsilon\|_{W^{1,\infty}}.$$

5.2 Additional results

In order to control the velocity of the vortices, the following quantitative version of the "product estimate" of [73] is needed; the proof is omitted, as it is a direct adaptation of [82, Appendix A] (further deforming the metric in a non-constant way in the time direction; see also [73, Section III]).

Lemma 5.4 (Product estimate)**.** *Denote by M_ε any quantity such that for all $q>0$,*

$$\lim_{\varepsilon\downarrow 0}\varepsilon^q M_\varepsilon=\lim_{\varepsilon\downarrow 0}|\log\varepsilon|M_\varepsilon^{-q}=\lim_{\varepsilon\downarrow 0}|\log\varepsilon|^{-1}\log M_\varepsilon=0.$$

Let $u_\varepsilon:[0,T]\times\mathbb{R}^2\to\mathbb{C}$, $\mathrm{v}_\varepsilon:[0,T]\times\mathbb{R}^2\to\mathbb{R}^2$, and $\mathrm{p}_\varepsilon:[0,T]\times\mathbb{R}^2\to\mathbb{R}$. Assume that $\mathcal{E}_{\varepsilon,R}^{*,t}\lesssim|\log\varepsilon|^2$ for all t, and that $\bar{\mathcal{E}}_{\varepsilon,R}^{*,T}\le M_\varepsilon$, where we have set

$$\bar{\mathcal{E}}_{\varepsilon,R}^{*,T}:=\sup_z\int_0^T\Big(\mathcal{E}_{\varepsilon,R}^{z,t}+\int\chi_R^z|\partial_t u_\varepsilon^t-iu_\varepsilon^tN_\varepsilon\mathrm{p}_\varepsilon^t|^2\Big)dt.$$

Then, for all $X\in W^{1,\infty}([0,T]\times\mathbb{R}^2)^2$ and $Y\in W^{1,\infty}([0,T]\times\mathbb{R}^2)$, we have for all $z\in\mathbb{R}^2$,

$$\begin{aligned}\Big|\int_0^T\int\chi_R^z\tilde V_\varepsilon\cdot XY\Big|\le\frac{1+C\frac{\log M_\varepsilon}{|\log\varepsilon|}}{|\log\varepsilon|}\Big(\int_0^T\int\chi_R^z|(\partial_t u_\varepsilon-iu_\varepsilon N_\varepsilon\mathrm{p}_\varepsilon)Y|^2+\int_0^T\int\chi_R^z|(\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon)\cdot X|^2\Big)\\+C\big(1+\|(X,Y)\|_{W^{1,\infty}([0,T]\times\mathbb{R}^2)}^5\big)\big(M_\varepsilon^{-1/8}+\varepsilon N_\varepsilon\big)\Big(\bar{\mathcal{E}}_{\varepsilon,R}^{*,T}+\sup_{0\le\tau\le T}\mathcal{E}_{\varepsilon,R}^{*,\tau}+N_\varepsilon^2\Big).\end{aligned}$$

We now turn to some useful a priori estimates on the solution u_ε of equation (1.5). We begin with the following (very suboptimal) a priori bound on the velocity of the vortices, adapted from [82, Lemma 4.1].

Lemma 5.5 (A priori bound on velocity)**.** *Let $\alpha\ge 0$, $\beta\in\mathbb{R}$, and let $h:\mathbb{R}^2\to\mathbb{R}$, $a:=e^h$, $F:\mathbb{R}^2\to\mathbb{R}^2$, $f:\mathbb{R}^2\to\mathbb{R}$ satisfy (2.1). Let $u_\varepsilon:[0,T)\times\mathbb{R}^2\to\mathbb{C}$ and $\mathrm{v}_\varepsilon:[0,T)\times\mathbb{R}^2\to\mathbb{R}^2$ be solutions of (1.5) and (3.2) as in Proposition 2.2(i) and in Proposition 3.1, respectively. Let $0<\varepsilon\ll 1$, $1\le N_\varepsilon\lesssim|\log\varepsilon|$, and $R\ge 1$ with $\varepsilon R^\theta\ll 1$ for some $\theta>0$, and assume that $\mathcal{E}_{\varepsilon,R}^{*,t}\lesssim_t N_\varepsilon|\log\varepsilon|$ for all t. Then, in each of the regimes considered,* (GL$_1$), (GL$_2$), (GL$_1'$), *and* (GL$_2'$), *we have for all $\theta>0$, for all t,*

$$\alpha^2\sup_z\int_0^t\int a\chi_R^z|\partial_t u_\varepsilon|^2\lesssim_{t,\theta}N_\varepsilon|\log\varepsilon|^3+R^\theta N_\varepsilon^2|\log\varepsilon|^2\lesssim R^\theta N_\varepsilon|\log\varepsilon|^3.$$

Proof. We focus on the non-decaying setting, as the other case is similar. Integrating identity (4.18) in time, reorganizing the terms, and setting $D_{\varepsilon,R}^{z,t}:=\int_0^t\int a\chi_R^z|\partial_t u_\varepsilon|^2$, we obtain

$$\begin{aligned}\lambda_\varepsilon\alpha D_{\varepsilon,R}^{z,t}=\hat{\mathcal{E}}_{\varepsilon,R}^{z,\circ}-\hat{\mathcal{E}}_{\varepsilon,R}^{z,t}-\int_0^t\int a\nabla\chi_R^z\cdot\langle\partial_t u_\varepsilon,\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon\rangle+\int_0^t\int N_\varepsilon\chi_R^z\langle\partial_t u_\varepsilon,iu_\varepsilon\rangle\operatorname{div}(a\mathrm{v}_\varepsilon)\\+\int\frac{aN_\varepsilon^2}{2}(1-|u_\varepsilon^t|^2)(\psi_{\varepsilon,R}^{z,t}-\chi_R^z|\mathrm{v}_\varepsilon^t|^2)-\int\frac{aN_\varepsilon^2}{2}(1-|u_\varepsilon^\circ|^2)(\psi_{\varepsilon,R}^{z,\circ}-\chi_R^z|\mathrm{v}_\varepsilon^\circ|^2)\\+\int_0^t\int a\chi_R^z\Big(N_\varepsilon(N_\varepsilon\mathrm{v}_\varepsilon-j_\varepsilon)\cdot\partial_t\mathrm{v}_\varepsilon-N_\varepsilon\mathrm{v}_\varepsilon\cdot V_\varepsilon-\frac{|\log\varepsilon|}{2}F^\perp\cdot V_\varepsilon\Big).\end{aligned}$$

Noting that $|\nabla\chi_R^z|\lesssim R^{-1}(\chi_R^z)^{1/2}$, using the pointwise estimates of Lemma 4.2 for V_ε and $j_\varepsilon-N_\varepsilon\mathrm{v}_\varepsilon$, and using assumptions (2.1), the properties of v_ε in Proposition 3.1, the bound (4.4) on $\psi_{\varepsilon,R}^z$, and Lemma 4.1 in the form $\hat{\mathcal{E}}^{z,t}_{\varepsilon,R} \lesssim \mathcal{E}^{*,t}_{\varepsilon,R} + o(N_\varepsilon^2) \lesssim_t N_\varepsilon|\log\varepsilon|$, we find for $\theta > 0$ small enough, in the considered regimes,

$$\begin{aligned}\lambda_\varepsilon \alpha D^{z,t}_{\varepsilon,R} &\lesssim_{t,\theta} N_\varepsilon|\log\varepsilon| + R^{-1}(N_\varepsilon|\log\varepsilon|)^{1/2}(D^{z,t}_{\varepsilon,R})^{1/2} + N_\varepsilon(1+\varepsilon(N_\varepsilon|\log\varepsilon|)^{1/2})(D^{z,t}_{\varepsilon,R})^{1/2} \\ &+ \varepsilon N_\varepsilon^2(N_\varepsilon|\log\varepsilon|)^{1/2}\Big(1+\frac{|\log\varepsilon|}{N_\varepsilon}(\lambda_\varepsilon R^\theta + \lambda_\varepsilon^{1/2} + R^{-1+\theta})\Big) + N_\varepsilon(N_\varepsilon|\log\varepsilon|)^{1/2}(1+\varepsilon N_\varepsilon) + \varepsilon\lambda_\varepsilon^{-1/2}N_\varepsilon^2|\log\varepsilon| \\ &\qquad + (N_\varepsilon + \lambda_\varepsilon|\log\varepsilon|)\big((1+\varepsilon N_\varepsilon)(N_\varepsilon|\log\varepsilon|)^{1/2} + N_\varepsilon R^\theta\big)(D^{z,t}_{\varepsilon,R})^{1/2} \\ &\qquad\qquad \lesssim_\theta N_\varepsilon|\log\varepsilon| + (N_\varepsilon + \lambda_\varepsilon|\log\varepsilon|)\big((N_\varepsilon|\log\varepsilon|)^{1/2} + N_\varepsilon R^\theta\big)(D^{z,t}_{\varepsilon,R})^{1/2} + o(1).\end{aligned}$$

Absorbing $(D^{z,t}_{\varepsilon,R})^{1/2}$ in the left-hand side, and noting that either $\lambda_\varepsilon = 1$ or $\lambda_\varepsilon = \frac{N_\varepsilon}{|\log\varepsilon|}$ in the considered regimes, the result follows. $\square$

The following optimal a priori estimate is also crucially needed in our analysis in the presence of pinning, due to the absence of a factor $\frac{1}{2}$ in front of the $\frac{a}{\varepsilon^2}(1-|u_\varepsilon|^2)^2$-part of the energy density as it appears in the term $I^H_{\varepsilon,\varrho,R}$ in Lemma 4.4. A simple computation based on the lower bound results of Proposition 5.2 yields a similar bound with N_ε replaced by N_ε^2 (see indeed (5.50)), but the optimal result below is much more subtle. It is proved as a combination of the Pohozaev vortex-balls construction of [74, Section 5], together with some careful cut-off techniques inspired by [74, Proof of Proposition 13.4].

Lemma 5.6. *Let $\alpha \ge 0$, $\beta \in \mathbb{R}$, and let $h : \mathbb{R}^2 \to \mathbb{R}$, $a := e^h$, $F : \mathbb{R}^2 \to \mathbb{R}^2$, $f : \mathbb{R}^2 \to \mathbb{R}$ satisfy* (2.1). *Let $u_\varepsilon : [0,T) \times \mathbb{R}^2 \to \mathbb{C}$ and $\mathrm{v}_\varepsilon : [0,T) \times \mathbb{R}^2 \to \mathbb{R}^2$ be solutions of* (1.5) *and* (3.2) *as in Proposition 2.2(i) and in Proposition 3.1, respectively. Let $0 < \varepsilon \ll 1$, $1 \le N_\varepsilon \lesssim |\log\varepsilon|$, and $R \ge 1$ with $\varepsilon R|\log\varepsilon|^3 \lesssim 1$, and assume that $\mathcal{E}^{*,t}_{\varepsilon,R} \lesssim_t N_\varepsilon|\log\varepsilon|$ for all t. Then, in each of the regimes considered,* (GL_1), (GL_2), (GL_1'), *and* (GL_2'), *we have for all t*

$$\alpha^2 \sup_z \int_0^t \int \frac{\chi_R^z}{\varepsilon^2}(1-|u_\varepsilon|^2)^2 \lesssim_t N_\varepsilon. \tag{5.56}$$

Proof. To simplify notation, we focus on the case $z = 0$, but the result of course holds uniformly with respect to the translation $z \in R\mathbb{Z}^2$. We split the proof into three steps.

Step 1: Pohozaev estimate on balls. In this step, we prove the following Pohozaev type estimate, adapted from [74, Theorem 5.1]: for any ball $B_r(x_0)$ with $r \le 1$, we have

$$\begin{aligned}\alpha^2 \int_0^t \int_{B_r(x_0)} \frac{a^2\chi_R}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 &\lesssim_t r\lambda_\varepsilon N_\varepsilon|\log\varepsilon|^3 \\ &+ r\int_0^t \int_{\partial B_r(x_0)} \frac{a\chi_R}{2}\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 + |1-|u_\varepsilon|^2|(N_\varepsilon^2|\mathrm{v}_\varepsilon|^2 + |f|)\Big).\end{aligned} \tag{5.57}$$

For any smooth vector field X and any bounded open set $U \subset \mathbb{R}^2$, we have by integration by parts

$$-\int_U \chi_R \nabla X : \tilde{S}_\varepsilon = \int_U \chi_R \,\mathrm{div}\, \tilde{S}_\varepsilon \cdot X + \int_U X \cdot \tilde{S}_\varepsilon \cdot \nabla\chi_R - \int_{\partial U} \chi_R X \cdot \tilde{S}_\varepsilon \cdot n,$$

and hence, for $U = B_r(x_0)$, $r > 0$, and $X = x - x_0$,

$$-\int_{B_r(x_0)} \chi_R \,\mathrm{Tr}\, \tilde{S}_\varepsilon = \int_{B_r(x_0)} \chi_R \,\mathrm{div}\, \tilde{S}_\varepsilon \cdot (x-x_0) + \int_{B_r(x_0)} (x-x_0) \cdot \tilde{S}_\varepsilon \cdot \nabla\chi_R - r\int_{\partial B_r(x_0)} \chi_R \,\tilde{S}_\varepsilon : n \otimes n.$$

By definition (4.13) of the modulated stress-energy tensor $\tilde{S}_\varepsilon$, this means

$$\int_{B_r(x_0)} a\chi_R\Big(\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2+(1-|u_\varepsilon|^2)f\Big)=\int_{B_r(x_0)}\chi_R\,\mathrm{div}\;\tilde{S}_\varepsilon\cdot(x-x_0)+\int_{B_r(x_0)}(x-x_0)\cdot\tilde{S}_\varepsilon\cdot\nabla\chi_R$$
$$+r\int_{\partial B_r(x_0)}\frac{a\chi_R}{2}\Big(|n^\perp\cdot(\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon)|^2-|n\cdot(\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon)|^2$$
$$+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2+(1-|u_\varepsilon|^2)\big(N_\varepsilon^2(|n^\perp\cdot \mathrm{v}_\varepsilon|^2-|n\cdot \mathrm{v}_\varepsilon|^2)+f\big)\Big),$$

so that we may simply estimate

$$\int_{B_r(x_0)}\frac{a^2\chi_R}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\le r\int_{B_r(x_0)}|\mathrm{div}\;\tilde{S}_\varepsilon|+r\int_{B_r(x_0)}|\nabla\chi_R||\tilde{S}_\varepsilon|+\int_{B_r(x_0)}a|1-|u_\varepsilon|^2||f|$$
$$+r\int_{\partial B_r(x_0)}\frac{a\chi_R}{2}\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2+|1-|u_\varepsilon|^2|(N_\varepsilon^2|\mathrm{v}_\varepsilon|^2+|f|)\Big).\quad(5.58)$$

It remains to estimate the first three right-hand side terms. Using the pointwise estimates of Lemma 4.2, using assumption (2.1) and the boundedness properties of $\mathrm{v}_\varepsilon,\mathrm{p}_\varepsilon$ (cf. Proposition 3.1), and noting that $\lambda_\varepsilon\lesssim 1$ holds in the regimes considered, Lemma 4.3 directly yields

$$|\mathrm{div}\;\tilde{S}_\varepsilon|\lesssim\lambda_\varepsilon|\log\varepsilon||\partial_t u_\varepsilon||\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|+N_\varepsilon(1+\lambda_\varepsilon^{1/2}|\log\varepsilon|)(1+|1-|u_\varepsilon|^2|)|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|$$
$$+\lambda_\varepsilon N_\varepsilon|\log\varepsilon||\partial_t u_\varepsilon|(1+|1-|u_\varepsilon|^2|)+(N_\varepsilon+\lambda_\varepsilon|\log\varepsilon|)|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\varepsilon^{-2}(1-|u_\varepsilon|^2)^2$$
$$+|1-|u_\varepsilon|^2|\Big(N_\varepsilon^2(N_\varepsilon+\lambda_\varepsilon|\log\varepsilon|)+\lambda_\varepsilon^2|\log\varepsilon|^2\Big)+N_\varepsilon^2(N_\varepsilon+\lambda_\varepsilon|\log\varepsilon|),$$

which gives, using $N_\varepsilon\lesssim|\log\varepsilon|$,

$$|\mathrm{div}\;\tilde{S}_\varepsilon|\lesssim\lambda_\varepsilon|\partial_t u_\varepsilon|^2+\lambda_\varepsilon|\log\varepsilon|^2|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\lambda_\varepsilon N_\varepsilon^2|\log\varepsilon|^2(1+(1-|u_\varepsilon|^2)^2)+\varepsilon^{-2}(1-|u_\varepsilon|^2)^2.$$

By Lemma 5.5 with $R=1$, we deduce for all $r\le 1$,

$$\alpha^2\int_0^t\int_{B_r(x_0)}|\mathrm{div}\;\tilde{S}_\varepsilon|\lesssim_t\lambda_\varepsilon N_\varepsilon|\log\varepsilon|^3+\lambda_\varepsilon N_\varepsilon^2|\log\varepsilon|^2(1+\varepsilon^2N_\varepsilon|\log\varepsilon|)\lesssim\lambda_\varepsilon N_\varepsilon|\log\varepsilon|^3.$$

Inserting this into (5.58), and noting that (2.1) in the form $\|f\|_{\mathrm{L}^\infty}\lesssim|\log\varepsilon|^2$ yields

$$\int_{B_r(x_0)}a|1-|u_\varepsilon|^2||f|\lesssim_t\varepsilon r(N_\varepsilon|\log\varepsilon|)^{1/2}\|f\|_{\mathrm{L}^\infty}\lesssim\varepsilon r|\log\varepsilon|^3,$$

and

$$\int_{B_r(x_0)}|\nabla\chi_R||\tilde{S}_\varepsilon|\lesssim R^{-1}\int_{B_r(x_0)}\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2+\frac{1}{\varepsilon^2}(1-|u_\varepsilon|^2)^2+\varepsilon^2(N_\varepsilon^4|\mathrm{v}_\varepsilon|^4+|f|^2)\Big)$$
$$\lesssim R^{-1}\big(N_\varepsilon|\log\varepsilon|+\varepsilon^2(N_\varepsilon^4+\|f\|_{\mathrm{L}^\infty}^2)\big)\lesssim N_\varepsilon|\log\varepsilon|,$$

the result (5.57) follows.

Step 2: estimate inside small balls. In this step, we prove the desired estimate (5.56) for the integral restricted to suitable small balls centered at the vortex locations. More precisely, since we have by assumption $\mathcal{E}^*_{\varepsilon,R}\lesssim N_\varepsilon|\log\varepsilon|\lesssim|\log\varepsilon|^2$, we may apply [74, Proposition 4.8] for any $\kappa\in(0,1)$. This yields a finite union

$\hat{\mathcal{B}}_{\varepsilon,0}$ of disjoint closed balls with total radius $r(\hat{\mathcal{B}}_{\varepsilon,0}) = \varepsilon^{\kappa/2}$, covering the set $\{x \in B_{2R} : ||u_\varepsilon(x)| - 1| \geq \varepsilon^{\kappa/4}\}$. We then prove that

$$\alpha^2 \int_0^t \int_{\hat{\mathcal{B}}_{\varepsilon,0}} \frac{a^2 \chi_R}{2\varepsilon^2}(1 - |u_\varepsilon|^2)^2 \lesssim_t N_\varepsilon. \tag{5.59}$$

For that purpose, we let the initial collection of balls $\hat{\mathcal{B}}_{\varepsilon,0}$ grow, and we use the Pohozaev estimate of Step 1 as in [74, Proof of Theorem 5.1]. By [74, Theorem 4.2], there exists a monotone family $(\hat{\mathcal{B}}_\varepsilon^s)_{s\geq 0}$ of unions of disjoint closed balls, such that $\hat{\mathcal{B}}_\varepsilon^0 = \hat{\mathcal{B}}_{\varepsilon,0}$, $\hat{\mathcal{B}}_\varepsilon^s$ has total radius $r(\hat{\mathcal{B}}_\varepsilon^s) = e^s r(\hat{\mathcal{B}}_{\varepsilon,0})$ for all $s \geq 0$, and $\hat{\mathcal{B}}_\varepsilon^s = e^{s-r}\hat{\mathcal{B}}_\varepsilon^r$ for all $0 \leq r \leq s$ with $[r,s] \subset \mathbb{R}^+ \setminus \mathcal{T}_\varepsilon$, for some finite set $\mathcal{T}_\varepsilon \subset \mathbb{R}^+$ (corresponding to the merging times in the growth process). For all $s \geq 0$ with $r(\hat{\mathcal{B}}_\varepsilon^s) \leq 1$, the result (5.57) of Step 1 gives the following estimate, for all $\theta > 0$,

$$\begin{aligned}\alpha^2 \int_0^t \int_{\hat{\mathcal{B}}_\varepsilon^s} \frac{a^2\chi_R}{2\varepsilon^2}(1 - |u_\varepsilon|^2)^2 &\lesssim_t r(\hat{\mathcal{B}}_\varepsilon^s) N_\varepsilon |\log \varepsilon|^3 \\ &+ \sum_{B_r(x) \in \hat{\mathcal{B}}_\varepsilon^s} r \int_0^t \int_{\partial B_r(x)} \frac{a\chi_R}{2}\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1 - |u_\varepsilon|^2)^2 + |1 - |u_\varepsilon|^2|(N_\varepsilon^2 |\mathrm{v}_\varepsilon|^2 + f)\Big).\end{aligned}$$

Integrating this estimate over s and applying [74, Proposition 4.1], we find, for all $s \geq 0$ with $r(\hat{\mathcal{B}}_\varepsilon(s)) \leq 1$,

$$\begin{aligned}&s\alpha^2 \int_0^t \int_{\hat{\mathcal{B}}_{\varepsilon,0}} \frac{a^2\chi_R}{2\varepsilon^2}(1 - |u_\varepsilon|^2)^2 \leq \alpha^2 \int_0^s dv \int_0^t \int_{\hat{\mathcal{B}}_\varepsilon^v} \frac{a^2\chi_R}{2\varepsilon^2}(1 - |u_\varepsilon|^2)^2 \\ &\lesssim_t s\, r(\hat{\mathcal{B}}_\varepsilon^s) N_\varepsilon |\log\varepsilon|^3 + \int_0^t \int_{\hat{\mathcal{B}}_\varepsilon^s \setminus \hat{\mathcal{B}}_{\varepsilon,0}} \frac{a\chi_R}{2}\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1 - |u_\varepsilon|^2)^2 + |1 - |u_\varepsilon|^2|(N_\varepsilon^2 |\mathrm{v}_\varepsilon|^2 + f)\Big),\end{aligned}$$

and hence, using assumption (2.1) and the boundedness of v_ε (cf. Proposition 2.2(i)), and the assumed energy bound,

$$s\alpha^2 \int_0^t \int_{\hat{\mathcal{B}}_{\varepsilon,0}} \frac{a^2\chi_R}{2\varepsilon^2}(1 - |u_\varepsilon|^2)^2 \lesssim_t s\, r(\hat{\mathcal{B}}_\varepsilon^s) N_\varepsilon |\log\varepsilon|^3 + N_\varepsilon |\log\varepsilon|.$$

Recalling that $r(\hat{\mathcal{B}}_\varepsilon^s) = e^s \varepsilon^{\kappa/2}$, this yields for all $s \geq 1$ with $r(\hat{\mathcal{B}}_\varepsilon^s) \leq 1$,

$$\alpha^2 \int_0^t \int_{\hat{\mathcal{B}}_{\varepsilon,0}} \frac{a^2\chi_R}{2\varepsilon^2}(1 - |u_\varepsilon|^2)^2 \lesssim_t e^s \varepsilon^{\kappa/2} N_\varepsilon |\log\varepsilon|^3 + \frac{N_\varepsilon |\log\varepsilon|}{s},$$

and the result (5.59) now follows for the choice $s = |\log \varepsilon^{\kappa/4}|$.

Step 3: estimate outside small balls. It remains to show that the desired estimate (5.56) also holds for the integral restricted to the complement of the small balls $\hat{\mathcal{B}}_{\varepsilon,0}$. More precisely, we prove in this step for all $\theta > 0$,

$$\alpha \int_0^t \int_{||u_\varepsilon| - 1| \leq \varepsilon^{\kappa/4}} \chi_R\Big(|\nabla |u_\varepsilon||^2 + \frac{a(1 - |u_\varepsilon|^2)^2}{2\varepsilon^2}\Big) \lesssim_{t,\theta} \varepsilon^{\kappa/4} R^\theta |\log\varepsilon|^2 + \varepsilon R |\log \varepsilon|^3 \tag{5.60}$$

The conclusion (5.56) of course follows from this together with (5.59), choosing $\theta > 0$ small enough. In order to prove (5.60), we adapt the argument of [74, Proof of Proposition 13.4]. For $0 < \varepsilon \leq 2^{-4/\kappa}$, we define a cut-off function ζ_ε as follows,

$$\zeta_\varepsilon(y) := \begin{cases} y, & \text{if } 0 \le y \le 1/2; \\ \frac{1}{2} + \frac{y-1/2}{1-2\varepsilon^{\kappa/4}}, & \text{if } 1/2 \le y \le 1-\varepsilon^{\kappa/4}; \\ 1, & \text{if } 1-\varepsilon^{\kappa/4} \le y \le 1+\varepsilon^{\kappa/4}; \\ 1 + \frac{y-1-\varepsilon^{\kappa/4}}{1-2\varepsilon^{\kappa/4}}, & \text{if } 1+\varepsilon^{\kappa/4} \le y \le 3/2; \\ y, & \text{if } y \ge 3/2. \end{cases}$$

Writing $u_\varepsilon := \rho_\varepsilon e^{i\varphi_\varepsilon}$ locally, the equation (1.5) for u_ε yields in particular

$$\alpha\lambda_\varepsilon\partial_t\rho_\varepsilon - \beta\lambda_\varepsilon|\log\varepsilon|\rho_\varepsilon\partial_t\varphi_\varepsilon = \triangle\rho_\varepsilon - \rho_\varepsilon|\nabla\varphi_\varepsilon|^2 + \frac{a\rho_\varepsilon}{\varepsilon^2}(1-\rho_\varepsilon^2) + \nabla h\cdot\nabla\rho_\varepsilon - \rho_\varepsilon|\log\varepsilon|F^\perp\cdot\nabla\varphi_\varepsilon + f\rho_\varepsilon. \quad (5.61)$$

Testing this equation against $\chi_R(\zeta_\varepsilon(\rho_\varepsilon)-\rho_\varepsilon)$, and rearranging the terms, we obtain

$$\begin{aligned} \int \chi_R(1-\zeta_\varepsilon'(\rho_\varepsilon))|\nabla\rho_\varepsilon|^2 + \int \frac{a\chi_R}{\varepsilon^2}\rho_\varepsilon(\zeta_\varepsilon(\rho_\varepsilon)-\rho_\varepsilon)(1-\rho_\varepsilon^2) &= \alpha\lambda_\varepsilon\int\chi_R(\zeta_\varepsilon(\rho_\varepsilon)-\rho_\varepsilon)\partial_t\rho_\varepsilon \\ - \beta\lambda_\varepsilon|\log\varepsilon|\int\chi_R\rho_\varepsilon(\zeta_\varepsilon(\rho_\varepsilon)-\rho_\varepsilon)\partial_t\varphi_\varepsilon &+ \int(\zeta_\varepsilon(\rho_\varepsilon)-\rho_\varepsilon)\nabla\chi_R\cdot\nabla\rho_\varepsilon + \int\chi_R(\zeta_\varepsilon(\rho_\varepsilon)-\rho_\varepsilon)\rho_\varepsilon|\nabla\varphi_\varepsilon|^2 \\ - \int\chi_R(\zeta_\varepsilon(\rho_\varepsilon)-\rho_\varepsilon)\nabla h\cdot\nabla\rho_\varepsilon &+ |\log\varepsilon|\int\chi_R\rho_\varepsilon(\zeta_\varepsilon(\rho_\varepsilon)-\rho_\varepsilon)F^\perp\cdot\nabla\varphi_\varepsilon - \int\chi_R(\zeta_\varepsilon(\rho_\varepsilon)-\rho_\varepsilon)f\rho_\varepsilon. \quad (5.62) \end{aligned}$$

Using that the cut-off function ζ_ε satisfies for all $y \ge 0$

$$|\zeta_\varepsilon(y)-y| \lesssim \varepsilon^{\kappa/4}\mathbb{1}_{|y-1|\le 1/2}, \qquad |\zeta_\varepsilon(y)-y| \le |1-y| \le |1-y^2|, \quad (5.63)$$

$$|\zeta_\varepsilon'(y)-1| \lesssim \mathbb{1}_{|y-1|\le\varepsilon^{\kappa/4}} + \varepsilon^{\kappa/4}\mathbb{1}_{|y-1|\le 1/2}, \qquad (\zeta_\varepsilon(y)-y)(1-y) \ge 0, \quad (5.64)$$

and noting that

$$\int_{|\rho_\varepsilon-1|\le\varepsilon^{\kappa/4}}\frac{a\chi_R}{5\varepsilon^2}(1-\rho_\varepsilon^2)^2 \le \int_{|\rho_\varepsilon-1|\le\varepsilon^{\kappa/4}}\frac{a\chi_R}{\varepsilon^2}\rho_\varepsilon(1-\rho_\varepsilon)(1-\rho_\varepsilon^2) \le \int\frac{a\chi_R}{\varepsilon^2}\rho_\varepsilon(\zeta_\varepsilon(\rho_\varepsilon)-\rho_\varepsilon)(1-\rho_\varepsilon^2),$$

we obtain from (2.1), (5.62) and (5.63),

$$\begin{aligned} \int_{|\rho_\varepsilon-1|\le\varepsilon^{\kappa/4}}\chi_R\Big(|\nabla\rho_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-\rho_\varepsilon^2)^2\Big) &\lesssim \varepsilon^{\kappa/4}\int_{|\rho_\varepsilon-1|\le 1/2}\chi_R(|\nabla\rho_\varepsilon|^2+\rho_\varepsilon^2|\nabla\varphi_\varepsilon|^2) \\ + \lambda_\varepsilon|\log\varepsilon|\int_{|\rho_\varepsilon-1|\le 1/2}\chi_R|1-\rho_\varepsilon^2|(|\partial_t\rho_\varepsilon|+\rho_\varepsilon|\partial_t\varphi_\varepsilon|) &+ (1+\lambda_\varepsilon|\log\varepsilon|)\int_{|\rho_\varepsilon-1|\le 1/2}\chi_R|1-\rho_\varepsilon^2|(|\nabla\rho_\varepsilon|+\rho_\varepsilon|\nabla\varphi_\varepsilon|) \\ &+ \int_{|\rho_\varepsilon-1|\le 1/2}\chi_R|f||1-\rho_\varepsilon^2| + \int_{|\rho_\varepsilon-1|\le 1/2}|\nabla\chi_R||1-\rho_\varepsilon^2||\nabla\rho_\varepsilon|. \end{aligned}$$

Since $|\nabla u_\varepsilon|^2 = |\nabla\rho_\varepsilon|^2 + \rho_\varepsilon^2|\nabla\varphi_\varepsilon|^2$, and $|\partial_t u_\varepsilon|^2 = |\partial_t\rho_\varepsilon|^2 + \rho_\varepsilon^2|\partial_t\varphi_\varepsilon|^2$, we obtain with assumption (2.1),

$$\begin{aligned} \int_{|\rho_\varepsilon-1|\le\varepsilon^{\kappa/4}}\chi_R\Big(|\nabla|u_\varepsilon||^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) &\lesssim \varepsilon^{\kappa/4}\|\nabla u_\varepsilon\|^2_{\mathrm{L}^2(B_{2R})} + \lambda_\varepsilon|\log\varepsilon|\|1-|u_\varepsilon|^2\|_{\mathrm{L}^2(B_{2R})}\|\partial_t u_\varepsilon\|_{\mathrm{L}^2(B_{2R})} \\ &+ (1+\lambda_\varepsilon|\log\varepsilon|)\|1-|u_\varepsilon|^2\|_{\mathrm{L}^2(B_{2R})}\|\nabla u_\varepsilon\|_{\mathrm{L}^2(B_{2R})} + R(1+\lambda_\varepsilon^2|\log\varepsilon|^2)\|1-|u_\varepsilon|^2\|_{\mathrm{L}^2(B_{2R})}. \end{aligned}$$

By the integrability properties of v_ε (cf. Proposition 3.1), we have for all $\theta > 0$

$$\|\nabla u_\varepsilon\|_{\mathrm{L}^2(B_{2R})} \lesssim_\theta \|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon\|_{\mathrm{L}^2(B_{2R})} + N_\varepsilon(R^\theta + \|1-|u_\varepsilon|^2\|_{\mathrm{L}^2(B_{2R})}),$$

hence, using also Lemma 5.5, with $\lambda_\varepsilon \lesssim 1$ and $N_\varepsilon \lesssim |\log\varepsilon|$,

$$\alpha \int_0^t \int_{|\rho_\varepsilon - 1| \le \varepsilon^{\kappa/4}} \chi_R \Big(|\nabla |u_\varepsilon||^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 \Big) \lesssim_{t,\theta} \varepsilon^{\kappa/4} R^\theta |\log\varepsilon|^2 + \varepsilon R |\log\varepsilon|^3,$$

and the result (5.60) follows. □

6 Dissipative case: main proof

In this section we prove Theorem 1.1, that is, the mean-field limit result in the dissipative case $\alpha > 0$, in both critical regimes (GL_1) and (GL_2) (that is, with $N_\varepsilon \ll |\log\varepsilon|$ and $N_\varepsilon \simeq |\log\varepsilon|$, respectively), and we further consider the subcritical regimes (GL_1') and (GL_2'). More precisely, the rescaled supercurrent density $N_\varepsilon^{-1} j_\varepsilon$ is shown to remain close to the solution v_ε of equation (3.2). Combining this with the results of Section 3.1 (in particular, with Lemma 3.2), the result of Theorem 1.1 follows.

6.1 Modulated energy argument

Using the various estimates and technical tools developed in Section 5, we may now turn to the estimation of the various terms in the decomposition of Lemma 4.4, and deduce the smallness of the modulated energy excess by a Grönwall argument. This is the main step in the proof of the mean-field limit result stated in Theorem 1.1. (In this section, as we assume $\alpha > 0$, multiplicative constants are allowed to additionally depend on an upper bound on α^{-1}.)

Proposition 6.1. *Let $\alpha > 0$, $\beta \in \mathbb{R}$, $\alpha^2 + \beta^2 = 1$, and let $h : \mathbb{R}^2 \to \mathbb{R}$, $a := e^h$, $F : \mathbb{R}^2 \to \mathbb{R}^2$, $f : \mathbb{R}^2 \to \mathbb{R}$ satisfy (2.1). Let $u_\varepsilon : [0,T) \times \mathbb{R}^2 \to \mathbb{C}$ and $\mathrm{v}_\varepsilon : [0,T) \times \mathbb{R}^2 \to \mathbb{R}^2$ be solutions of (1.5) and (3.2) as in Proposition 2.2(i) and in Proposition 3.1, respectively, for some $T > 0$. Let $0 < \varepsilon \ll 1$ and $R \ge 1$ satisfy $1 \ll N_\varepsilon \lesssim |\log\varepsilon|$, $|\log\varepsilon|/N_\varepsilon \ll R \lesssim |\log\varepsilon|^n$ for some $n \ge 1$, and assume that the initial modulated energy excess satisfies $\mathcal{D}^{*,\circ}_{\varepsilon,R} \ll N_\varepsilon^2$. Then,*

- *(i) if $\log|\log\varepsilon| \ll N_\varepsilon \lesssim |\log\varepsilon|$, we have $\mathcal{D}^{*,t}_{\varepsilon,R} \ll_t N_\varepsilon^2$ for all $t \in [0,T)$, in each of the regimes (GL_1), (GL_2), (GL_1'), and (GL_2');*
- *(ii) if $1 \ll N_\varepsilon \lesssim \log|\log\varepsilon|$, in the pure dissipative case $\alpha = 1$, $\beta = 0$, the same conclusion $\mathcal{D}^{*,t}_{\varepsilon,R} \ll_t N_\varepsilon^2$ holds for all $t \in [0,T)$ in the regime (GL_1), as well as in the regime (GL_2') with $\lambda_\varepsilon \lesssim e^{o(N_\varepsilon)}/|\log\varepsilon|$.*

In particular, in both cases, we deduce $N_\varepsilon^{-1} j_\varepsilon - \mathrm{v}_\varepsilon \to 0$ in $\mathrm{L}^\infty_{\mathrm{loc}}([0,T); \mathrm{L}^1_{\mathrm{uloc}}(\mathbb{R}^2)^2)$ as $\varepsilon \downarrow 0$. If we further assume $\mathcal{D}^{*,\circ}_{\varepsilon,\infty} \ll N_\varepsilon^2$, then for any $\ell \ge 1$ we obtain more precisely, for all $t \in [0,T)$ and $L \ge 1$,

$$\sup_z \|N_\varepsilon^{-1} j_\varepsilon - \mathrm{v}_\varepsilon\|_{(\mathrm{L}^1 + \mathrm{L}^2)(B_L(z))} \ll_{t,\ell} \Big(1 + \frac{L}{|\log\varepsilon|^\ell}\Big)^2. \tag{6.1}$$

Remarks 6.2.

(a) If we further assume $\|u_\varepsilon^t\|_{\mathrm{L}^\infty} \lesssim_t 1$ for all t, we note that the convergence $N_\varepsilon^{-1} j_\varepsilon - \mathrm{v}_\varepsilon \to 0$ actually holds in $\mathrm{L}^\infty_{\mathrm{loc}}([0,T); \mathrm{L}^p_{\mathrm{uloc}}(\mathbb{R}^2)^2)$ for all $p < 2$. In the parabolic case $\beta = 0$ without forcing $F = f = 0$, a maximum principle type argument gives that $\|u_\varepsilon^\circ\|_{\mathrm{L}^\infty} \le 1$ implies $\|u_\varepsilon^t\|_{\mathrm{L}^\infty} \le 1$ for all $t \ge 0$ (see e.g. [21, Proposition 4.4]). However, the same argument fails in the presence of forcing $F, f \ne 0$. Moreover, such a uniform L^∞-bound on u_ε is expected to fail in the Gross-Pitaevski case $\alpha = 0$, due to the time reversibility of the equation in that case, and it is also expected to fail in the dissipative case $\alpha > 0$, $\beta \ne 0$. We therefore systematically avoid to use such L^∞-estimates here.

(b) The main reason why we obtain no result in the case $N_\varepsilon \gg |\log\varepsilon|$ with our computations is as follows: the estimate (6.18) at the end of Step 3 yields

$$\hat{\mathcal{D}}^t_{\varepsilon,R} \le o_t(N_\varepsilon^2) + C_t(1+\lambda_\varepsilon\alpha)\int_0^t \hat{\mathcal{D}}_{\varepsilon,R},$$

where in the case $N_\varepsilon \gg |\log\varepsilon|$ we need to choose a rescaling $\lambda_\varepsilon = N_\varepsilon/|\log\varepsilon| \gg 1$, so that in the case $\alpha > 0$ the conclusion fails to hold due to the prefactor $\lambda_\varepsilon\alpha \gg 1$. This problem formally disappears in the Gross-Pitaevskii case $\alpha = 0$, for which the regime $N_\varepsilon \gg |\log\varepsilon|$ is indeed treated in Section 7.

Proof. We choose $R \gg |\log\varepsilon|/N_\varepsilon$ with $R^{\theta_0} \lesssim |\log\varepsilon|$ for some $\theta_0 > 0$. Given the assumption $\mathcal{D}^{*,\circ}_{\varepsilon,R} \ll N_\varepsilon^2$ on the initial data, for all $\varepsilon > 0$ we define $T_\varepsilon > 0$ the maximum time $\le T$ such that $\mathcal{D}^{*,t}_{\varepsilon,R} \le N_\varepsilon^2$ holds for all $t \le T_\varepsilon$. By Lemma 4.1 and Proposition 5.2, we deduce $\hat{\mathcal{D}}^{*,\circ}_{\varepsilon,R} \ll N_\varepsilon^2$ and for all $t \le T_\varepsilon$,

$$\mathcal{E}^{*,t}_{\varepsilon,R} \lesssim_t N_\varepsilon|\log\varepsilon|, \quad \hat{\mathcal{E}}^{*,t}_{\varepsilon,R} \lesssim_t N_\varepsilon|\log\varepsilon|, \quad \hat{\mathcal{D}}^{*,t}_{\varepsilon,R} \lesssim_t N_\varepsilon^2, \quad \mathcal{D}^{*,t}_{\varepsilon,R} \lesssim \hat{\mathcal{D}}^{*,t}_{\varepsilon,R} + o_t(N_\varepsilon^2). \tag{6.2}$$

The strategy of the proof consists in showing that for all $t \le T_\varepsilon$,

$$\hat{\mathcal{D}}^{*,t}_{\varepsilon,R} \lesssim_t o(N_\varepsilon^2) + \int_0^t \hat{\mathcal{D}}^{*}_{\varepsilon,R}. \tag{6.3}$$

By the Grönwall inequality, this implies $\hat{\mathcal{D}}^{*,t}_{\varepsilon,R} \ll_t N_\varepsilon^2$, hence $\mathcal{D}^{*,t}_{\varepsilon,R} \ll_t N_\varepsilon^2$ for all $t \le T_\varepsilon$. This gives in particular $T_\varepsilon = T$, and the main conclusion follows.

To simplify notation, we focus on (6.3) with the left-hand side $\hat{\mathcal{D}}^t_{\varepsilon,R}$ centered at $z = 0$, but the result of course holds uniformly with respect to the translation. We split the proof of (6.3) into four steps. We begin with the general mixed-flow case in the regime $\log|\log\varepsilon| \ll N_\varepsilon \lesssim |\log\varepsilon|$, while Step 4 describes the modifications needed in the proof for the purely parabolic case in the regime $1 \ll N_\varepsilon \lesssim \log|\log\varepsilon|$. The additional stated consequences are deduced in Step 5.

Let us first introduce some notation. For all $t \le T_\varepsilon$, as we are in the framework of Proposition 5.2 with $u^t_\varepsilon, \mathrm{v}^t_\varepsilon$, we let $\mathcal{B}^t_\varepsilon := \mathcal{B}^t_{\varepsilon,R}$ denote the constructed collection of disjoint closed balls $\mathcal{B}^{r_\varepsilon}_{\varepsilon,R}(u^t_\varepsilon, \mathrm{v}^t_\varepsilon)$ with total radius $r_\varepsilon := |\log\varepsilon|^{-4}e^{-\sqrt{N_\varepsilon}}$. Let then $\bar\Gamma^t_\varepsilon$ denote the corresponding approximation of Γ^t_ε given by Lemma 5.3. We decompose $\Gamma_\varepsilon := \alpha\Gamma_{\varepsilon,0} - \beta\Gamma^\perp_{\varepsilon,0}$ with

$$\Gamma_{\varepsilon,0} := \lambda_\varepsilon^{-1}\Big(\nabla^\perp h - F^\perp - \frac{2N_\varepsilon}{|\log\varepsilon|}\mathrm{v}_\varepsilon\Big).$$

Step 1: time-derivative of the modulated energy excess. Lemma 4.4 yields the following decomposition,

$$\partial_t\hat{\mathcal{D}}_{\varepsilon,R} = I^S_{\varepsilon,R} + I^V_{\varepsilon,R} + I^E_{\varepsilon,R} + I^D_{\varepsilon,R} + I^H_{\varepsilon,R} + I^d_{\varepsilon,R} + I^g_{\varepsilon,R} + I^n_{\varepsilon,R} + I'_{\varepsilon,R}, \tag{6.4}$$

where the eight first terms are as in the statement of Lemma 4.4, and where the error $I'_{\varepsilon,R}$ is estimated as follows (cf. (4.16)), in the considered regimes,

$$\int_0^t |I'_{\varepsilon,R}| \lesssim_t \varepsilon R(N_\varepsilon|\log\varepsilon|)^{1/2}|\log\varepsilon|^2 = o(N_\varepsilon^2).$$

Step 2: estimating the error terms. In this step, we study the three error terms $I^d_{\varepsilon,R}$, $I^g_{\varepsilon,R}$, and $I^n_{\varepsilon,R}$, and we prove

$$\int_0^t (I^d_{\varepsilon,R} + I^g_{\varepsilon,R} + I^n_{\varepsilon,R}) \lesssim_t o(N_\varepsilon^2) + o\Big(\frac{N_\varepsilon}{|\log\varepsilon|}\Big)\int_0^t\int \chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_\varepsilon|^2. \tag{6.5}$$

We begin with the estimation of $I^n_{\varepsilon,R}$. Using (6.2), Lemma 5.5, and the boundedness properties of p_ε (cf. Proposition 3.1), the quantity $\bar{\mathcal{E}}^*_{\varepsilon,R}$ defined in Lemma 5.4 is estimated as follows, in the regimes considered, for all $\theta > 0$,

$$\begin{aligned}\bar{\mathcal{E}}^{*,t}_{\varepsilon,R} &\lesssim \sup_z \int_0^t \mathcal{E}^z_{\varepsilon,R} + \sup_z \int_0^t \int \chi^z_R\big(|\partial_t u_\varepsilon|^2 + N_\varepsilon^2|\mathrm{p}_\varepsilon|^2 + N_\varepsilon^2|1-|u_\varepsilon|^2||\mathrm{p}_\varepsilon|^2\big)\\ &\lesssim_{t,\theta} R^\theta N_\varepsilon|\log\varepsilon|^3 + \lambda_\varepsilon^{-1}N_\varepsilon^2(1+\varepsilon(N_\varepsilon|\log\varepsilon|)^{1/2}) \lesssim R^\theta|\log\varepsilon|^4,\end{aligned}$$

hence, choosing $\theta > 0$ small enough, $\bar{\mathcal{E}}^{*,t}_{\varepsilon,R} \lesssim_t |\log\varepsilon|^5$. Choosing e.g. $M_\varepsilon = |\log\varepsilon|^{40}$, and using the obvious estimate $|\nabla\chi_R| \lesssim R^{-1}\chi_R^{1/2}$, Lemma 5.4 then yields

$$\Big|\int_0^t\int a\tilde{V}_\varepsilon\cdot\nabla^\perp\chi_R\Big| \lesssim o_t(1) + R^{-1}|\log\varepsilon|^{-1}\Big(\int_0^t\int\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\,\mathrm{p}_\varepsilon|^2 + \int_0^t\int_{B_{2R}}|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2\Big),$$

and hence,

$$\begin{aligned}\Big|\int_0^t I^n_{\varepsilon,R}\Big| \lesssim_t o(1) + R^{-1}\int_0^t\int_{B_{2R}}\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 + |1-|u_\varepsilon|^2|(N_\varepsilon^2|\mathrm{v}_\varepsilon|^2 + |f|)\Big)\\ + R^{-1}\Big(\int_0^t\int\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\,\mathrm{p}_\varepsilon|^2 + \int_0^t\int_{B_{2R}}|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|^2\Big).\end{aligned}$$

By (6.2), (2.1), and the integrability properties of v_ε (cf. Proposition 3.1), with the choice $R \gg |\log\varepsilon|/N_\varepsilon$,

$$\Big|\int_0^t I^n_{\varepsilon,R}\Big| \lesssim_t o(1) + R^{-1}N_\varepsilon|\log\varepsilon| + R^{-1}\int_0^t\int\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\,\mathrm{p}_\varepsilon\,|^2 \tag{6.6}$$

$$\lesssim o(N_\varepsilon^2) + o\Big(\frac{N_\varepsilon}{|\log\varepsilon|}\Big)\int_0^t\int\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\,\mathrm{p}_\varepsilon\,|^2. \tag{6.7}$$

We now turn to the estimation of $I^g_{\varepsilon,R}$. Using (2.1) and the pointwise estimates of Lemma 4.2, we find

$$\begin{aligned}|I^g_{\varepsilon,R}| \lesssim \|\Gamma_\varepsilon - \bar\Gamma_\varepsilon\|_{\mathrm{L}^\infty}\Big(N_\varepsilon\int_{B_{2R}}(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon| + N_\varepsilon|1-|u_\varepsilon|^2|)|\mathrm{curl}\,\mathrm{v}_\varepsilon| + N_\varepsilon\int_{B_{2R}}|1-|u_\varepsilon|^2||\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|\\ + \lambda_\varepsilon\int\chi_R\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|^2 + \frac{a}{\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) + \lambda_\varepsilon|\log\varepsilon|\int\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{p}_{\varepsilon,\varrho}||\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|\\ + (N_\varepsilon + \lambda_\varepsilon|\log\varepsilon|)\int\chi_R(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|^2 + N_\varepsilon^2|1-|u_\varepsilon|^2||\mathrm{v}_\varepsilon|^2) + N_\varepsilon^2\int\chi_R|\mathrm{v}_\varepsilon|^2(N_\varepsilon|\mathrm{v}_\varepsilon| + |\log\varepsilon||F|)\\ + \lambda_\varepsilon N_\varepsilon|\log\varepsilon||\beta|\int\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{p}_\varepsilon|(|\mathrm{v}_\varepsilon| + |1-|u_\varepsilon|^2|)\Big).\end{aligned}$$

By (6.2), by Lemma 5.3 in the form $\|\Gamma_\varepsilon - \bar\Gamma_\varepsilon\|_{\mathrm{L}^\infty} \lesssim r_\varepsilon = |\log\varepsilon|^{-4}e^{-\sqrt{N_\varepsilon}}$, and by the integrability properties of v_ε (cf. Proposition 3.1), we deduce in the considered regimes for all $\theta > 0$,

$$|I^g_{\varepsilon,R}| \lesssim_{t,\theta} \frac{e^{-\sqrt{N_\varepsilon}}}{|\log\varepsilon|^4}R^\theta N_\varepsilon|\log\varepsilon|^2\Big(1 + \int\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{p}_\varepsilon|^2\Big)^{1/2}, \tag{6.8}$$

and hence, for $\theta > 0$ small enough such that $R^\theta \lesssim |\log\varepsilon|$, we conclude

$$|I^g_{\varepsilon,R}| \lesssim_t o(N_\varepsilon^2) + o\Big(\frac{N_\varepsilon}{|\log\varepsilon|}\Big)\int\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{p}_\varepsilon|^2. \tag{6.9}$$

Regarding the last term $I^d_{\varepsilon,R}$, the definition of the pressure in (3.2) simply yields $I^d_{\varepsilon,R} = 0$, and the conclusion (6.5) follows.

Step 3: estimating the dominant terms. We now turn to the estimation of the five first terms in (6.4), showing more precisely that

$$\hat{\mathcal{D}}^t_{\varepsilon,R} \lesssim_t o(N_\varepsilon^2) + \int_0^t \hat{\mathcal{D}}_{\varepsilon,R}. \tag{6.10}$$

As this result obviously holds uniformly with respect to translations of the cut-off functions, the conclusion (6.3) follows. We begin with the estimation of the first term $I^S_{\varepsilon,R}$. Since for all t the field $\bar\Gamma^t_\varepsilon$ is by definition constant in each ball of the collection $\mathcal{B}^t_\varepsilon$ and satisfies $\|\nabla\bar\Gamma^t_\varepsilon\|_{\mathrm{L}^\infty} \lesssim \|\nabla\Gamma^t_\varepsilon\|_{\mathrm{L}^\infty}$, we obtain

$$|I^S_{\varepsilon,R}| \lesssim \int_{\mathbb{R}^2\setminus\mathcal{B}_\varepsilon} \chi_R|\tilde S_\varepsilon| \lesssim \int_{\mathbb{R}^2\setminus\mathcal{B}_\varepsilon} a\chi_R\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) + \int \chi_R|1-|u_\varepsilon|^2|(N_\varepsilon^2|\mathrm{v}_\varepsilon|^2+|f|).$$

Since $\mathcal{B}_\varepsilon$ has total radius $r_\varepsilon := |\log\varepsilon|^{-4}e^{-\sqrt{N_\varepsilon}}$, and since the choice $N_\varepsilon \gg \log|\log\varepsilon|$ ensures $r_\varepsilon \ge e^{-o(N_\varepsilon)}$, we may apply Proposition 5.2(v), which shows that the first integral in the above right-hand side is bounded by $\mathcal{D}^*_{\varepsilon,R} + o(N_\varepsilon^2)$. Further using (6.2), (2.1), and the integrability properties of v_ε (cf. Proposition 3.1), we obtain in the considered regimes, with $\varepsilon R|\log\varepsilon|^3 \lesssim 1$,

$$|I^S_{\varepsilon,R}| \lesssim \mathcal{D}^*_{\varepsilon,R} + o(N_\varepsilon^2) + \varepsilon(N_\varepsilon|\log\varepsilon|)^{1/2}(N_\varepsilon^2 + R\lambda_\varepsilon^2|\log\varepsilon|^2) \lesssim \hat{\mathcal{D}}^*_{\varepsilon,R} + o(N_\varepsilon^2). \tag{6.11}$$

We turn to $I^H_{\varepsilon,R}$. Since $\|(\Gamma_\varepsilon,\nabla h)\|_{\mathrm{L}^\infty} \lesssim_t 1$, Lemma 5.6 yields

$$\int_0^t I^H_{\varepsilon,R} = O_t(N_\varepsilon) + \int_0^t\int \frac{a\chi_R}{2}\Gamma_\varepsilon^\perp\cdot\nabla h\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|^2 + \frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 - |\log\varepsilon|\mu_\varepsilon\Big),$$

and hence by Proposition 5.2(iv) and Lemma 4.1, with $\|(\Gamma_\varepsilon,\nabla h)\|_{W^{1,\infty}} \lesssim_t 1$,

$$\int_0^t I^H_{\varepsilon,R} \lesssim_t o(N_\varepsilon^2) + \int_0^t \mathcal{D}_{\varepsilon,R} \lesssim_t o(N_\varepsilon^2) + \int_0^t \hat{\mathcal{D}}_{\varepsilon,R}. \tag{6.12}$$

The term $I^D_{\varepsilon,R}$ is simply estimated by

$$I^D_{\varepsilon,R} \le -\frac{\lambda_\varepsilon\alpha}{2}\int a\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_\varepsilon|^2 + \frac{\lambda_\varepsilon\alpha}{2}\int a\chi_R|(\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon)\cdot\Gamma_\varepsilon^\perp|^2. \tag{6.13}$$

We finally turn to $I^V_{\varepsilon,R}$. Using $\alpha^2+\beta^2=1$, we have by definition

$$\Gamma_{\varepsilon,0} - \beta\Gamma_\varepsilon^\perp = \Gamma_{\varepsilon,0} - \beta(\alpha\Gamma_{\varepsilon,0}^\perp + \beta\Gamma_{\varepsilon,0}) = \alpha^2\Gamma_{\varepsilon,0} - \alpha\beta\Gamma_{\varepsilon,0}^\perp = \alpha\Gamma_\varepsilon,$$

and hence $I^V_{\varepsilon,R}$ takes on the following guise, in terms of Γ_ε, $\Gamma_{\varepsilon,0}$,

$$I^V_{\varepsilon,R} = \lambda_\varepsilon|\log\varepsilon|\int\frac{a\chi_R}{2}\tilde V_\varepsilon\cdot(\Gamma_{\varepsilon,0}-\beta\Gamma_\varepsilon^\perp) = \lambda_\varepsilon\alpha|\log\varepsilon|\int\frac{a\chi_R}{2}\tilde V_\varepsilon\cdot\Gamma_\varepsilon.$$

As shown in Step 2, the quantity $\bar{\mathcal{E}}^*_{\varepsilon,R}$ defined in Lemma 5.4 satisfies $\bar{\mathcal{E}}^{*,t}_{\varepsilon,R} \lesssim_t |\log\varepsilon|^5$. Choosing e.g. $M_\varepsilon = |\log\varepsilon|^{40}$, Lemma 5.4 then yields, for any $\Lambda \simeq 1$,

$$\begin{aligned}\Big|\int_0^t I^V_{\varepsilon,R}\Big| &\le o_t(1)\\ &+ \lambda_\varepsilon\alpha\Big(1+\frac{C\log|\log\varepsilon|}{|\log\varepsilon|}\Big)\Big(\frac{1}{\Lambda}\int_0^t\int a\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{p}_\varepsilon|^2 + \frac{\Lambda}{4}\int_0^t\int a\chi_R|(\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon)\cdot\Gamma_\varepsilon|^2\Big),\end{aligned}$$

and thus, using the optimal energy bound (6.2), and noting that $\lambda_\varepsilon N_\varepsilon \log|\log\varepsilon| \ll N_\varepsilon^2$ holds in the considered regimes,

$$\Big|\int_0^t I^V_{\varepsilon,R}\Big| \le o_t(N_\varepsilon^2) + \Big(\lambda_\varepsilon + o\Big(\frac{N_\varepsilon}{|\log\varepsilon|}\Big)\Big)\frac{\alpha}{\Lambda}\int_0^t\int a\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_\varepsilon|^2 + \frac{\lambda_\varepsilon\alpha\Lambda}{4}\int_0^t\int a\chi_R|(\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon)\cdot\Gamma_\varepsilon|^2. \quad (6.14)$$

We now distinguish between two cases:

$$\text{(Case 1)} \qquad \int_0^t\int a\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_\varepsilon|^2 \le 5\int_0^t\int a\chi_R|(\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon)\cdot\Gamma_\varepsilon|^2, \quad (6.15)$$

$$\text{(Case 2)} \qquad \int_0^t\int a\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_\varepsilon|^2 > 5\int_0^t\int a\chi_R|(\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon)\cdot\Gamma_\varepsilon|^2. \quad (6.16)$$

In Case 1, choosing $\Lambda = 2$ in (6.14) yields

$$\Big|\int_0^t I^V_{\varepsilon,R}\Big| \le o_t(N_\varepsilon^2) + \Big(\lambda_\varepsilon + o\Big(\frac{N_\varepsilon}{|\log\varepsilon|}\Big)\Big)\frac{\alpha}{2}\int_0^t\int a\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_\varepsilon|^2 + \frac{\lambda_\varepsilon\alpha}{2}\int_0^t\int a\chi_R|(\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon)\cdot\Gamma_\varepsilon|^2.$$

In Case 2, the condition (6.16) can be rewritten as

$$\frac14\int_0^t\int a\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_\varepsilon|^2 + \int_0^t\int a\chi_R|(\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon)\cdot\Gamma_\varepsilon|^2 \le \Big(\frac14+\frac1{10}\Big)\int_0^t\int a\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_\varepsilon|^2 + \frac12\int_0^t\int a\chi_R|(\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon)\cdot\Gamma_\varepsilon|^2,$$

and choosing $\Lambda = 4$ in (6.14) then yields, with $N_\varepsilon/|\log\varepsilon| \lesssim \lambda_\varepsilon$ in the considered regimes,

$$\Big|\int_0^t I^V_{\varepsilon,R}\Big| \le o_t(N_\varepsilon^2) + \lambda_\varepsilon\alpha\bigg(\Big(\frac14+\frac1{10}+o(1)\Big)\int_0^t\int a\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_\varepsilon|^2 + \frac12\int_0^t\int a\chi_R|(\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon)\cdot\Gamma_\varepsilon|^2\bigg).$$

Further noting that in Case 1 the condition (6.15) together with the energy bound (6.2) yields

$$o\Big(\frac{N_\varepsilon}{|\log\varepsilon|}\Big)\int a\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_\varepsilon|^2 \le o\Big(\frac{N_\varepsilon}{|\log\varepsilon|}\Big)\int_0^t\int a\chi_R|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 \lesssim_t o(N_\varepsilon^2),$$

and combining this with (6.5) and (6.13), we observe an exact recombination of the terms, and obtain in Case 1

$$\int_0^t (I^V_{\varepsilon,R} + I^D_{\varepsilon,R} + I^d_{\varepsilon,R} + I^g_{\varepsilon,R} + I^n_{\varepsilon,R} + I'_{\varepsilon,R}) \le \frac{\lambda_\varepsilon\alpha}{2}\int_0^t\int a\chi_R|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2|\Gamma_\varepsilon|^2 + o_t(N_\varepsilon^2), \quad (6.17)$$

and in Case 2

$$\int_0^t (I^V_{\varepsilon,R}+I^D_{\varepsilon,R}+I^d_{\varepsilon,R}+I^g_{\varepsilon,R}+I^n_{\varepsilon,R}+I'_{\varepsilon,R})$$
$$\le -\frac{\lambda_\varepsilon\alpha}{2}\Big(\frac12-\frac15-o(1)\Big)\int_0^t\int a\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_{\varepsilon,\varrho}|^2+\frac{\lambda_\varepsilon\alpha}{2}\int_0^t\int a\chi_R|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2|\Gamma_\varepsilon|^2+o_t(N_\varepsilon^2),$$

so that (6.17) holds in both cases for $\varepsilon>0$ small enough. Using $\alpha^2+\beta^2=1$, we have by definition $\Gamma_\varepsilon\cdot\Gamma_{\varepsilon,0}=\alpha|\Gamma_{\varepsilon,0}|^2=\alpha|\Gamma_\varepsilon|^2$, and hence the term $I^E_{\varepsilon,R}$ takes on the following guise, in terms of Γ_ε, $\Gamma_{\varepsilon,0}$,

$$I^E_{\varepsilon,R}=-\frac{\lambda_\varepsilon}{2}|\log\varepsilon|\int a\chi_R\Gamma_\varepsilon\cdot\Gamma_{\varepsilon,0}\,\mu_\varepsilon=-\frac{\lambda_\varepsilon\alpha}{2}|\log\varepsilon|\int a\chi_R|\Gamma_\varepsilon|^2\mu_\varepsilon.$$

Together with (6.17), this yields

$$\int_0^t (I^V_{\varepsilon,R}+I^E_{\varepsilon,R}+I^D_{\varepsilon,R}+I^d_{\varepsilon,R}+I^g_{\varepsilon,R}+I^n_{\varepsilon,R}+I'_{\varepsilon,R})$$
$$\le\frac{\lambda_\varepsilon\alpha}{2}\int_0^t\int a\chi_R\big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|^2-|\log\varepsilon|\mu_\varepsilon\big)|\Gamma_\varepsilon|^2+o_t(N_\varepsilon^2).$$

Combining this with (6.4), (6.11), (6.12), and with $\hat{\mathcal{D}}^{*,\circ}_{\varepsilon,R}\ll N_\varepsilon^2$, we conclude

$$\hat{\mathcal{D}}^t_{\varepsilon,R}\le o_t(N_\varepsilon^2)+C_t\int_0^t\hat{\mathcal{D}}_{\varepsilon,R}+\frac{\lambda_\varepsilon\alpha}{2}\int_0^t\int a\chi_R\big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|^2-|\log\varepsilon|\mu_\varepsilon\big)|\Gamma_\varepsilon|^2, \tag{6.18}$$

and the result (6.10) now follows from Proposition 5.2(iv).

Step 4: refinement in the purely parabolic case. We consider the parabolic case $\alpha=1$, $\beta=0$, in both the critical regime (GL$_1$), and the subcritical regime (GL$'_2$) with $\lambda_\varepsilon\le e^{o(N_\varepsilon)}/|\log\varepsilon|$, and we show that the assumption $N_\varepsilon\gg\log|\log\varepsilon|$ can be dropped. In the proof in Steps 1–3 above, the limitation comes from the fact that we need to use balls $\mathcal{B}_\varepsilon$ with a particularly small total radius r_ε in order to obtain smallness of the error term $I^g_{\varepsilon,\varrho,R}$ in (6.8), while on the other hand the term $I^S_{\varepsilon,\varrho,R}$ corresponds to the energy outside the small balls $\mathcal{B}_\varepsilon$, so that we need to have $r_\varepsilon\ge e^{-o(N_\varepsilon)}$ to be allowed to apply Proposition 5.2(v). However, in the parabolic case, the worst terms in $I^g_{\varepsilon,\varrho,R}$ vanish, and the total radius r_ε may then be chosen to be much larger.

Let us thus consider the regime (GL$_1$) or the subcritical regime (GL$'_2$) with $\lambda_\varepsilon\lesssim e^{o(N_\varepsilon)}/|\log\varepsilon|$, in the parabolic case $\alpha=1$, $\beta=0$, and with a "small" number of vortices $1\ll N_\varepsilon\lesssim\log|\log\varepsilon|$. Let $\tilde r_\varepsilon:=(\lambda_\varepsilon|\log\varepsilon|)^{-2}\ge e^{-o(N_\varepsilon)}$ and choose $\varepsilon^{1/2}<\tilde r^0_\varepsilon\ll N_\varepsilon|\log\varepsilon|^{-1}$. For all $t\le T_\varepsilon$, as we are in the framework of Proposition 5.2 with $u^t_\varepsilon,\mathrm{v}^t_\varepsilon$, we let $\tilde{\mathcal{B}}^t_\varepsilon:=\tilde{\mathcal{B}}^t_{\varepsilon,R}$ denote the corresponding collection of disjoint closed balls $\mathcal{B}^{\tilde r^0_\varepsilon,\tilde r_\varepsilon}_{\varepsilon,R}(u^t_\varepsilon,\mathrm{v}^t_\varepsilon)$. Let then $\tilde\Gamma^t_\varepsilon$ denote the associated approximation of Γ^t_ε given by Lemma 5.3. As in Step 1, Lemma 4.4 yields the following decomposition, with $\bar\Gamma_\varepsilon$ replaced by $\tilde\Gamma_\varepsilon$,

$$\partial_t\hat{\mathcal{D}}_{\varepsilon,R}=I^S_{\varepsilon,R}+I^V_{\varepsilon,R}+I^E_{\varepsilon,R}+I^D_{\varepsilon,R}+I^H_{\varepsilon,R}+I^d_{\varepsilon,R}+I^g_{\varepsilon,R}+I^n_{\varepsilon,R}+I'_{\varepsilon,R},$$

where all the terms are estimated just as before, except $I^g_{\varepsilon,R}$ and $I^S_{\varepsilon,R}$. We begin with the discussion of $I^g_{\varepsilon,R}$.

For $\alpha = 1$, $\beta = 0$, this term takes on the following simpler form,

$$I^g_{\varepsilon,R} = \int a\chi_R N_\varepsilon (N_\varepsilon \mathrm{v}_\varepsilon - j_\varepsilon)\cdot(\Gamma_\varepsilon - \tilde\Gamma_\varepsilon)\mathrm{curl}\ \mathrm{v}_\varepsilon + \int \lambda_\varepsilon a\chi_R(\Gamma_\varepsilon - \tilde\Gamma_\varepsilon)^\perp\cdot\langle \partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_\varepsilon, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon\rangle$$
$$+\int \frac{a\chi_R}{2}(\tilde\Gamma_\varepsilon - \Gamma_\varepsilon)^\perp\cdot\nabla h\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)$$
$$+\int a\chi_R(\tilde\Gamma_\varepsilon - \Gamma_\varepsilon)\cdot(N_\varepsilon \mathrm{v}_\varepsilon + |\log\varepsilon| F^\perp/2)\mu_\varepsilon. \quad (6.19)$$

We estimate each of the four right-hand side terms separately. We begin with the first term. Using the pointwise estimates of Lemma 4.2 and the integrability properties of v_ε (cf. Proposition 3.1), we find

$$\int a\chi_R N_\varepsilon(N_\varepsilon \mathrm{v}_\varepsilon - j_\varepsilon)\cdot(\Gamma_\varepsilon - \tilde\Gamma_\varepsilon)\mathrm{curl}\ \mathrm{v}_\varepsilon$$
$$\lesssim N_\varepsilon\|\Gamma_\varepsilon - \tilde\Gamma_\varepsilon\|_{\mathrm{L}^\infty}\Big(\int_{\tilde{\mathcal{B}}^t_\varepsilon}\chi_R|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon| + \Big(\int_{\mathbb{R}^2\setminus\tilde{\mathcal{B}}^t_\varepsilon}\chi_R|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2\Big)^{1/2}\Big)$$
$$+ N_\varepsilon\|\Gamma_\varepsilon - \tilde\Gamma_\varepsilon\|_{\mathrm{L}^\infty}\Big(\int\chi_R|1-|u_\varepsilon|^2||\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon| + N_\varepsilon\int\chi_R|1-|u_\varepsilon|^2||\mathrm{curl}\ \mathrm{v}_\varepsilon|\Big),$$

and hence, using (6.2) and Proposition 5.2(v)–(vi) with $p=1$ to estimate the first two integrals in the right-hand side, and using Lemma 5.3 in the form $\|\Gamma^t_\varepsilon - \tilde\Gamma^t_\varepsilon\|_{\mathrm{L}^\infty}\lesssim_t \tilde r_\varepsilon \ll 1$,

$$\int a\chi_R N_\varepsilon(N_\varepsilon \mathrm{v}_\varepsilon - j_\varepsilon)\cdot(\Gamma_\varepsilon - \tilde\Gamma_\varepsilon)\mathrm{curl}\ \mathrm{v}_\varepsilon \lesssim N_\varepsilon^2\|\Gamma_\varepsilon - \tilde\Gamma_\varepsilon\|_{\mathrm{L}^\infty}\ll_t N_\varepsilon^2.$$

For the second term in (6.19), using (6.2) and again Lemma 5.3, with $\tilde r_\varepsilon\lambda_\varepsilon \ll N_\varepsilon|\log\varepsilon|^{-1}$, we obtain

$$\int\lambda_\varepsilon a\chi_R(\Gamma_\varepsilon - \tilde\Gamma_\varepsilon)^\perp\cdot\langle\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_\varepsilon, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon\rangle$$
$$\lesssim \lambda_\varepsilon(N_\varepsilon|\log\varepsilon|)^{1/2}\|\Gamma_\varepsilon - \tilde\Gamma_\varepsilon\|_{\mathrm{L}^\infty}\Big(\int\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_\varepsilon|^2\Big)^{1/2}$$
$$\lesssim o(N_\varepsilon^2) + o\Big(\frac{N_\varepsilon}{|\log\varepsilon|}\Big)\int\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_\varepsilon|^2.$$

For the third term in (6.19), using (6.2), and (2.1) together with Lemma 5.3 in the form $\|(\tilde\Gamma_\varepsilon - \Gamma_\varepsilon)^\perp\cdot\nabla h\|_{\mathrm{L}^\infty}\lesssim_t \tilde r_\varepsilon\lambda_\varepsilon \ll N_\varepsilon|\log\varepsilon|^{-1}$, we find

$$\int\frac{a\chi_R}{2}(\tilde\Gamma_\varepsilon - \Gamma_\varepsilon)^\perp\cdot\nabla h\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big)\ll_t N_\varepsilon^2.$$

It remains to estimate the fourth term in (6.19). Using (6.2), Proposition 5.2(iii) in the form (5.13) with $\gamma = 1/2$, the regularity properties of v_ε (cf. Proposition 3.1), (2.1) in the form $\|F\|_{C^{1/2}}\lesssim\lambda_\varepsilon$, and Lemma 5.3 in the form $\|\tilde\Gamma_\varepsilon - \Gamma_\varepsilon\|_{C^{1/2}}\lesssim_t \tilde r_\varepsilon^{1/2} = (\lambda_\varepsilon|\log\varepsilon|)^{-1}$, we obtain

$$\int a\chi_R(\tilde\Gamma_\varepsilon - \Gamma_\varepsilon)\cdot(N_\varepsilon \mathrm{v}_\varepsilon + |\log\varepsilon|F^\perp/2)\mu_\varepsilon \lesssim N_\varepsilon\|a\chi_R(\tilde\Gamma_\varepsilon - \Gamma_\varepsilon)\cdot(N_\varepsilon \mathrm{v}_\varepsilon + |\log\varepsilon|F^\perp/2)\|_{C^{1/2}}$$
$$\lesssim N_\varepsilon(N_\varepsilon + \lambda_\varepsilon|\log\varepsilon|)\|\tilde\Gamma_\varepsilon - \Gamma_\varepsilon\|_{C^{1/2}}\ll_t N_\varepsilon^2.$$

Inserting these various estimates into (6.19) now yields

$$I^g_{\varepsilon,R}\lesssim_t o(N_\varepsilon^2) + o\Big(\frac{N_\varepsilon}{|\log\varepsilon|}\Big)\int\chi_R|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_\varepsilon|^2,$$

proving that (6.9) again holds in the present setting. On the other hand, since the total radius satisfies $\tilde{r}_\varepsilon \geq e^{-o(N_\varepsilon)}$, we may apply Proposition 5.2(v), so that the estimate (6.11) for $I^S_{\varepsilon,\varrho,R}$ can again be obtained just as in Step 3, and the conclusion (6.3) follows.

Step 5: consequences. In the previous steps, the result $\mathcal{D}^{*,t}_{\varepsilon,R} \ll_t N_\varepsilon^2$ for $t \in [0,T)$ is proved in both cases (i) and (ii) of the statement. We now show that it implies the convergence $N_\varepsilon^{-1} j_\varepsilon - \mathrm{v}_\varepsilon \to 0$. For all $t \in [0,T)$, Proposition 5.2 yields $\mathcal{E}^{*,t}_{\varepsilon,R} \lesssim_t N_\varepsilon |\log \varepsilon|$, and for $r \in (\varepsilon^{1/2}, \bar{r})$ we denote by $\mathcal{B}^{r,t}_{\varepsilon,R}$ the constructed collection of disjoint closed balls corresponding to $u^t_\varepsilon, \mathrm{v}^t_\varepsilon, R$ and total radius r. Let $e^{-o(N_\varepsilon)} \leq r < \bar{r}$. For all $t \in [0,T)$, Proposition 5.2(v)–(vi) gives

$$\sup_z \int_{\mathbb{R}^2 \setminus \mathcal{B}^{r,t}_{\varepsilon,R}} \chi^z_R |\nabla u_\varepsilon - i u_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 \ll_t N_\varepsilon^2,$$

and for all $1 \leq p < 2$,

$$\sup_z \int_{\mathcal{B}^{r,t}_{\varepsilon,R}} \chi^z_R |\nabla u_\varepsilon - i u_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^p \ll_t N_\varepsilon^p.$$

Using the pointwise estimates of Lemma 4.2, we deduce

$$\begin{aligned}\sup_z \int_{B(z)} |j_\varepsilon - N_\varepsilon \mathrm{v}_\varepsilon| &\lesssim_t \sup_z \int_{B(z)} |\nabla u_\varepsilon - i u_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon| + \varepsilon N_\varepsilon |\log \varepsilon| \\ &\lesssim_t \sup_z \int_{\mathcal{B}^{r,t}_{\varepsilon,R}} \chi^z_R |\nabla u_\varepsilon - i u_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon| + \sup_z \Big(\int_{B(z) \setminus \mathcal{B}^{r,t}_{\varepsilon,R}} |\nabla u_\varepsilon - i u_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 \Big)^{1/2} + o(N_\varepsilon) \ll_t N_\varepsilon,\end{aligned}$$

hence $N_\varepsilon^{-1} j_\varepsilon - \mathrm{v}_\varepsilon \to 0$ in $\mathrm{L}^\infty_{\mathrm{loc}}([0,T); \mathrm{L}^1_{\mathrm{uloc}}(\mathbb{R}^2)^2)$. More precisely, we may decompose for all $L \geq 1$,

$$\begin{aligned}\sup_z \|j_\varepsilon - N_\varepsilon \mathrm{v}_\varepsilon\|_{(\mathrm{L}^1 + \mathrm{L}^2)(B_L(z))} &\lesssim_t \sup_z \int_{\mathcal{B}^{r,t}_{\varepsilon,R}} \chi^z_R |\nabla u_\varepsilon - i u_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon| + \sup_z \|\nabla u_\varepsilon - i u_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon\|_{\mathrm{L}^2(B_L(z) \setminus \mathcal{B}^{r,t}_{\varepsilon,R})} \\ &\quad + N_\varepsilon \sup_z \|1 - |u_\varepsilon|^2\|_{\mathrm{L}^2(B_L(z))} + \sup_z \|1 - |u_\varepsilon|^2\|_{\mathrm{L}^2(B_L(z))} \|\nabla u_\varepsilon - i u_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon\|_{\mathrm{L}^2(B_L(z))},\end{aligned}$$

hence

$$\sup_z \|j_\varepsilon - N_\varepsilon \mathrm{v}_\varepsilon\|_{(\mathrm{L}^1 + \mathrm{L}^2)(B_L(z))} \lesssim_t o(N_\varepsilon)(1 + L/R) + \varepsilon N_\varepsilon (N_\varepsilon |\log \varepsilon|)^{1/2} (1 + L/R) + \varepsilon N_\varepsilon |\log \varepsilon| (1 + L/R)^2,$$

and the result (6.1) follows. As stated in Remark 6.2(a), under the additional assumption that $\|u^t_\varepsilon\|_{\mathrm{L}^\infty} \lesssim_t 1$, the convergence $N_\varepsilon^{-1} j_\varepsilon - \mathrm{v}_\varepsilon \to 0$ also holds in $\mathrm{L}^\infty_{\mathrm{loc}}([0,T); \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^2)^2)$ for all $1 \leq p < 2$; this follows from a similar argument as above, replacing the pointwise estimate of Lemma 4.2 for $j_\varepsilon - N_\varepsilon \mathrm{v}_\varepsilon$ by

$$|j_\varepsilon - N_\varepsilon \mathrm{v}_\varepsilon| \leq |u_\varepsilon| |\nabla u_\varepsilon - i u_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon| + N_\varepsilon |1 - |u_\varepsilon|^2| |\mathrm{v}_\varepsilon|. \qquad \square$$

7 Gross-Pitaevskii case

In this section, we prove Theorem 1.3, that is, the mean-field limit result in the Gross-Pitaevskii case in the regime (GP) (in particular, with $N_\varepsilon \gg |\log \varepsilon|$). More precisely, the rescaled supercurrent density $N_\varepsilon^{-1} j_\varepsilon$ is shown to be very close to the solution v_ε of equation (3.3). Combining this with the results of Section 3.2 (in particular, with Lemma 3.4), the result of Theorem 1.3 follows.

7.1 Preliminary: vorticity estimate

Although the vortex-balls construction and the localized lower bound of Lemma 5.1 could be adapted to the present setting with $N_\varepsilon \gg |\log\varepsilon|$, we only need the following optimal estimate on the number of vortices based on an estimate on the modulated energy excess. Since the vector field ∇h is assumed to decay at infinity, the proof is considerably reduced. (Note that in the absence of pinning and forcing no cut-off is needed and the corresponding property is completely trivial: the excess is then indeed simply defined by $\mathcal{D}_\varepsilon = \mathcal{E}_\varepsilon - \pi N_\varepsilon|\log\varepsilon|$, cf. [82].)

Lemma 7.1. *Let $h:\mathbb{R}^2\to\mathbb{R}$, $a:=e^h$, with $a\le 1$ and $\|\nabla h\|_{\mathrm{L}^2\cap\mathrm{L}^\infty}\lesssim 1$, let $u_\varepsilon:\mathbb{R}^2\to\mathbb{C}$, $\mathrm{v}_\varepsilon:\mathbb{R}^2\to\mathbb{R}^2$, with $\|\mathrm{curl}\ \mathrm{v}_\varepsilon\|_{\mathrm{L}^1\cap\mathrm{L}^\infty}\lesssim 1$ and $\|\mathrm{v}_\varepsilon\|_{\mathrm{L}^\infty}\lesssim 1$. Assume that $0<\varepsilon\ll 1$, $|\log\varepsilon|\ll N_\varepsilon\lesssim\varepsilon^{-1}$, $R\ge 1$, and assume that the modulated energy excess satisfies $\mathcal{D}^*_{\varepsilon,R}\lesssim N_\varepsilon^2$. Then,*

$$\sup_z \|\mu_\varepsilon\|_{(\dot H^1\cap W^{1,\infty}(B_R(z)))^*}\lesssim N_\varepsilon,$$

hence in particular

$$\sup_z |\mathcal{E}^z_{\varepsilon,R}-\mathcal{D}^z_{\varepsilon,R}|\lesssim N_\varepsilon|\log\varepsilon|\ll N_\varepsilon^2.$$

Proof. Let $\phi\in\dot H^1\cap W^{1,\infty}(\mathbb{R}^2)$ be supported in a ball of radius R. We decompose

$$\int\phi\mu_\varepsilon=\int\phi\big(N_\varepsilon\mathrm{curl}\ \mathrm{v}_\varepsilon+\mathrm{curl}\,(j_\varepsilon-N_\varepsilon\mathrm{v}_\varepsilon)\big)=N_\varepsilon\int\phi\mathrm{curl}\ \mathrm{v}_\varepsilon-\int\nabla^\perp\phi\cdot(j_\varepsilon-N_\varepsilon\mathrm{v}_\varepsilon),$$

hence, using the pointwise estimates of Lemma 4.2,

$$\int\phi\mu_\varepsilon\lesssim N_\varepsilon\|\phi\|_{\mathrm{L}^\infty}+(\mathcal{E}^*_{\varepsilon,R})^{1/2}\|\nabla\phi\|_{\mathrm{L}^2}+\varepsilon\mathcal{E}^*_{\varepsilon,R}\|\nabla\phi\|_{\mathrm{L}^\infty}. \tag{7.1}$$

In particular, using the assumptions $\mathcal{D}^*_{\varepsilon,R}\lesssim N_\varepsilon^2$ and $\|\nabla h\|_{\mathrm{L}^2\cap\mathrm{L}^\infty}\lesssim 1$, we obtain

$$\mathcal{E}^z_{\varepsilon,R}=\mathcal{D}^z_{\varepsilon,R}+|\log\varepsilon|\int a\chi^z_R\mu_\varepsilon\lesssim N_\varepsilon^2+|\log\varepsilon|(\mathcal{E}^*_{\varepsilon,R})^{1/2}+\varepsilon\mathcal{E}^*_{\varepsilon,R},$$

and hence, taking the supremum in z and absorbing $\mathcal{E}^*_{\varepsilon,R}$ in the left-hand side, for $\varepsilon>0$ small enough,

$$\mathcal{E}^*_{\varepsilon,R}\lesssim N_\varepsilon^2+(1+\varepsilon N_\varepsilon)^2|\log\varepsilon|^2\lesssim N_\varepsilon^2.$$

Inserting this into (7.1) yields $\int\phi\mu_\varepsilon\lesssim N_\varepsilon\|\phi\|_{\dot H^1\cap W^{1,\infty}}$, and the result follows. ☐

7.2 Modulated energy argument

Using the estimates of the previous section, we may now turn to the estimation of the different terms in the decomposition of Lemma 4.4, and deduce the smallness of the modulated energy excess by a Grönwall argument. This is the main step in the proof of the mean-field limit result stated in Theorem 1.3.

Proposition 7.2. *Let $\alpha=0$, $\beta=1$, and let $h:\mathbb{R}^2\to\mathbb{R}$, $a:=e^h$, $F:\mathbb{R}^2\to\mathbb{R}^2$, $f:\mathbb{R}^2\to\mathbb{R}$ satisfy (2.2). Let $u_\varepsilon:[0,T]\times\mathbb{R}^2\to\mathbb{C}$ and $\mathrm{v}_\varepsilon:[0,T]\times\mathbb{R}^2\to\mathbb{R}^2$ be solutions of (1.5) and (3.2) as in Proposition 2.2(ii) and in Proposition 3.3, respectively, for some $T>0$. Let $0<\varepsilon\ll 1$, $|\log\varepsilon|\ll N_\varepsilon\ll\varepsilon^{-1}$, $R\gtrsim\|\partial_t u_\varepsilon\|_{\mathrm{L}^\infty_T\mathrm{L}^2}+|\log\varepsilon|^2$, and assume that the initial modulated energy satisfies $\mathcal{E}^{*,\circ}_{\varepsilon,R}\ll N_\varepsilon^2$.*
Then, in the regime (GP)*, we have $\mathcal{E}^{*,t}_{\varepsilon,R}\ll_t N_\varepsilon^2$ for all $t\in[0,T]$, and in particular $N_\varepsilon^{-1}j_\varepsilon-\mathrm{v}_\varepsilon\to 0$ holds in $\mathrm{L}^\infty([0,T];\mathrm{L}^1_{\mathrm{uloc}}(\mathbb{R}^2)^2)$ as $\varepsilon\downarrow 0$. Under the stronger assumption $\mathcal{E}^{*,\circ}_\varepsilon\ll N_\varepsilon^2$, the same convergence holds in $\mathrm{L}^\infty([0,T];(\mathrm{L}^1+\mathrm{L}^2)(\mathbb{R}^2)^2)$.*

In the present regime $N_\varepsilon \gg |\log\varepsilon|$, Lemma 7.1 states that $|\mathcal{D}_{\varepsilon,R} - \mathcal{E}_{\varepsilon,R}| \ll N_\varepsilon^2$. Hence, as opposed to the more difficult situation treated in Section 6, the different terms appearing in the decomposition in Lemma 4.4 now only need to be estimated by means of the modulated energy $\mathcal{E}_{\varepsilon,R}$, without having to take care of the renormalization on the small balls around the vortex locations. In particular, the vector field Γ_ε does no more need to be truncated on the small balls, and we simply set $\bar\Gamma_\varepsilon = \Gamma_\varepsilon$. For this choice, all terms involving the vortex velocity $\tilde V_{\varepsilon,\varrho}$ are easily seen to vanish. This simplification is crucial since in the present conservative case we have no good a priori control on the velocity (apart from rough a priori estimates of the form $\|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_{\varepsilon,\varrho}\|_{\mathrm{L}^2} \lesssim \varepsilon^{-2}$), and this prevents us from treating the case $N_\varepsilon \lesssim |\log\varepsilon|$ in the Gross-Pitaevskii case.

Proof. In the sequel, we choose $1 \ll \varrho \le R$ with $\varrho^{\theta_0} \ll (\varepsilon N_\varepsilon)^{-1}$ for some $\theta_0 > 0$. Regarding the global truncation at the scale R, it is not really needed in the present context (as a consequence of the decay assumption on the fields $\nabla h, F, f$), and can be sent to infinity arbitrarily fast; here it suffices to choose $R \ge \sup_{t\in[0,T]} \|\partial_t u_\varepsilon\|_{\mathrm{L}^2} + |\log\varepsilon|^2$ (where the right-hand side is indeed finite by Proposition 2.2(ii)). Given the assumption $\mathcal{E}^{*,\circ}_{\varepsilon,R} \ll N_\varepsilon^2$ on the initial data, for all $\varepsilon > 0$, we define $T_\varepsilon > 0$ as the maximum time $\le T$ such that $\mathcal{E}^{*,t}_{\varepsilon,R} \le N_\varepsilon^2$ holds for all $t \le T_\varepsilon$. By Lemmas 4.1 and 7.1, we deduce $\hat{\mathcal{D}}^{*,\circ}_{\varepsilon,\varrho,R} \ll N_\varepsilon^2$ and for all $t \le T_\varepsilon$,

$$\mathcal{D}^{*,t}_{\varepsilon,R} \lesssim_t N_\varepsilon^2, \quad \hat{\mathcal{E}}^{*,t}_{\varepsilon,\varrho,R} \lesssim_t N_\varepsilon^2, \quad \hat{\mathcal{D}}^{*,t}_{\varepsilon,\varrho,R} \lesssim_t N_\varepsilon^2, \quad \mathcal{E}^{*,t}_{\varepsilon,R} \lesssim \hat{\mathcal{E}}^{*,t}_{\varepsilon,\varrho,R} + o_t(N_\varepsilon^2), \quad \hat{\mathcal{E}}^{*,t}_{\varepsilon,\varrho,R} \lesssim \hat{\mathcal{D}}^{*,t}_{\varepsilon,\varrho,R} + o_t(N_\varepsilon^2). \tag{7.2}$$

The strategy of the proof now consists in showing that for all $t \le T_\varepsilon$

$$\hat{\mathcal{E}}^{*,t}_{\varepsilon,\varrho,R} \lesssim_t o(N_\varepsilon^2) + \int_0^t \hat{\mathcal{E}}^{*}_{\varepsilon,\varrho,R}. \tag{7.3}$$

This estimate is proved in Step 1 below. To simplify notation, we focus on (7.3) with the left-hand side $\hat{\mathcal{E}}^{t}_{\varepsilon,\varrho,R}$ centered at $z = 0$, but the result of course holds uniformly with respect to the translation. By the Grönwall inequality, it implies $\hat{\mathcal{E}}^{*,t}_{\varepsilon,\varrho,R} \ll_t N_\varepsilon^2$, hence $\mathcal{E}^{*,t}_{\varepsilon,R} \ll_t N_\varepsilon^2$ for all $t \le T_\varepsilon$. This gives in particular $T_\varepsilon = T$, and the main conclusion follows, while the additional stated consequences are deduced in Step 2.

Step 1: proof of (7.3). Using the constraint $0 = a^{-1}\operatorname{div}(a\mathrm{v}_\varepsilon) = \operatorname{div}\mathrm{v}_\varepsilon + \mathrm{v}_\varepsilon\cdot\nabla h$, and choosing $\bar\Gamma_\varepsilon := \Gamma_\varepsilon$, the result of Lemma 4.4 takes the following simpler form,

$$\partial_t \hat{\mathcal{D}}_{\varepsilon,\varrho,R} = I^S_{\varepsilon,\varrho,R} + I^V_{\varepsilon,\varrho,R} + I^E_{\varepsilon,\varrho,R} + I^H_{\varepsilon,\varrho,R} + I^n_{\varepsilon,\varrho,R} + I'_{\varepsilon,\varrho,R}, \tag{7.4}$$

where we have set

$$\begin{aligned}
I^S_{\varepsilon,\varrho,R} &:= -\int \chi_R \nabla\Gamma_\varepsilon^\perp : \tilde S_\varepsilon,\\
I^V_{\varepsilon,\varrho,R} &:= \int \frac{a\chi_R}{2}\tilde V_{\varepsilon,\varrho}\cdot(-\lambda_\varepsilon|\log\varepsilon|\Gamma_\varepsilon^\perp + |\log\varepsilon|(\nabla^\perp h - F^\perp) - 2N_\varepsilon \mathrm{v}_\varepsilon),\\
I^E_{\varepsilon,\varrho,R} &:= -\int \frac{a\chi_R}{2}\Gamma_\varepsilon\cdot(|\log\varepsilon|(\nabla^\perp h - F^\perp) - 2N_\varepsilon\mathrm{v}_\varepsilon)\mu_\varepsilon,\\
I^H_{\varepsilon,\varrho,R} &:= \int \frac{a\chi_R}{2}\Gamma_\varepsilon^\perp\cdot\nabla h\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{\varepsilon^2}(1-|u_\varepsilon|^2)^2 - |\log\varepsilon|\mu_\varepsilon\Big),\\
I^n_{\varepsilon,\varrho,R} &:= -\int \nabla\chi_R\cdot\tilde S_\varepsilon\cdot\Gamma_\varepsilon^\perp - \int a\nabla\chi_R\cdot\Big(\langle\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{p}_\varepsilon, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon\rangle + \frac{|\log\varepsilon|}{2}\tilde V^\perp_{\varepsilon,\varrho}\Big),
\end{aligned}$$

and where the error $I'_{\varepsilon,\varrho,R}$ is estimated as follows (cf. (4.17)),

$$|I'_{\varepsilon,\varrho,R}| \lesssim_{t,\theta} \varepsilon N_\varepsilon \mathcal{E}^*_{\varepsilon,R} + N_\varepsilon(\mathcal{E}^*_{\varepsilon,R})^{1/2}\|\nabla(\mathrm{p}_\varepsilon - \mathrm{p}_{\varepsilon,\varrho})\|_{\mathrm{L}^2} + \varepsilon N_\varepsilon^2\varrho^\theta(\mathcal{E}^*_{\varepsilon,R})^{1/2}.$$

Choosing $\theta > 0$ small enough, and using Proposition 3.3 in the form $\|\nabla(\mathrm{p}_\varepsilon^t - \mathrm{p}_{\varepsilon,\varrho}^t)\|_{\mathrm{L}^2} \ll_t 1$ (cf. (3.22)), we obtain

$$|I'_{\varepsilon,\varrho,R}| \lesssim_{t,\theta} \mathcal{E}^*_{\varepsilon,R} + o(N_\varepsilon)(\mathcal{E}^*_{\varepsilon,R})^{1/2}. \tag{7.5}$$

The choice (3.3) for Γ_ε gives $I^V_{\varepsilon,\varrho,R} = I^E_{\varepsilon,\varrho,R} = 0$, hence

$$\partial_t \hat{\mathcal{D}}_{\varepsilon,\varrho,R} = I^S_{\varepsilon,\varrho,R} + I^H_{\varepsilon,\varrho,R} + I^n_{\varepsilon,\varrho,R} + I'_{\varepsilon,\varrho,R}. \tag{7.6}$$

It remains to estimate the first three right-hand side terms. By (2.2) in the form $\|f\|_{\mathrm{L}^2} \lesssim N_\varepsilon^2$, and by the integrability properties of v_ε (cf. Proposition 3.3), the first right-hand side term is estimated by

$$\begin{aligned} I^S_{\varepsilon,\varrho,R} &\lesssim \|\nabla\Gamma_\varepsilon\|_{\mathrm{L}^\infty} \int \chi_R\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{\varepsilon^2}(1-|u_\varepsilon|^2)^2 + |1-|u_\varepsilon|^2|(N_\varepsilon^2|\mathrm{v}_\varepsilon|^2 + |f|)\Big) \\ &\lesssim_t \mathcal{E}_{\varepsilon,R} + \varepsilon N_\varepsilon^2(\mathcal{E}_{\varepsilon,R})^{1/2} \lesssim \mathcal{E}_{\varepsilon,R} + o(N_\varepsilon^2). \end{aligned} \tag{7.7}$$

We turn to the second right-hand side term in (7.6). Lemma 7.1 yields

$$\begin{aligned} I^H_{\varepsilon,\varrho,R} &\le \|\Gamma_\varepsilon^\perp \cdot \nabla h\|_{\mathrm{L}^\infty} \int \chi_R\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon|^2 + \frac{a}{\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big) + |\log\varepsilon|\Big|\int a\chi_R\Gamma_\varepsilon^\perp \cdot \nabla h\, \mu_\varepsilon\Big| \\ &\lesssim \mathcal{E}_{\varepsilon,R}\|\Gamma_\varepsilon^\perp\cdot\nabla h\|_{\mathrm{L}^\infty} + N_\varepsilon|\log\varepsilon|\|a\chi_R\Gamma_\varepsilon^\perp\cdot\nabla h\|_{\dot H^1\cap W^{1,\infty}}, \end{aligned}$$

and hence, using (2.2) and the properties of v_ε (cf. Proposition 3.3),

$$I^H_{\varepsilon,\varrho,R} \lesssim_t \mathcal{E}_{\varepsilon,R} + N_\varepsilon|\log\varepsilon| \le \mathcal{E}_{\varepsilon,R} + o(N_\varepsilon^2). \tag{7.8}$$

It remains to estimate the third right-hand side term in (7.6). Arguing as above, we find

$$\begin{aligned} I^n_{\varepsilon,\varrho,R} &\lesssim R^{-1}\|\Gamma_\varepsilon\|_{\mathrm{L}^\infty}\int_{B_{2R}}\Big(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon|^2 + \frac{a}{\varepsilon^2}(1-|u_\varepsilon|^2)^2 + |1-|u_\varepsilon|^2|(N_\varepsilon^2|\mathrm{v}_\varepsilon|^2+|f|)\Big) \\ &\qquad + R^{-1}|\log\varepsilon|\int_{B_{2R}}|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{p}_{\varepsilon,\varrho}||\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon| \\ &\lesssim_t \mathcal{E}^*_{\varepsilon,R} + o(N_\varepsilon^2) + R^{-1}|\log\varepsilon|(\mathcal{E}^*_{\varepsilon,R})^{1/2}\|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{p}_{\varepsilon,\varrho}\|_{\mathrm{L}^2(B_{2R})}. \end{aligned}$$

The properties of p_ε (cf. Proposition 3.3) yield for all $\theta > 0$,

$$\begin{aligned} \|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{p}_{\varepsilon,\varrho}\|_{\mathrm{L}^2(B_{2R})} &\lesssim \|\partial_t u_\varepsilon\|_{\mathrm{L}^2(B_{2R})} + N_\varepsilon\|\mathrm{p}_{\varepsilon,\varrho}\|_{\mathrm{L}^2(B_{2R})} + N_\varepsilon\|\mathrm{p}_{\varepsilon,\varrho}\|_{\mathrm{L}^\infty(B_{2R})}\|1-|u_\varepsilon|^2\|_{\mathrm{L}^2(B_{2R})} \\ &\lesssim_{t,\theta} \|\partial_t u_\varepsilon\|_{\mathrm{L}^2(B_{2R})} + N_\varepsilon\varrho^\theta + \varepsilon N_\varepsilon(\mathcal{E}^*_{\varepsilon,R})^{1/2}, \end{aligned}$$

so that the above takes the form

$$I^n_{\varepsilon,\varrho,R} \lesssim_{t,\theta} \mathcal{E}^*_{\varepsilon,R} + R^{-2}|\log\varepsilon|^2\|\partial_t u_\varepsilon\|^2_{\mathrm{L}^2(B_{2R})} + R^{-2(1-\theta)}N_\varepsilon^2|\log\varepsilon|^2 + o(N_\varepsilon^2).$$

Using the choice $R \gtrsim \|\partial_t u_\varepsilon\|_{\mathrm{L}^2} + |\log\varepsilon|^2$, and choosing $\theta > 0$ small enough, we deduce $I^n_{\varepsilon,\varrho,R} \lesssim_t \mathcal{E}^*_{\varepsilon,R} + o(N_\varepsilon^2)$. Combining this with (7.5), (7.6), (7.7), and (7.8), we conclude

$$\partial_t\hat{\mathcal{D}}_{\varepsilon,\varrho,R} \lesssim_t \mathcal{E}^*_{\varepsilon,R} + o(N_\varepsilon^2).$$

Integrating this in time with $\hat{\mathcal{D}}^{*,\circ}_{\varepsilon,\varrho,R} \ll N_\varepsilon^2$, using (7.2), and noting that the same result holds uniformly with respect to translations of the cut-off functions, the conclusion (7.3) follows.

Step 2: conclusion. As explained, the result of Step 1 implies $T_\varepsilon = T$ and $\mathcal{E}^{*,t}_{\varepsilon,R} \ll_t N_\varepsilon^2$ for all $t \in [0,T)$. We now show that it implies the convergence $N_\varepsilon^{-1} j_\varepsilon - \mathrm{v}_\varepsilon \to 0$. Using the pointwise estimates of Lemma 4.2, we obtain

$$\|j_\varepsilon - N_\varepsilon \mathrm{v}_\varepsilon\|_{(\mathrm{L}^1+\mathrm{L}^2)(B_R(z))} \lesssim \|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon\|_{\mathrm{L}^2(B_R(z))}\big(1 + \|1-|u_\varepsilon|^2\|_{\mathrm{L}^2(B_R(z))}\big) + N_\varepsilon \|1-|u_\varepsilon|^2\|_{\mathrm{L}^2(B_R(z))}$$
$$\ll_t N_\varepsilon(1+\varepsilon N_\varepsilon) \lesssim N_\varepsilon,$$

and the conclusion follows, letting $R \uparrow \infty$. □

Part II
Homogenization questions

8 Small pin separation limit

In this section, we aim to combine the mean-field limit with the homogenization limit of a small pin separation $\eta_\varepsilon \downarrow 0$. Only partial results are obtained here for this double limit. We focus on the dissipative case, and for simplicity we restrict to the periodic setting, that is, $\hat{h}(x) = \eta_\varepsilon \hat{h}^0(x, x/\eta_\varepsilon)$ with $\hat{h}^0$ periodic in its second variable.

8.1 Modulated energy argument

In this section, we adapt the result of Proposition 6.1 to the case with fast oscillating pinning. Since for simplicity we have not been looking for precise rates of convergence in Proposition 6.1 (that is, refinements of the $o(N_\varepsilon^2)$ error in (6.3)), we are only in position to treat inexplicit diagonal regimes $\eta_{\varepsilon,0} \ll \eta_\varepsilon \ll 1$, for some suitable $\eta_{\varepsilon,0}$. Further refinements are left to the interested reader.

Proposition 8.1. *Given a fast oscillating pinning potential* (1.22)*, we consider the regimes* (GL_1)*,* (GL_2)*,* (GL_1')*, and* (GL_2')*. Then, the solution v_ε of the corresponding limiting equation* (3.2) *exists up to time $\eta_\varepsilon T$, where $T > 0$ is as in Proposition 3.1. In particular, except in the regime* (GL_2) *with $\beta \neq 0$, the time T can be chosen either infinite, or at least arbitrary large for $\varepsilon > 0$ small enough (independently of η_ε).*

Moreover, there exists some exponent $\sigma > 0$ and some increasing bijection $\theta : \mathbb{R}^+ \to \mathbb{R}^+$ such that, if the initial modulated energy satisfies $\mathcal{D}^{*,\circ}_{\varepsilon,R} \ll N_\varepsilon^2$, then we have in the corresponding regimes, with the same restrictions as in Proposition 6.1, for all $0 \le t < \eta_\varepsilon T$,

$$\sup_{0\le s\le t} \hat{\mathcal{D}}^s_{\varepsilon,R} \le N_\varepsilon^2 \qquad \Longrightarrow \qquad \hat{\mathcal{D}}^t_{\varepsilon,R} \le \theta(t/\eta_\varepsilon)\Big(\eta_\varepsilon^{-\sigma} o(N_\varepsilon^2) + \eta_\varepsilon^{-1}\int_0^t \hat{\mathcal{D}}_{\varepsilon,R}\Big). \tag{8.1}$$

Proof. We adapt the proof of Proposition 6.1 to the present case with fast oscillating pinning. For that purpose we first need to check how the solution v_ε of the limiting equations (3.2) depends on the small parameter η_ε, thus adapting the result of Proposition 3.1. A scaling argument shows that the solution v_ε exists up to time $\eta_\varepsilon T$, where T is as in Proposition 3.1. Moreover, an inspection of the proofs in [37] together with a scaling argument shows that all the estimates in Proposition 3.1 still hold up to multiplicative constants of the form $\eta_\varepsilon^{-\sigma}\theta(t/\eta_\varepsilon)$, for all $0 \le t < \eta_\varepsilon T$, for some exponent $\sigma \ge 0$ and some increasing bijection $\theta : \mathbb{R}^+ \to \mathbb{R}^+$. (Of course this is but a rough estimate, but it is enough for our purposes here.) Note that a scaling argument yields more precisely, for all $0 \le t < \eta_\varepsilon T$,

$$\|\Gamma^t_\varepsilon\|_{\mathrm{L}^\infty} \le \theta(t/\eta_\varepsilon), \qquad \|\nabla\Gamma^t_\varepsilon\|_{\mathrm{L}^\infty} \le \eta_\varepsilon^{-1}\theta(t/\eta_\varepsilon),$$

for some increasing bijection $\theta : \mathbb{R}^+ \to \mathbb{R}^+$. Repeating the proof of Proposition 6.1, but now taking into account this η_ε-dependence, the conclusion follows. $\square$

8.2 Local relaxation for slowed-down dynamics

The result of Proposition 8.1 a priori prevents us from applying a Grönwall argument beyond times of order η_ε. As the following shows, in this short timescale, in each (mesoscopic) periodicity cell, the vorticity gets projected onto the invariant measure for the cell dynamics associated with the initial vector field Γ_ε° (where Γ_ε is the vector field driving the limiting equation (3.2)). This initial-boundary layer is captured in the framework of 2-scale convergence. The proof of this short-time result is very easy since the non-linearity does not play any role in this timescale. In contrast, in the next sections, we give formal arguments that on larger timescales the effective vector field is given by the cell vector field projected onto the corresponding invariant measure (which is indeed in agreement with the present short-time result), but on such large timescales the nonlinearity truly enters into play and a rigorous justification is still missing.

Proposition 8.2. *Let Assumption A(a) prevail, with the initial data $(u_\varepsilon^\circ, \mathrm{v}_\varepsilon^\circ, \mathrm{v}^\circ)$ satisfying the well-preparedness condition (1.14). We consider the regimes (GL_1), (GL_2), (GL_1'), and (GL_2'), with fast oscillating pinning potential (1.22). Let $u_\varepsilon : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{C}$ be the solution of (1.5) as in Proposition 2.2(i). Let $T > 0$ denote the finite existence time given by Proposition 3.1 in the regime (GL_2) in the mixed-flow case $\beta \neq 0$, and set $T := \infty$ otherwise. Let also $\hat{\mathrm{m}}_0$ denote the unique solution of the following transport equation on $\mathbb{R}^+ \times Q$, for all $x \in \mathbb{R}^d$,*

$$\partial_t \hat{\mathrm{m}}_0(x,\cdot) = \mathrm{div}_y\big(\Gamma^\circ(x,\cdot)\hat{\mathrm{m}}_0(x,\cdot)\big), \qquad \hat{\mathrm{m}}_0(x,\cdot)|_{t=0} = \mathrm{curl}\,\mathrm{v}^\circ(x), \tag{8.2}$$
$$\Gamma^\circ(x,y) := (\alpha - \mathbb{J}\beta)\big(\nabla_2 \hat{h}^0(x,y) - \hat{F}(x) + 2\kappa \mathrm{v}^\circ(x)^\perp\big),$$

where $\kappa := 1$ in the regime (GL_1), $\kappa := \lambda$ in the regime (GL_2), and $\kappa := 0$ in the regimes (GL_1')–(GL_2'). Then, there exists $\eta_{\varepsilon,0} \ll 1$ (depending on all the data of the problem), such that for any choice $\eta_{\varepsilon,0} \le \eta_\varepsilon \ll 1$ the rescaled vorticity $N_\varepsilon^{-1}\mu_\varepsilon^{\eta_\varepsilon t}$ 2-scale converges to $\hat{\mathrm{m}}_0^t$, in the sense that, for all $\phi \in C_c^\infty([0,T)\times\mathbb{R}^2; C_{\mathrm{per}}^\infty(Q))$,

$$\lim_{\varepsilon \downarrow 0} \iint \phi(t,x,x/\eta_\varepsilon) N_\varepsilon^{-1}\mu_\varepsilon^{\eta_\varepsilon t}(x)\,dxdt = \iiint \phi(t,x,y)\hat{\mathrm{m}}_0^t(x,y)\,dydxdt.$$

Proof. Let $\mathrm{v}_\varepsilon : [0,\eta_\varepsilon T) \times \mathbb{R}^2 \to \mathbb{R}^2$ denote the solution of the limiting equations (3.2) with oscillating pinning (1.22), as given by Proposition 8.1. Now using Proposition 8.1 in the form of (8.1), and choosing $\eta_{\varepsilon,0} \ll 1$ large enough such that $\eta_{\varepsilon,0}^{-\sigma} o(N_\varepsilon^2) \ll N_\varepsilon^2$, the Grönwall inequality yields for any choice $\eta_{\varepsilon,0} \le \eta_\varepsilon \ll 1$ that $\mathcal{D}_{\varepsilon,R}^{*,\eta_\varepsilon t} \lesssim_t o(N_\varepsilon^2)$ holds for all $0 \le t < T$. Hence, arguing as in Step 5 of the proof of Proposition 6.1, we deduce $N_\varepsilon^{-1} j_\varepsilon^{\eta_\varepsilon t}(x) - \mathrm{v}_\varepsilon^{\eta_\varepsilon t}(x) \to 0$ in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^1_{\mathrm{uloc}}(\mathbb{R}^2)^2)$ as $\varepsilon \downarrow 0$. It now remains to determine the asymptotic behaviour of $\mathrm{v}_\varepsilon^{\eta_\varepsilon t}$.

Step 1: 2-scale convergence of $\mathrm{curl}\,\mathrm{v}_\varepsilon^{\eta_\varepsilon t}$. Let $\hat{\mathrm{v}}_\varepsilon^t := \mathrm{v}_\varepsilon^{\eta_\varepsilon t}$ and $\hat{\mathrm{m}}_\varepsilon := \mathrm{curl}\,\hat{\mathrm{v}}_\varepsilon$. Taking the curl in both sides of (3.2), we deduce

$$\partial_t \hat{\mathrm{m}}_\varepsilon = \eta_\varepsilon\,\mathrm{div}\,(\Gamma_\varepsilon \hat{\mathrm{m}}_\varepsilon), \qquad \hat{\mathrm{m}}_\varepsilon|_{t=0} = \mathrm{curl}\,\mathrm{v}_\varepsilon^\circ, \tag{8.3}$$
$$\Gamma_\varepsilon := \lambda_\varepsilon^{-1}(\alpha - \mathbb{J}\beta)\Big(\nabla h - F + \frac{2N_\varepsilon}{|\log\varepsilon|}\hat{\mathrm{v}}_\varepsilon^\perp\Big).$$

By [37, Lemma 4.1(iii)] in the dissipative case $\alpha > 0$, with $\|h\|_{W^{1,\infty}}$, $\|\lambda_\varepsilon^{-1}(\nabla^\perp h - F^\perp)\|_{\mathrm{L}^\infty}$, $\|\mathrm{v}_\varepsilon^\circ\|_{\mathrm{L}^\infty}$, $\|\mathrm{div}\,(a\mathrm{v}_\varepsilon^\circ)\|_{\mathrm{L}^2} \lesssim 1$, we deduce that $\int |\mathrm{v}_\varepsilon^t - \mathrm{v}_\varepsilon^\circ|^2 \lesssim t$ for all $t \in [0,\eta_\varepsilon T)$. On the other hand, by [37, Lemma 4.2], with further $\|\mathrm{curl}\ \mathrm{v}_\varepsilon^\circ\|_{\mathrm{L}^\infty} \lesssim 1$ and $\|\lambda_\varepsilon^{-1}\nabla(\nabla^\perp h - F^\perp)\|_{\mathrm{L}^\infty} \lesssim \eta_\varepsilon^{-1}$, we find $\|\mathrm{curl}\ \mathrm{v}_\varepsilon^t\|_{\mathrm{L}^\infty} \lesssim_{t/\eta_\varepsilon} 1$. After time rescaling, this implies for all $t \in [0,T)$,

$$\int |\hat{\mathrm{v}}_\varepsilon^t - \mathrm{v}_\varepsilon^\circ|^2 \lesssim_t \eta_\varepsilon, \qquad \text{and} \qquad \|\hat{\mathrm{m}}_\varepsilon^t\|_{\mathrm{L}^\infty} \lesssim_t 1. \tag{8.4}$$

Nguetseng's 2-scale compactness theorem [68, 4] (e.g. in the form of [38, Theorem 3.2]) then states that there exists $\tilde{\mathrm{m}}_0 \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^\infty(\mathbb{R}^2 \times Q))$ such that (up to a subsequence) $\hat{\mathrm{m}}_\varepsilon$ 2-scale converges to $\tilde{\mathrm{m}}_0$, in the sense that for all $\phi \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R}^2; C^\infty_{\mathrm{per}}(Q))$ we have

$$\lim_{\varepsilon \downarrow 0} \iint \phi(t,x,x/\eta_\varepsilon)\hat{\mathrm{m}}_\varepsilon^t(x)dxdt = \iiint \phi(t,x,y)\tilde{\mathrm{m}}_0^t(x,y)dydxdt.$$

Testing equation (8.3) against $\phi^t(x,x/\eta_\varepsilon)$, we find

$$\int \phi^\circ(x,x/\eta_\varepsilon)\mathrm{curl}\ \mathrm{v}^\circ(x)dx + \iint \partial_t\phi(t,x,x/\eta_\varepsilon)\hat{\mathrm{m}}_\varepsilon^t(x)dxdt$$
$$= \iint \hat{\mathrm{m}}_\varepsilon^t(x)(\eta_\varepsilon\nabla_1\phi(t,x,x/\eta_\varepsilon) + \nabla_2\phi(t,x,x/\eta_\varepsilon)) \cdot \Gamma_\varepsilon^t(x)dxdt,$$

and hence, passing to the limit $\varepsilon \downarrow 0$ (up to a subsequence), using that $\hat{\mathrm{v}}_\varepsilon \to \mathrm{v}^\circ$ in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^2_{\mathrm{uloc}}(\mathbb{R}^2))$ (cf. (8.4)), we obtain in the considered regimes,

$$\iint \phi(0,x,y)\mathrm{curl}\ \mathrm{v}^\circ(x)dydx + \iiint \partial_t\phi(t,x,y)\tilde{\mathrm{m}}_0^t(x,y)dydxdt = \iiint \tilde{\mathrm{m}}_0^t(x,y)\nabla_2\phi(t,x,y) \cdot \Gamma^\circ(x,y)dydxdt.$$

This proves that $\tilde{\mathrm{m}}_0$ satisfies the weak formulation of the linear transport equation (8.2), and therefore coincides with its unique solution, $\tilde{\mathrm{m}}_0 = \hat{\mathrm{m}}_0$.

Step 2: conclusion. Let $\phi \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R}^2; C^\infty_{\mathrm{per}}(Q))$, with $\phi(t,x,y) = 0$ for $t > T_0$ or $|x| > R_0$. Integration by parts yields

$$\Big|\iint \phi(t,x,x/\eta_\varepsilon)\mathrm{curl}\,(j_\varepsilon^{\eta_\varepsilon t}/N_\varepsilon)(x)dxdt - \iiint \phi(t,x,y)\hat{\mathrm{m}}_0^t(x,y)dydxdt\Big| \tag{8.5}$$
$$\le \eta_\varepsilon^{-1}\|\nabla_{x,y}\phi\|_{\mathrm{L}^\infty}\int_0^{T_0}\int_{B_{R_0}} |j_\varepsilon^{\eta_\varepsilon t}/N_\varepsilon - \hat{\mathrm{v}}_\varepsilon^t| + \Big|\iint \phi(t,x,x/\eta_\varepsilon)\mathrm{curl}\,\hat{\mathrm{v}}_\varepsilon^t(x)dxdt - \iiint \phi(t,x,y)\hat{\mathrm{m}}_0^t(x,y)dydxdt\Big|.$$

By Step 1, the second right-hand side term goes to 0. It remains to estimate the first term. In the very beginning of the proof, we have shown that $\int_0^{T_0}\int_{B_{R_0}} |j_\varepsilon^{\eta_\varepsilon t}/N_\varepsilon - \hat{\mathrm{v}}_\varepsilon^t| \to 0$ holds uniformly with respect to the choice of $\eta_{\varepsilon,0} \le \eta_\varepsilon \ll 1$. Now choosing $\eta_{\varepsilon,0} \ll 1$ large enough ensures that for any $\eta_{\varepsilon,0} \le \eta_\varepsilon \ll 1$ the first right-hand side term in (8.5) also goes to 0. □

8.3 Homogenization diagonal result

Although the result of Proposition 8.1 a priori prevents us from applying a Grönwall argument beyond times of order η_ε, it is possible to find some perturbative diagonal regime where the conclusion holds for all times. (While this regime is still denoted by $\eta_{\varepsilon,0} \le \eta_\varepsilon \ll 1$ for some large enough $\eta_{\varepsilon,0} \ll 1$, it should be emphasized that the sequence $\eta_{\varepsilon,0}$ needs here to be taken in practice incomparably much larger than in Propositions 8.1–8.2.) In such a regime, the homogenization limit may simply be performed *after* the mean-field limit.

Corollary 8.3. *Given a fast oscillating pinning potential* (1.22)*, we consider the regimes* (GL_1)*,* (GL_2)*,* (GL_1')*, and* (GL_2')*, and in the regime* (GL_2) *we restrict to the parabolic case* $\beta = 0$*. Then there exists* $\eta_{\varepsilon,0} \ll 1$ *(depending on all the data of the problem), such that for all* $\eta_{\varepsilon,0} \ll \eta_\varepsilon \ll 1$ *the conclusions of Proposition* 6.1 *hold in each of the corresponding regimes.*

Proof. Since the regime (GL_2) is excluded here in the mixed-flow case $\beta \neq 0$, Proposition 8.1 asserts that the solution v_ε of (3.2) with oscillating pinning exists up to time $\eta_\varepsilon T$, where $T > 0$ can be chosen arbitrarily large for $\varepsilon > 0$ small enough (independently of η_ε). Hence, choosing $\eta_{\varepsilon,0} \le \eta_\varepsilon \ll 1$ with $\eta_{\varepsilon,0}$ large enough, the existence time $\eta_\varepsilon T$ can itself be taken arbitrarily large. Now given the assumption $\mathcal{D}^{*,\circ}_{\varepsilon,R} \ll N_\varepsilon^2$ on the initial data, for all $\varepsilon > 0$ we define $T_\varepsilon > 0$ as the maximum time such that $\mathcal{D}^{*,t}_{\varepsilon,R} \le N_\varepsilon^2$ holds for all $t \le T_\varepsilon$, so that Proposition 8.1 yields for all $0 \le t \le T_\varepsilon$,

$$\hat{\mathcal{D}}^t_{\varepsilon,R} \le \theta(t/\eta_\varepsilon)\Big(\eta_\varepsilon^{-\sigma} o(N_\varepsilon^2) + \eta_\varepsilon^{-1} \int_0^t \hat{\mathcal{D}}_{\varepsilon,R}\Big),$$

for some exponent $\sigma \ge 0$ and some increasing bijection $\theta : \mathbb{R}^+ \to \mathbb{R}^+$. Hence we find by the Grönwall inequality, for all $0 \le t \le T_\varepsilon$,

$$\hat{\mathcal{D}}^t_{\varepsilon,R} \lesssim \theta\Big(\frac{t+1}{\eta_\varepsilon^\sigma}\Big) o(N_\varepsilon^2),$$

for some exponent $\sigma \ge 1$ and some other increasing bijection $\theta : \mathbb{R}^+ \to \mathbb{R}^+$. Choosing $[\theta^{-1}(N_\varepsilon/\sqrt{o(N_\varepsilon^2)})]^{-1/\sigma} \le \eta_{\varepsilon,0} \ll 1$, for any $\eta_{\varepsilon,0} \ll \eta_\varepsilon \ll 1$ we deduce $\hat{\mathcal{D}}^t_{\varepsilon,R} \ll N_\varepsilon^2$ for all $t \ge 0$, and the conclusion now follows as in Step 5 of the proof of Proposition 6.1. ☐

In this diagonal regime, the problem is thus reduced to determining the asymptotic behavior as $\varepsilon \downarrow 0$ of the solution v_ε of the limiting equation (3.2) with fast oscillating pinning potential (1.22). As the following shows, we may further replace v_ε by the solution $\bar{\mathrm{v}}_\varepsilon$ of the simpler corresponding equations in Lemma 3.2 with fast oscillating pinning potential. Determining the asymptotic behavior of $\bar{\mathrm{v}}_\varepsilon$ exactly coincides with an homogenization problem; this is precisely the content of Corollary 1.5.

Corollary 8.4. *Given a fast oscillating pinning potential* (1.22)*, we consider the regimes* (GL_1)*,* (GL_2)*,* (GL_1')*, and* (GL_2')*, and in the regime* (GL_2) *we restrict to the parabolic case* $\beta = 0$*. Let* v_ε *be the solution of* (3.2) *with fast oscillating pinning as in Proposition* 8.1*, and let* $\bar{\mathrm{v}}_\varepsilon$ *be the solution of the corresponding equation* (3.13)*–*(3.16) *in Lemma* 3.2 *with* $\nabla\hat{h}(x)$ *replaced by* $\nabla_2\hat{h}^0(x, x/\eta_\varepsilon)$*. Then there exists* $\eta_{\varepsilon,0} \ll 1$ *(depending on all the data of the problem) such that for all* $\eta_{\varepsilon,0} \ll \eta_\varepsilon \ll 1$*, choosing the fast oscillating pinning potential* (1.22)*, the solutions* v_ε *and* $\bar{\mathrm{v}}_\varepsilon$ *exist on arbitrarily large time intervals as* $\varepsilon \downarrow 0$*, and the same conclusions hold as in Lemma* 3.2 *in the form* $\mathrm{v}_\varepsilon - \bar{\mathrm{v}}_\varepsilon \to 0$*.*

Note that here the correct choice of the diagonal regime $\eta_{\varepsilon,0} \ll 1$ could be made completely explicit in terms of the rate of convergence of $N_\varepsilon/|\log\varepsilon|$ to its limit. This is however not made precise here, as anyway we are limited to some unclear diagonal regime when combining this with Corollary 8.3.

Proof. This convergence result directly follows from the computations in the proof of Lemma 3.2, taking into account the η_ε-dependence of v_ε and $\bar{\mathrm{v}}_\varepsilon$, and applying the Grönwall inequality in a diagonal regime as in the proof of Corollary 8.3. ☐

In the next sections 8.4–8.5, we examine the various homogenization problems arising in the above result. Although the justification of the homogenization of the nonlinear equation arising in the critical regimes seems out of reach, the situation is simpler in the subcritical regimes.

8.4 Critical regimes: formal asymptotics

In this section, we investigate the asymptotic behavior of the mean-field equations in the critical regimes (GL_1)–(GL_2) with fast oscillating pinning (1.22). In order to extract the effective equations that should rule the system in the limit $\eta_\varepsilon \downarrow 0$, we use a formal 2-scale expansion (see [7] for a general presentation), which yields the result of Heuristics 1.7. However, as emphasized in Remark 8.5 below, due to both the nonlinear nonlocal character of the mean-field equations and their instability as $\eta_\varepsilon \downarrow 0$, the rigorous justification of this homogenization limit seems to be a very difficult task, and is not pursued here. Regarding the interpretation of the formal limiting equations as a stick-slip model, we refer to Section 1.3.1.

Formal justification of Heuristics 1.7. We focus on the regime (GL_1), while the formal justification is easily adapted to the regime (GL_2). The only difference is that in the regime (GL_2) it is further needed to restrict to the parabolic case $\beta = 0$ in order to get global existence for the solution $\bar{\mathrm{v}}_\varepsilon$ of (3.14) with fast oscillating pinning, since otherwise the finite existence time would a priori shrink to 0 as $\eta_\varepsilon \downarrow 0$ (cf. Proposition 8.1). Let $\bar{\mathrm{v}}_\varepsilon : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}^2$ denote the unique (global) smooth solution of (3.13) with $\nabla \hat{h}(x)$ replaced by $\nabla_2 \hat{h}^0(x, x/\eta_\varepsilon)$ (see [37]),

$$
\begin{gathered}
\partial_t \bar{\mathrm{v}}_\varepsilon = \nabla \bar{\mathrm{p}}_\varepsilon + \Gamma_\varepsilon \operatorname{curl} \bar{\mathrm{v}}_\varepsilon, \qquad \operatorname{div} \bar{\mathrm{v}}_\varepsilon = 0, \qquad \bar{\mathrm{v}}_\varepsilon|_{t=0} = \mathrm{v}_\varepsilon^\circ, \\
\Gamma_\varepsilon := (\alpha - \mathbb{J}\beta)\big(\nabla_2^\perp \hat{h}^0(\cdot, \cdot/\eta_\varepsilon) - \hat{F}^\perp - 2\bar{\mathrm{v}}_\varepsilon\big),
\end{gathered}
$$

with $\hat{h}^0$ and $\hat{F}$ independent of ε. Let us recall the more convenient vorticity formulation of this equation: the vorticity $\bar{\mathrm{m}}_\varepsilon := \operatorname{curl} \bar{\mathrm{v}}_\varepsilon$ satisfies

$$
\partial_t \bar{\mathrm{m}}_\varepsilon = -\operatorname{div}(\Gamma_\varepsilon^\perp \bar{\mathrm{m}}_\varepsilon), \qquad \bar{\mathrm{v}}_\varepsilon = \nabla^\perp \bar{g}_\varepsilon, \qquad \triangle \bar{g}_\varepsilon = \bar{\mathrm{m}}_\varepsilon. \tag{8.6}
$$

As a consequence of [37, Lemmas 4.1(iii) and 4.2], we find $\|\bar{\mathrm{v}}_\varepsilon^t - \mathrm{v}^\circ\|_{\mathrm{L}^2} \lesssim_t 1$ and $\|\bar{\mathrm{m}}_\varepsilon^t\|_{\mathrm{L}^\infty} \lesssim_{t/\eta_\varepsilon} 1$. In order to obtain the effective equations satisfied by v_ε in the limit $\eta_\varepsilon \downarrow 0$, we use a formal 2-scale expansion (see [7] for a general presentation): we assume that v_ε satisfies the following natural 2-scale Ansatz,

$$
\begin{aligned}
\bar{\mathrm{v}}_\varepsilon^t(x) &= \bar{\mathrm{v}}_0(t, t/\eta_\varepsilon, x, x/\eta_\varepsilon) + \eta_\varepsilon \bar{\mathrm{v}}_1(t, t/\eta_\varepsilon, x, x/\eta_\varepsilon) + O(\eta_\varepsilon^2), \\
\bar{\mathrm{m}}_\varepsilon^t(x) &= \bar{\mathrm{m}}_0(t, t/\eta_\varepsilon, x, x/\eta_\varepsilon) + \eta_\varepsilon \bar{\mathrm{m}}_1(t, t/\eta_\varepsilon, x, x/\eta_\varepsilon) + O(\eta_\varepsilon^2), \\
\bar{g}_\varepsilon^t(x) &= \bar{g}_0(t, t/\eta_\varepsilon, x, x/\eta_\varepsilon) + \eta_\varepsilon \bar{g}_1(t, t/\eta_\varepsilon, x, x/\eta_\varepsilon) + \eta_\varepsilon^2 \bar{g}_2(t, t/\eta_\varepsilon, x, x/\eta_\varepsilon) + O(\eta_\varepsilon^3).
\end{aligned} \tag{8.7}
$$

We denote by (t, τ, x, y) the coordinates corresponding with $(t, t/\eta_\varepsilon, x, x/\eta_\varepsilon)$. Injecting the above ansatz into equation (8.6), and formally identifying the powers of η_ε, we derive the following equations,

$$
\begin{gathered}
\partial_\tau \bar{\mathrm{m}}_0 = \operatorname{div}_y(\Gamma^0[\bar{\mathrm{v}}_0]\bar{\mathrm{m}}_0), \\
\partial_t \bar{\mathrm{m}}_0 + \partial_\tau \bar{\mathrm{m}}_1 = \operatorname{div}_x(\Gamma^0[\bar{\mathrm{v}}_0]\bar{\mathrm{m}}_0) + \operatorname{div}_y(\Gamma^0[\bar{\mathrm{v}}_0]\bar{\mathrm{m}}_1) + \operatorname{div}_y(\Gamma^1[\bar{\mathrm{v}}_1]\bar{\mathrm{m}}_0), \\
\bar{\mathrm{v}}_0 = \nabla_x^\perp \bar{g}_0 + \nabla_y^\perp \bar{g}_1, \\
\nabla_y \bar{g}_0 = 0, \quad \triangle_y \bar{g}_1 = 0, \quad \triangle_x \bar{g}_0 + 2\nabla_x \cdot \nabla_y \bar{g}_1 + \triangle_y \bar{g}_2 = \bar{\mathrm{m}}_0,
\end{gathered} \tag{8.8}
$$

where for any vector field w we have defined for simplicity the following vector fields,

$$
\begin{aligned}
\Gamma^0[w] &:= (\alpha - \mathbb{J}\beta)(\nabla_2 \hat{h}^0 - \hat{F} + 2w^\perp), \\
\Gamma^1[w] &:= 2(\alpha - \mathbb{J}\beta)w^\perp.
\end{aligned}
$$

The first two equations in the last line of (8.8) imply that both $\bar{g}_0$ and $\bar{g}_1$ are independent of the y-variable. The third equation then ensures that $\bar{\mathrm{v}}_0 = \nabla_x^\perp \bar{g}_0$ is also independent of the y-variable. Averaging both the first and the last equations on the periodicity cell Q, and denoting for simplicity $\langle\cdot\rangle := \int_Q dy$ the averaging operator, we find

$$\partial_\tau\langle\bar{\mathrm{m}}_0\rangle = 0, \qquad \bar{\mathrm{v}}_0 = \nabla_x^\perp \bar{g}_0, \qquad \triangle_x \bar{g}_0 = \langle\bar{\mathrm{m}}_0\rangle,$$

which implies that $\langle\bar{\mathrm{m}}_0\rangle$ is independent of the τ-variable, hence the same holds for $\bar{g}_0$ and $\bar{\mathrm{v}}_0$. The 2-scale Ansatz (8.7) then takes the more precise form

$$\begin{aligned}\bar{\mathrm{v}}_\varepsilon^t(x) &= \bar{\mathrm{v}}_0^t(x) + \eta_\varepsilon \bar{\mathrm{v}}_1(t, t/\eta_\varepsilon, x, x/\eta_\varepsilon) + O(\eta_\varepsilon^2),\\ \bar{\mathrm{m}}_\varepsilon^t(x) &= \bar{\mathrm{m}}_0(t, t/\eta_\varepsilon, x, x/\eta_\varepsilon) + \eta_\varepsilon \bar{\mathrm{m}}_1(t, t/\eta_\varepsilon, x, x/\eta_\varepsilon) + O(\eta_\varepsilon^2).\end{aligned}$$

Further averaging the second equation in (8.8) on the periodicity cell Q, we obtain

$$\begin{aligned}\partial_\tau \bar{\mathrm{m}}_0 &= \mathrm{div}_y(\Gamma^0[\bar{\mathrm{v}}_0]\bar{\mathrm{m}}_0),\\ \partial_t\langle\bar{\mathrm{m}}_0\rangle + \partial_\tau\langle\bar{\mathrm{m}}_1\rangle &= \mathrm{div}_x(\langle\Gamma^0[\bar{\mathrm{v}}_0]\bar{\mathrm{m}}_0\rangle),\\ \bar{\mathrm{v}}_0 &= \nabla_x^\perp \triangle_x^{-1}\langle\bar{\mathrm{m}}_0\rangle.\end{aligned} \tag{8.9}$$

Let us now take a closer look at these equations (8.9). For any $x \in \mathbb{R}^2$ and $t \in \mathbb{R}^+$, consider the periodic flow $\phi_{x,t} : \mathbb{R}^+ \times Q \to Q$ associated with the periodic vector field $\Gamma^0[\bar{\mathrm{v}}_0^t](x,\cdot) : Q \to \mathbb{R}^2$,

$$\partial_\tau \phi_{x,t}^\tau(y) = -\Gamma^0[\bar{\mathrm{v}}_0^t](x, \phi_{x,t}^\tau(y)).$$

The first equation in (8.9) then yields

$$\bar{\mathrm{m}}_0(t,\tau,x,y) = \big((\phi_{x,t}^\tau)_* \bar{\mathrm{m}}_0(t,0,x,\cdot)\big)(y).$$

Now applying $s^{-1}\int_0^s d\tau$ to both sides of the second equation in (8.9), passing to the limit $s \uparrow \infty$, and recalling that $\langle\bar{\mathrm{m}}_0\rangle$ is independent of the τ-variable, we formally deduce

$$\partial_t\langle\bar{\mathrm{m}}_0\rangle(t,x) = \mathrm{div}_x \int_Q \Big(\lim_{s\uparrow\infty} s^{-1}\int_0^s \Gamma^0[\bar{\mathrm{v}}_0^t](x, \phi_{x,t}^\tau(y))d\tau\Big)\bar{\mathrm{m}}_0(t,0,x,y)dy. \tag{8.10}$$

By assumption, the periodic vector field $\Gamma^0[\bar{\mathrm{v}}_0^t](x,\cdot)$ admits a unique stable (normalized) invariant measure $\mu_x[\bar{\mathrm{v}}_0^t] \in \mathcal{P}(Q)$. By the ergodic theorem, for any $\psi \in C_{\mathrm{per}}(Q)$, we deduce for $\mu_x[\bar{\mathrm{v}}_0^t]$-almost all $y \in Q$,

$$\lim_{s\uparrow\infty} s^{-1}\int_0^s \psi(\phi_{x,t}^\tau(y))d\tau = \langle\psi\, \mu_x[\bar{\mathrm{v}}_0^t]\rangle.$$

In view of the unique stability assumption, it is most natural to admit that the above also holds for $\bar{\mathrm{m}}_0(t,0,x,\cdot)$-almost all $y \in Q$, in which case we find

$$\begin{aligned}\lim_{s\uparrow\infty}\int_Q \psi(y)\Big(s^{-1}\int_0^s \bar{\mathrm{m}}_0(t,\tau,x,y)d\tau\Big)dy &= \lim_{s\uparrow\infty}\int_Q \Big(s^{-1}\int_0^s \psi(\phi_{x,t}^\tau(y))d\tau\Big)\bar{\mathrm{m}}_0(t,0,x,y)dy\\ &= \langle\bar{\mathrm{m}}_0\rangle(t,x)\langle\psi\,\mu_x[\bar{\mathrm{v}}_0^t]\rangle,\end{aligned}$$

that is,

$$\lim_{s\uparrow\infty} s^{-1}\int_0^s \bar{\mathrm{m}}_0(t,\tau,x,y)d\tau = \langle\bar{\mathrm{m}}_0\rangle(t,x)\mu_x[\bar{\mathrm{v}}_0^t],$$

in the weak-* sense of measures. In particular, the limit in the right-hand side of (8.10) is explicitly computed,

$$\partial_t\langle\bar{\mathrm{m}}_0\rangle(t,x) = \mathrm{div}_x\big(\langle\Gamma^0[\bar{\mathrm{v}}_0^t](x,\cdot)\mu_x[\bar{\mathrm{v}}_0^t]\rangle\langle\bar{\mathrm{m}}_0\rangle(t,x)\big). \tag{8.11}$$

Combining this with the first and the last equations in (8.9), the heuristics follows. ☐

Remark 8.5 (Obstacles to a rigorous justification)**.** As described below, there are essentially three distinct weaknesses in the above formal justification of Heuristics 1.7.

(a) The first part of the justification consists in formally deriving the relations (8.9) for the 2-scale expansion of $\bar{\mathrm{v}}_\varepsilon$. This derivation is based on formally inserting the 2-scale Ansatz in the equation for $\bar{\mathrm{v}}_\varepsilon$ and identifying the powers of η_ε. However, due to both the nonlinear nonlocal character of the equation for $\bar{\mathrm{v}}_\varepsilon$ and its instability as $\eta_\varepsilon \downarrow 0$, a rigorous justification seems difficult to obtain, as we explain here.

   In order to justify formal 2-scale expansions, a powerful tool is given by Nguetseng's 2-scale weak compactness theorem [68, 4]. Since the equation for v_ε is nonlinear, this technique is of course not well suited, and since the nonlinearity is in addition nonlocal, E's technique of 2-scale Young measures [38] is also of no use.

   Another way to proceed (see e.g. [25, Section 3.1]) consists in approximating the solution $\bar{\mathrm{v}}_\varepsilon$ by the first terms of its formal 2-scale expansion (8.7): by definition this approximation satisfies the very same equation as $\bar{\mathrm{v}}_\varepsilon$ up to a small error, and this could be combined with a quantitative uniqueness principle to ensure that $\bar{\mathrm{v}}_\varepsilon$ remains close to its expansion. However, the linear part of the equation with fast oscillating forcing and the nonlinear interaction part are difficult to conciliate, and we do not know of any stability estimate which does not blow up in the homogenization limit. On the one hand, the L^1-contraction principle for the vorticity holds in the linear case but interacts badly with the nonlinearity. On the other hand, the nonlinear interaction part calls for energy-type estimates (that is, estimates on the L^2-distance between supercurrent densities), but the evolution of such metrics (as well as the 2-Wasserstein distance) is sensitive to the blowing Lipschitz norm of the oscillating forcing vector field. This issue is linked with the particularly strong instability of the equation upon perturbations as $\eta_\varepsilon \downarrow 0$.

(b) The last part of the justification consists in checking that the relations (8.9) imply the closed equation (8.11) for the averaged vorticity $\langle \mathrm{m}_0 \rangle$. If the (normalized) invariant measure $\mu_x[\bar{\mathrm{v}}^t]$ was truly unique for all x, t, then the given justification would be perfectly rigorous. Unfortunately, in the periodic setting, due to the gradient structure, this uniqueness (or *unique ergodicity*) is impossible, while the uniqueness assumption for a *stable* invariant measure seems more reasonable. The flaw in the above justification then lies in the assumption that unstable invariant measures do not play any role in the limit in (8.10), which is however not obvious and would require some argument.

(c) Finally, the well-posedness of the limiting equation (1.24) or (1.25) is unclear. The main difficulty is that the map $\mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d : (x, Z) \mapsto \Gamma_{\mathrm{hom}}[Z](x)$ is not even expected to be Lipschitz-continuous in Z: indeed, as explained in Remark 8.8 and Proposition 8.10, for fixed x, this map typically vanishes for Z in some bounded domain (pinning phenomenon), and is expected to have a power-law behavior with some power < 1 at the boundary of this domain (fractional depinning rate). Note that no comparison principle is expected to hold here (compare indeed with [85, Section 6.5]), so that a good notion of viscosity solutions seems unavailable.

Remark 8.6 (Toy model with vanishing viscosity)**.** For simplicity, we may consider the corresponding homogenization problems with a vanishing viscosity, that is, adding in the right-hand side of equation (3.13) or (3.14) for $\bar{\mathrm{v}}_\varepsilon$ a term $+D\eta_\varepsilon \triangle \bar{\mathrm{v}}_\varepsilon$, for some $D > 0$. A similar formal 2-scale expansion as above then yields the following modification of the relations (8.9), in the case of the regime (GL_1),

$$\begin{aligned} \partial_\tau \bar{\mathrm{m}}_0 &= D \triangle_y \bar{\mathrm{m}}_0 + \mathrm{div}_y(\Gamma^0[\bar{\mathrm{v}}_0]\bar{\mathrm{m}}_0), \\ \partial_t \langle \bar{\mathrm{m}}_0 \rangle + \partial_\tau \langle \bar{\mathrm{m}}_1 \rangle &= \mathrm{div}_x(\langle \Gamma^0[\bar{\mathrm{v}}_0]\bar{\mathrm{m}}_0 \rangle), \\ \bar{\mathrm{v}}_0 &= \nabla_x^\perp \triangle_x^{-1} \langle \bar{\mathrm{m}}_0 \rangle. \end{aligned} \tag{8.12}$$

From these relations the interpretation is now much easier: the first equation implies the (exponential) convergence of $\bar{\mathrm{m}}_0(t, \tau, x, \cdot)$ to $\langle \bar{\mathrm{m}}_0 \rangle (t, x) \tilde{\mu}_x^D[\bar{\mathrm{v}}_0^t]$ as $\tau \uparrow \infty$, where the viscous invariant measure $\tilde{\mu}_x^D[\bar{\mathrm{v}}_0^t] \in$

$\mathcal{P}_{\mathrm{per}}(Q)$ is the unique (smooth) solution to the following equation on the periodicity cell Q,

$$D\triangle_y \tilde{\mu}_x^D[\bar{\mathrm{v}}_0^t] + \mathrm{div}_y(\Gamma^0[\bar{\mathrm{v}}_0^t]\tilde{\mu}_x^D[\bar{\mathrm{v}}_0^t]) = 0.$$

The formal limiting equation then takes exactly the same form as in Heuristics 1.7, but with $\Gamma_{\mathrm{hom}}[w]$ replaced by its better-behaved viscous analogue,

$$\tilde{\Gamma}_{\mathrm{hom}}^D[w](x) := \int_Q \Gamma_x[w](y) d\tilde{\mu}_x^D[w](y).$$

In this case, the last two difficulties (b) and (c) pointed out in Remark 8.5 above disappear: the viscous invariant measure is easily checked to be always uniquely defined, and the corresponding limiting equation for $\bar{\mathrm{m}}$ is (locally) well-posed. Nevertheless, the difficulty (a) remains unchanged (that is, the rigorous derivation of the relations (8.12) for the 2-scale limit), and finding a rigorous proof still seems challenging.

8.5 Subcritical regimes

In the subcritical regimes (GL_1')–(GL_2'), the interaction of the vortices vanishes in the limit, and we are left with a much simpler *linear* transport equation with fast oscillating forcing for the vorticity $\bar{\mathrm{m}}_\varepsilon := \mathrm{curl}\, \bar{\mathrm{v}}_\varepsilon$ (obtained as the curl of equations (3.15)–(3.16) in the form of Corollary 8.4),

$$\partial_t \bar{\mathrm{m}}_\varepsilon = \mathrm{div}\, \big(\bar{\Gamma}_\varepsilon \bar{\mathrm{m}}_\varepsilon\big), \qquad \bar{\mathrm{m}}_\varepsilon|_{t=0} = \mathrm{curl}\, \mathrm{v}_\varepsilon^\circ,$$
$$\bar{\Gamma}_\varepsilon(x) := \bar{\Gamma}(x, x/\eta_\varepsilon), \qquad \bar{\Gamma} := (\alpha - \mathbb{J}\beta)\bar{\Gamma}_0 \qquad \bar{\Gamma}_0(x,y) := \nabla_2 \hat{h}^0(x,y) - \hat{F}(x).$$

With its fast oscillating gradient part, this linear transport equation is referred to as a *washboard* or *wiggly system*. Obviously the macroscopic dynamics strongly depends on microstructural events, for instance if some mass gets stuck in local minima: the typical mental picture is that of a particle sliding down a rough slope (like a washboard), thus taking a jerky path downwards, sometimes getting stuck along the way. Due to its gradient part, the corresponding vertical flow $\bar{\Gamma}(x,\cdot)$ on the periodicity cell Q cannot be uniquely ergodic, so that the problem of determining the asymptotic behavior of the solution $\bar{\mathrm{m}}_\varepsilon$ lies outside the classical theory of averaging. This problem was first studied in dimension 1 by [1], and later investigated in dimension 2 by Menon [61].

Menon's results [61] show that the space $\mathbb{R}^2$ splits into three regions associated with different dynamical properties: (1) an open set where the mass gets stuck (pinning region), (2) a transition region with a combination of sticking and slipping, and (3) the rest of the plane with only slipping. The slipping region is actually further split into countably many resonance zones where the limiting vector field has a constant direction given by the (rational) rotation number of the underlying microscopic cell flow, and the direction of the vector field varies continuously but not smoothly across the boundary of the resonance zones: given an initial position far from the pinning region, its path downwards is typically rough like a Cantor function. The dynamics is indeed particularly rich in dimensions $d \geq 2$: through the forcing $\hat{F}$, the macroscopic variable x acts as a bifurcation parameter for the topology of the underlying microscopic cell flow, and the bifurcations in the topology generate changes in the macroscopic motion between stick and slip, as well as between (rational) slipping directions. Note that Menon's results [61] are only partially justified, and are restricted to dimension $d = 2$ (due to some key topological arguments).

Simplified model. In order to exemplify the complexity of the structure of the limiting motion described above, let us consider (in general dimension d, say) the easier case of a constant forcing $F \in \mathbb{R}^d$ together with a wiggly potential $\tilde{h}^0$ that only depends on the microscopic variable; we thus consider the linear transport equation

$$\partial_t \tilde{\mathrm{m}}_\varepsilon = \mathrm{div}\, (\Gamma_\varepsilon^F \tilde{\mathrm{m}}_\varepsilon), \qquad \tilde{\mathrm{m}}_\varepsilon|_{t=0} = \tilde{\mathrm{m}}_\varepsilon^\circ, \tag{8.13}$$
$$\Gamma_\varepsilon^F(x) = \Gamma^F(x/\eta_\varepsilon), \qquad \Gamma^F(y) = (\alpha - \mathbb{J}\beta)(\nabla \tilde{h}^0(y) - F).$$

In this context, there is a true separation of scales in the limit $\eta_\varepsilon \downarrow 0$, and we may simply study the bifurcation of the limiting motion with respect to the *constant* forcing F. This system is a very particular case of the general nonlinear systems studied in [27] under additional well-preparedness conditions, but a more precise result is obtained here (see also [38] for the easier incompressible case, and [42, 26] for the corresponding Hamiltonian setting).

We first introduce some notation and make some regularity assumptions. The periodic vector field $-\Gamma^F$ on the unit cell $Q \subset \mathbb{R}^d$ defines a dynamical system on the d-torus Q. Assume that $\tilde{h}^0$ is smooth and non-degenerate, in the sense that for $F \neq 0$ this dynamical system admits a finite number of (normalized) ergodic invariant measures $(\mu_k^F)_{k=1}^{L_F} \subset \mathcal{P}(Q)$, $1 \le L_F < \infty$. For $F = 0$ we only assume that the dynamical system admits a finite number of (normalized) ergodic invariant measures on $\operatorname{int} Q$, while the boundary ∂Q is assumed to be made of unstable fixed points of the dynamics, thus yielding an infinite family $(\delta_p)_{p\in\partial Q}$ of ergodic measures on this boundary. (This assumption is motivated by the typical choice $\tilde{h}_0 \le 0$, $(\tilde{h}_0)^{-1}(\{0\}) = \partial Q$; cf. the explicit example in Figure 4.) For all $1 \le k \le L_F$ we define the minimal invariant sets $A_k^F := \operatorname{supp} \mu_k^F$, and we let B_k^F denote the set of μ_k^F-generic points. We order the ergodic measures in such a way that $|B_k^F| > 0$ holds for all $1 \le k \le K_F$, and $|B_k^F| = 0$ for all $K_F + 1 \le k \le L_F$, with $1 \le K_F \le L_F$. By construction,

$$\Big| Q \setminus \biguplus_{k=1}^{K_F} B_k \Big| = 0.$$

Note that in dimension $d = 2$ the dynamical picture is particularly simple, as Denjoy's version of the Poincaré-Bendixson on the 2-torus [30] (see also [81]) asserts that minimal invariant sets are either fixed points, periodic orbits, or the whole torus.

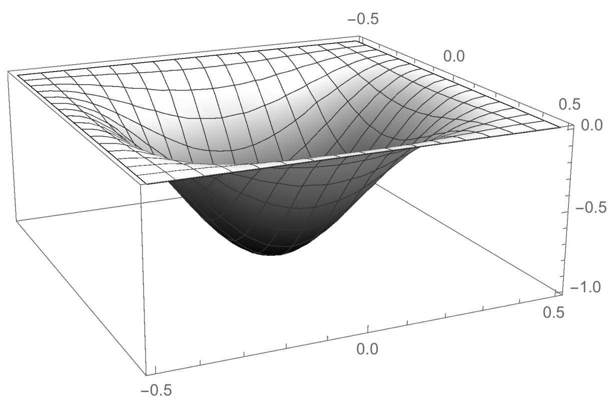


Figure 4 – In dimension 2, a typical choice for the pinning potential is e.g. $\tilde{h}^0(x) := -\cos(\pi x_1)^2 \cos(\pi x_2)^2$ for $x \in Q = [-\frac{1}{2}, \frac{1}{2})^2$.

The limiting behavior of the solution $\tilde{\mathrm{m}}_\varepsilon$ of (8.13) is then characterized as follows; note that the result is much simpler in the case $K_F = 1$, that is, if there exists a unique stable (normalized) invariant measure.

Theorem 8.7. *Let the above notation and assumptions hold, and let* $\tilde{\mathrm{m}}_\varepsilon^\circ \in \mathcal{P} \cap \mathrm{L}^\infty(\mathbb{R}^d)$ *satisfy*

$$\tilde{\mathrm{m}}_\varepsilon^\circ(x) - \omega^\circ(x, x/\eta_\varepsilon) \to 0, \tag{8.14}$$

strongly in $\mathrm{L}^1(\mathbb{R}^d)$ *as* $\varepsilon \downarrow 0$*, for some* $\omega^\circ \in \mathrm{L}^1(\mathbb{R}^d; C_{\mathrm{per}}(Q))$*. Let* $F \in \mathbb{R}^d$*, and denote by* $\tilde{\mathrm{m}}_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{P}(\mathbb{R}^d))$

the unique solution to the transport equation (8.13) with initial data $\tilde{m}_\varepsilon^\circ$. Then we have for all $t \geq 0$,

$$\tilde{m}_\varepsilon^t \overset{*}{\rightharpoonup} \tilde{m}^t := \sum_{k=1}^{K_F} \tilde{m}_k^t,$$

where for all k we denote by $\tilde{m}_k \in L^\infty_{loc}(\mathbb{R}^+; \mathcal{P}(\mathbb{R}^d))$ the unique solution of the (constant-coefficient) transport equation

$$\partial_t \tilde{m}_k = \operatorname{div}(\Gamma_k^F \tilde{m}_k), \qquad \Gamma_k^F := \int_Q \Gamma^F(y) d\mu_k^F(y), \qquad \tilde{m}_k|_{t=0} = \tilde{m}_k^\circ := \int_{B_k^F} \omega^\circ(\cdot, y) dy.$$

In particular, if the stable invariant sets of the dynamical system generated by $-\Gamma^F$ are all reduced to a point (that is, if A_k^F is a point for all $1 \leq k \leq K_F$), then we have $\tilde{m}_\varepsilon^t \overset{*}{\rightharpoonup} \tilde{m}^\circ := \int_Q \omega^\circ(\cdot, y) dy = \sum_{k=1}^K \tilde{m}_k^\circ$ for all $t \geq 0$.

Remarks 8.8.

(a) *Stick-slip motion.* In this remark, we consider the behavior of the limit $\tilde{m}$ as a function of the forcing F, and we argue that the space $\mathbb{R}^d$ of values of F splits into three regions: (1) an open bounded set around 0 for which the limiting solution is stuck $\tilde{m} = \tilde{m}^\circ$ (pinning phenomenon), (2) a transition region for which a part of the mass is stuck and another part is transported, and (3) the rest of $\mathbb{R}^d$ for which there is only transport (with possibly a superposition of different effective velocities). The link with Menon's results [61] is thus clear. A natural question consists in studying the precise behavior of the effective velocity as a function of F beyond the pinning region. The behavior at the depinning threshold, that is, for forcing F just across the boundary of the pinning region, is shortly addressed in the sequel of this section (see Proposition 8.10 below). On the other hand, for very large $|F| \gg 1$, the deviation of the effective velocity due to the wiggly potential $\tilde{h}^0$ naturally tends to 0,

$$-\Gamma_k^F = (\alpha - \mathbb{J}\beta) F - (\alpha - \mathbb{J}\beta) \int_Q \nabla \tilde{h}^0 d\mu_k = (\alpha - \mathbb{J}\beta) F(1 + o(1)).$$

We first consider the case $F = 0$, hence $-\Gamma^0 = -\alpha \nabla \tilde{h}^0 + \beta \nabla^\perp \tilde{h}^0$. For energy reasons, we note that the only invariant sets are then necessarily made of unions of fixed points of the dynamics. The last part of Theorem 8.7 then allows to conclude that the limiting solution $\tilde{m}$ is constant in time. Now for F close enough to 0, the stable invariant sets of $-\Gamma^F$ are still made of stable fixed points, which are simply deformations of the stable fixed points of $-\Gamma^0$, and we conclude that the limiting solution $\tilde{m}$ still remains constant. In contrast, for larger values of F, the topological nature of the stable invariant sets may change, yielding a possible combination of both stable fixed points and other types of stable sets, hence by Theorem 8.7 a combination of pinning and transport. Finally, for $|F| > \|\nabla \tilde{h}^0\|_{L^\infty}$, we note that the map $-\Gamma^F$ no longer has a fixed point (since the condition on F implies $|\Gamma^F|^2 = (\alpha^2 + \beta^2)|\nabla \tilde{h}^0 - F|^2 > 0$), so that Theorem 8.7 yields pure transport in that case.

(b) *Initial-boundary layer.* While the initial data $\tilde{m}_\varepsilon^\circ$ may have some microscopic heterogeneities, which are assumed to be given by $\omega^\circ(\cdot, \cdot/\eta_\varepsilon)$, it is instantaneously relaxed to an invariant measure $\sum_{k=1}^{K_F} \mu_k(\cdot/\eta_\varepsilon) \tilde{m}_k^\circ$ in a timescale of order $O(\eta_\varepsilon)$. This initial-boundary layer at the microscopic scale could be described in similar terms as in Proposition 8.2.

We now turn to the proof of Theorem 8.7. It is obtained from a suitable version of 2-scale convergence methods. More precisely, we use the following L^1-version of Nguetseng's 2-scale compactness theorem [68, 4]; as it is not standard in this form, we include a short proof.

Lemma 8.9 (à la Nguetseng)**.** *Let $(g_\eta)_\eta$ be a bounded sequence in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^1(\mathbb{R}^d))$. Further assume that it is tight, in the sense that for all $T>0$,*

$$\lim_{L\uparrow\infty}\sup_{t\in[0,T]}\sup_{\eta>0}\int_{|x|>L}|g_\eta^t|=0. \tag{8.15}$$

Then, there exists a subsequence, still denoted by $(g_\eta)_\eta$, and an element $g_0\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{M}(\mathbb{R}^d;\mathcal{M}_{\mathrm{per}}(Q)))$ (where $\mathcal{M}$ (resp. $\mathcal{M}_{\mathrm{per}}$) denotes the space of Radon measures (resp. periodic Radon measures)), such that we have for all $T>0$ and all $\psi\in\mathrm{L}^1([0,T];C_b(\mathbb{R}^d;C_{\mathrm{per}}(Q)))$,

$$\lim_{\eta\downarrow 0}\int_0^T\int\psi^t(x,x/\eta)g_\eta^t(x)dxdt=\int_0^T\iint_{\mathbb{R}^d\times Q}\psi^t(x,y)dg_0^t(x,y)dt, \tag{8.16}$$

We say that g_η 2-scale converges weakly- to g_0. Moreover, if $\psi_\eta\to\psi$ holds strongly in $\mathrm{L}^1([0,T];C_b(\mathbb{R}^d;C_{\mathrm{per}}(Q)))$, then we find*

$$\lim_{\eta\downarrow 0}\int_0^T\int\psi_\eta^t(x,x/\eta)g_\eta^t(x)dxdt=\int_0^T\iint_{\mathbb{R}^d\times Q}\psi^t(x,y)dg_0^t(x,y)dt.$$

Proof. Let $T>0$ be fixed. The boundedness assumption on g_η gives $\sup_\eta\|g_\eta\|_{\mathrm{L}^\infty([0,T];\mathrm{L}^1(\mathbb{R}^d))}\le C_T$, so that we find for all $\psi\in\mathrm{L}^1([0,T];C_0(\mathbb{R}^d;C_{\mathrm{per}}(Q)))$,

$$\left|\int_0^T\int\psi^t(x,x/\eta)g_\eta^t(x)dxdt\right|\le C_T\|\psi\|_{\mathrm{L}^1([0,T];C_0(\mathbb{R}^d;C_{\mathrm{per}}(Q)))}.$$

The sequence $(g_\eta)_\eta$ may thus be seen as a bounded sequence of elements in the dual of the Banach space $\mathrm{L}^1([0,T];C_0(\mathbb{R}^d;C_{\mathrm{per}}(Q)))$, that is, a bounded sequence in the space $\mathrm{L}^\infty([0,T];\mathcal{M}(\mathbb{R}^d;\mathcal{M}_{\mathrm{per}}(Q)))$. Combining this with the additional tightness assumption (8.15), we deduce that there is a subsequence, still denoted by $(g_\eta)_\eta$, and an element $g_0\in\mathrm{L}^\infty([0,T];\mathcal{M}(\mathbb{R}^d;\mathcal{M}_{\mathrm{per}}(Q)))$ such that g_η converges weakly-* to g_0 in that space, in the sense of (8.16). □

With this compactness result at hand, we now sketch a proof of Theorem 8.7.

Sketch of the proof of Theorem 8.7. Let F be fixed, and write for simplicity $A_k:=A_k^F$, $B_k:=B_k^F$, and $\mu_k:=\mu_k^F$. We split the proof into four steps.

Step 1: 2-scale compactness argument. In this step, we show that up to a subsequence the solution $\tilde{\mathrm{m}}_\varepsilon$ of (8.13) 2-scale converges weakly-* (in the sense of Lemma 8.9) to some limit $\tilde{\mathrm{m}}_0\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{M}^+(\mathbb{R}^d;\mathcal{M}^+_{\mathrm{per}}(Q)))$. Moreover, denoting for simplicity by $\langle\cdot\rangle:=\int_Q dy$ the averaging operator, the limit $\tilde{\mathrm{m}}_0$ satisfies the following equations:

$$-\operatorname{div}_y(\Gamma^F\tilde{\mathrm{m}}_0)=0, \tag{8.17}$$

$$\partial_t\langle\tilde{\mathrm{m}}_0\rangle=\operatorname{div}_x\langle\Gamma^F\tilde{\mathrm{m}}_0\rangle,\qquad\langle\tilde{\mathrm{m}}_0\rangle|_{t=0}=\langle\omega^\circ\rangle=\tilde{\mathrm{m}}^\circ. \tag{8.18}$$

Equation (8.17) means that $\tilde{\mathrm{m}}_0^t(x,\cdot)$ is an invariant measure for the vector field $-\Gamma^F$ on Q for almost all t,x. For $F\neq 0$, by assumption, we may then decompose $\tilde{\mathrm{m}}_0$ as a linear combination

$$\tilde{\mathrm{m}}_0^t(x,y)=\sum_{k=1}^{L_F}\xi_k^t(x)\mu_k(y). \tag{8.19}$$

For $F=0$, by assumption, a similar decomposition holds in $\operatorname{int} Q$: there exists $\hat{\mathrm{m}}_0 \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{M}(\mathbb{R}^d;\mathcal{M}_{\mathrm{per}}(Q)))$ such that for all t,x the measure $\hat{\mathrm{m}}_0^t(x,\cdot)$ is supported in ∂Q, and such that

$$\tilde{\mathrm{m}}_0^t(x,y) = \hat{\mathrm{m}}_0^t(x,y) + \sum_{k=1}^{L_0} \xi_k^t(x)\mu_k(y).$$

Since $\tilde{\mathrm{m}}_\varepsilon$ is nonnegative and has constant mass 1, it is bounded in $\mathrm{L}^\infty(\mathbb{R}^+;\mathrm{L}^1(\mathbb{R}^d))$. Moreover, as the velocity field Γ_ε^F is bounded in $\mathrm{L}^\infty(\mathbb{R}^d)^d$, the tightness of the initial data $(\tilde{\mathrm{m}}_\varepsilon^\circ)_\varepsilon$ easily implies the tightness of the solutions $(\tilde{\mathrm{m}}_\varepsilon)_\varepsilon$ in the sense of (8.15). Therefore, by Lemma 8.9, up to a subsequence, $\tilde{\mathrm{m}}_\varepsilon$ 2-scale converges weakly-* to some $\tilde{\mathrm{m}}_0 \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{M}^+(\mathbb{R}^d;\mathcal{M}^+_{\mathrm{per}}(Q)))$. We now prove that this limit satisfies equations (8.17)–(8.18). Testing the equation for $\tilde{\mathrm{m}}_\varepsilon$ against a test function $\psi^t(x,x/\eta_\varepsilon)$, with $\psi \in C^1_c(\mathbb{R}^+\times\mathbb{R}^d;C^1_{\mathrm{per}}(Q))$, we find

$$\begin{aligned}\int_{\mathbb{R}^+}\int_{\mathbb{R}^d} \partial_t\psi^t(x,x/\eta_\varepsilon)d\tilde{\mathrm{m}}_\varepsilon^t(x)dt + \int_{\mathbb{R}^d}\psi^0(x,x/\eta_\varepsilon)d\tilde{\mathrm{m}}_\varepsilon^\circ(x)\\ = \int_{\mathbb{R}^+}\int_{\mathbb{R}^d}(\eta_\varepsilon^{-1}\nabla_y\psi^t(x,x/\eta_\varepsilon)+\nabla_x\psi^t(x,x/\eta_\varepsilon))\cdot\Gamma^F(x/\eta_\varepsilon)d\tilde{\mathrm{m}}_\varepsilon^t(x)dt.\end{aligned}$$

Choosing $\psi^t(x,y) := \eta_\varepsilon\phi^t(x,y)$ with $\phi\in C^1_c(\mathbb{R}^+\times\mathbb{R}^d;C^1_{\mathrm{per}}(Q))$, and letting $\varepsilon\downarrow 0$ (along the subsequence), we find

$$\int_{\mathbb{R}^+}\iint_{\mathbb{R}^d\times Q}\nabla_y\phi^t(x,y)\cdot\Gamma^F(y)d\tilde{\mathrm{m}}_0^t(x,y)dt = 0,$$

that is (8.17). Now choosing $\psi^t(x,y) := \phi^t(x)$ with $\phi\in C^1_c(\mathbb{R}^+\times\mathbb{R}^d)$, letting $\varepsilon\downarrow 0$ (along the subsequence), and using assumption (8.14), we obtain

$$\int_{\mathbb{R}^+}\iint_{\mathbb{R}^d\times Q}\partial_t\phi^t(x)d\tilde{\mathrm{m}}_0^t(x,y)dt + \iint_{\mathbb{R}^d\times Q}\phi^0(x)d\omega^\circ(x,y) = \int_{\mathbb{R}^+}\iint_{\mathbb{R}^d\times Q}\nabla\phi(t,x)\cdot\Gamma^F(y)d\tilde{\mathrm{m}}_0^t(x,y)dt,$$

that is (8.18).

Step 2: localization. Let $1\le k\le K_F$ be fixed. Denote by B_k' the 1-periodic extension of B_k on $\mathbb{R}^d$. In this step, we show that, if $\tilde{\mathrm{m}}_\varepsilon^\circ(\mathbb{R}^d\setminus\eta_\varepsilon B_k')=0$ for all ε, then $\xi_j^t(x)=0$ holds for all $j\neq k$ for almost all t,x. In particular, this implies $\tilde{\mathrm{m}}_0^t(x,y)=\xi_k^t(x)\mu_k(y)$ almost everywhere.

Given the smoothness assumptions, viewing B_k as the attraction basin associated with A_k, it follows that we must have $n\cdot\Gamma^F=0$ on the boundary ∂B_k. Note that the method of propagation along characteristics together with the Liouville-Ostrogradski formula yields the following estimate for the solution $\tilde{\mathrm{m}}_\varepsilon$ of (8.13),

$$\|\tilde{\mathrm{m}}_\varepsilon^t\|_{\mathrm{L}^\infty}\le\|\tilde{\mathrm{m}}_\varepsilon^\circ\|_{\mathrm{L}^\infty}\exp(t\|\mathrm{div}\ \Gamma_\varepsilon^F\|_{\mathrm{L}^\infty})\le\|\tilde{\mathrm{m}}_\varepsilon^\circ\|_{\mathrm{L}^\infty}\exp(\alpha\eta_\varepsilon^{-1}t\|\triangle\tilde{h}^0\|_{\mathrm{L}^\infty}),$$

hence $\tilde{\mathrm{m}}_\varepsilon\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^\infty(\mathbb{R}^d))$ (although of course no ε-uniform bound holds in that space). We may then deduce by integration by parts, for all $t\ge 0$,

$$\partial_t\int_{\eta_\varepsilon B_k'}d\tilde{\mathrm{m}}_\varepsilon^t = \int_{\eta_\varepsilon\partial B_k'} n\cdot\Gamma_\varepsilon^F(x)\tilde{\mathrm{m}}_\varepsilon^t(x)d\sigma(x)=0,$$

that is, $\tilde{\mathrm{m}}_\varepsilon^t(\eta_\varepsilon B_k')=\tilde{\mathrm{m}}_\varepsilon^\circ(\eta_\varepsilon B_k')=1$, and the conclusion follows from the decomposition (8.19).

Step 3: convergence of partitioned initial data. Decompose $\tilde{\mathrm{m}}_\varepsilon^\circ=\sum_{k=1}^{K_F}\tilde{\mathrm{m}}_{\varepsilon,k}^\circ$ with $\tilde{\mathrm{m}}_{\varepsilon,k}^\circ := \tilde{\mathrm{m}}_\varepsilon^\circ\mathbb{1}_{\eta_\varepsilon B_k'}$. In this step, for all k, we show that $\tilde{\mathrm{m}}_{\varepsilon,k}^\circ$ converges weakly in $\mathrm{L}^1(\mathbb{R}^2)$ to $\tilde{\mathrm{m}}_k^\circ := \int_{B_k}\omega^\circ(\cdot,y)dy$.

For any test function $\phi \in \mathrm{L}^\infty(\mathbb{R}^2)$, assumption (8.14) yields

$$\limsup_{k\uparrow\infty} \Big| \int \phi d\tilde{\mathrm{m}}^\circ_{\varepsilon,k} - \int_{\eta_\varepsilon B'_k} \phi(x)\omega^\circ(x,x/\eta_\varepsilon)dx \Big| \le \limsup_{k\uparrow\infty} \int |\phi(x)|\,|\tilde{\mathrm{m}}^\circ_\varepsilon(x) - \omega^\circ(x,x/\eta_\varepsilon)|dx = 0,$$

while by periodicity we may compute (see e.g. [4, proof of Lemma 5.2])

$$\lim_{\varepsilon\downarrow 0} \int \phi(x)\omega^\circ(x,x/\eta_\varepsilon)\mathbb{1}_{x/\eta_\varepsilon \in B'_k} dx = \iint_{\mathbb{R}^d\times B_k} \phi(x)\omega^\circ(x,y)dxdy = \int \phi d\tilde{\mathrm{m}}^\circ_k,$$

and the result follows.

Step 4: conclusion. By linearity, with the choice of the $\tilde{\mathrm{m}}^\circ_{\varepsilon,k}$'s in Step 3, we may decompose $\tilde{\mathrm{m}}_\varepsilon = \sum_{k=1}^{K_F} \tilde{\mathrm{m}}_{\varepsilon,k}$, where for all k the function $\tilde{\mathrm{m}}_{\varepsilon,k} \in \mathrm{L}^\infty(\mathbb{R}^+;\mathcal{M}^+(\mathbb{R}^d))$ is the unique solution of the following equation,

$$\partial_t \tilde{\mathrm{m}}_{\varepsilon,k} = \operatorname{div}(\Gamma^F_\varepsilon \tilde{\mathrm{m}}_{\varepsilon,k}), \qquad \tilde{\mathrm{m}}_{\varepsilon,k}|_{t=0} = \tilde{\mathrm{m}}^\circ_{\varepsilon,k}.$$

Up to a subsequence, for all k, we know by Step 1 that $\tilde{\mathrm{m}}_{\varepsilon,k}$ 2-scale converges weakly-* to some $\tilde{\mathrm{m}}_{0,k} \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{M}^+(\mathbb{R}^2;\mathcal{M}^+_{\mathrm{per}}(Q)))$, which satisfies

$$\begin{aligned} -\operatorname{div}_y(\Gamma^F \tilde{\mathrm{m}}_{0,k}) &= 0, \\ \partial_t \langle \tilde{\mathrm{m}}_{0,k}\rangle = \operatorname{div}_x \langle \Gamma^F \tilde{\mathrm{m}}_{0,k}\rangle&, \qquad \langle \tilde{\mathrm{m}}_{0,k}\rangle|_{t=0} = \tilde{\mathrm{m}}^\circ_k, \end{aligned}$$

where the first equation implies for $\tilde{\mathrm{m}}_{0,k}$ a similar decomposition (8.19) as in Step 1. By Step 2, since we have by construction $\tilde{\mathrm{m}}^\circ_{\varepsilon,k}(\mathbb{R}^2 \setminus \eta_\varepsilon B'_k) = 0$ for all ε, we deduce $\tilde{\mathrm{m}}^t_{0,k}(x,y) = \langle \tilde{\mathrm{m}}^t_{0,k}(x,\cdot)\rangle \mu_k(y)$. Inserting this form into the above equations, we find

$$\partial_t \langle \tilde{\mathrm{m}}_{0,k}\rangle = \operatorname{div}(\Gamma^F_k \langle \tilde{\mathrm{m}}_{0,k}\rangle), \qquad \Gamma^F_k := \langle \Gamma^F \mu_k\rangle, \qquad \langle \tilde{\mathrm{m}}_{0,k}\rangle|_{t=0} = \tilde{\mathrm{m}}^\circ_k.$$

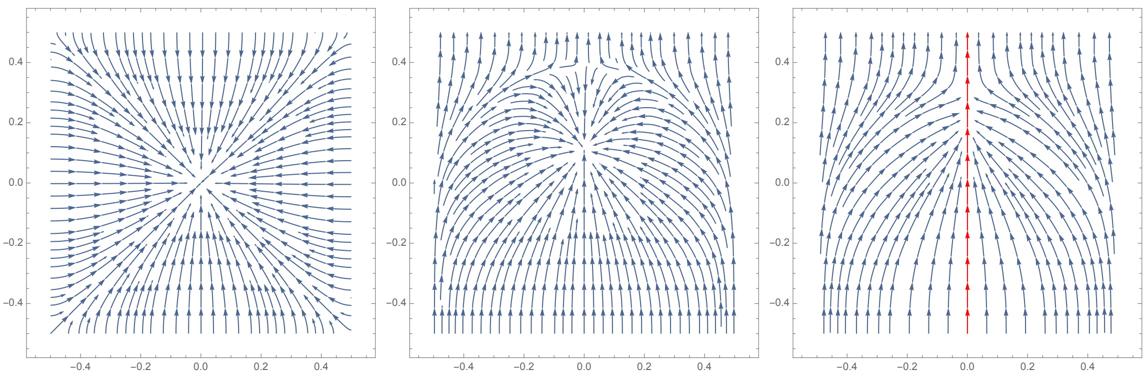
This is now a linear transport equation for $\langle \tilde{\mathrm{m}}_{0,k}\rangle$. Uniqueness allows us to get rid of all extractions of subsequences, and the conclusion follows, since by linearity we necessarily have $\tilde{\mathrm{m}}_0 = \sum_{k=1}^{K_F} \tilde{\mathrm{m}}_{0,k}$, where $\tilde{\mathrm{m}}_0$ is the weak limit extracted in Step 1. ☐

As noticed in Remark 8.8(a), the question of determining the depinning rate at the depinning threshold is of particular interest. While obtaining a complete answer seems difficult due to the variety of possible dynamical behaviors, we consider the simplest situation when the depinning is due to the bifurcation of a unique stable fixed point into a stable periodic orbit. A square-root power law is then obtained under a non-degeneracy condition. An additional assumption is made for simplicity, which reduces the computation to a 1D setting (being then comparable to some explicit computations in [1, 50]). This assumption is satisfied for $\beta = 0$ and for a forcing F that is parallel to a coordinate axis when the pinning potential h has similar symmetries as in the example of Figure 4 (see indeed Figure 5). Yet, we believe that the same result holds in more general situations.

Proposition 8.10. *Let e, $|e| = 1$, be some direction, and consider the system (8.13) with $F = \kappa e$. Assume that the vector field $-\Gamma^{\kappa e}$ has a unique stable invariant set for all $\kappa \ge 0$, and assume that there exists a critical value $\kappa_c > 0$ such that this invariant set is a fixed point for $0 \le \kappa < \kappa_c$, and is a periodic orbit for $\kappa > \kappa_c$. Further assume that the image of the periodic orbit $\mathcal{O} \subset Q$ remains the same for all $\kappa > \kappa_c$. Assume that $\tilde{h}^0$ is smooth, and is non-degenerate in the following sense: for all x and all $|v| = 1$, if $v\cdot\nabla(\alpha\nabla - \beta\nabla^\perp)\tilde{h}^0(x) = 0$ holds, then $(v\cdot\nabla)^2(\alpha\nabla - \beta\nabla^\perp)\tilde{h}^0(x) \neq 0$. Then, the effective velocity $\Gamma^{\kappa e}_1$ defined in Theorem 8.7 satisfies, as $\kappa \downarrow \kappa_c$,*

$$\Gamma^{\kappa e}_1 = C(1 + o(1))(\kappa - \kappa_c)^{1/2} e,$$

for some constant $C > 0$ depending on the shape of $\tilde{h}^0$.



(a) $\kappa = 0$ (b) $0 < \kappa < \kappa_c = \pi$ (c) $\kappa > \kappa_c = \pi$

Figure 5 – In dimension $d = 2$, for the typical example of pinning potential $\tilde{h}^0$ given in Figure 4, with $\alpha = 1$, $\beta = 0$, we plot the stream lines of the vector field $-\Gamma^{(0,\kappa)}$ for growing values of κ. The assumptions of Proposition 8.10 are clearly seen to be satisfied: for $\kappa < \kappa_c = \pi$ there is a unique stable fixed point, while for $\kappa > \kappa_c = \pi$ the stable fixed point gives way to a periodic orbit with image $\mathcal{O} = \{0\} \times [-1/2, 1/2)$.

Remarks 8.11.

(a) While Proposition 8.10 above is proved in the particularly simple situation of the bifurcation of a fixed point into a periodic orbit, it would be interesting to determine the best general lower bound on the Hölder regularity of the multivalued map $F \mapsto \{\Gamma_1^F, \dots, \Gamma_{K_F}^F\}$ at the depinning threshold, for smooth $\tilde{h}^0$. We do not pursue this question here, but note that at least the continuity of this map essentially follows from the argument in [61, Section 7.2] together with the result on circle maps in [69, Theorem I.1].

(b) Without the non-degeneracy assumption for the pinning potential $\tilde{h}^0$, the behavior can be very different: if $\tilde{h}^0$ is degenerate at order k for some $0 \le k \le \infty$, in the sense that the power 2 in the expansion (8.21) near critical points is replaced by a power $k + 2$, then we indeed rather obtain $\Gamma_1^{\kappa e} \sim (\kappa - \kappa_c)^{1-1/(k+2)} e$ as $\kappa \downarrow \kappa_c$. (Although in this case the effective velocity $\Gamma_1^{\kappa e}$ is still a Hölder function of κ, and is at least of class $C^{1/2}$, examples of non-smooth pinning potentials $\tilde{h}^0 \in C^{0,1}(\mathbb{R}^d)$ can be constructed for which the Hölder property fails at $\kappa = \kappa_c$; see e.g. [50, Example 1.3].)

Proof. Choose an arc-length parametrization $(\phi^t)_{0 \le t \le T}$ of the periodic orbit $\mathcal{O}$, where $|\partial_t \phi^t| = 1$ for all $t \ge 0$, and where the period $T \in \mathbb{R}^+$ is the total length of the orbit. Since $\mathcal{O}$ is the image of the (unique stable) periodic orbit of $-\Gamma^{\kappa e}$ for all $\kappa > \kappa_c$, we find $\partial_t \phi^t = -\Gamma^{\kappa e}(\phi^t)/|\Gamma^{\kappa e}(\phi^t)|$ for all $t \ge 0$. We then deduce that for all $\kappa > \kappa_c$ the unique stable ergodic invariant measure $\mu_\kappa \in \mathcal{P}_{\mathrm{per}}(Q)$ takes the form

$$\int f d\mu_\kappa = \Big(\int_0^T f(\phi^t) |\Gamma^{\kappa e}(\phi^t)|^{-1} dt \Big) \Big(\int_0^T |\Gamma^{\kappa e}(\phi^t)|^{-1} dt \Big)^{-1},$$

so that the effective velocity is given by

$$\Gamma_1^F = \Big(\int_0^T \Gamma^{\kappa e}(\phi^t) |\Gamma^{\kappa e}(\phi^t)|^{-1} dt \Big) \Big(\int_0^T |\Gamma^{\kappa e}(\phi^t)|^{-1} dt \Big)^{-1} = (\phi^0 - \phi^T) \Big(\int_0^T |\Gamma^{\kappa e}(\phi^t)|^{-1} dt \Big)^{-1}.$$

Now setting $\tilde{e} := \phi^T - \phi^0$, we obtain

$$-\Gamma_1^F = \Big(\int_0^T |\Gamma^{\kappa e}(\phi^t)|^{-1} dt \Big)^{-1} \tilde{e}.$$

Consider the finite collection $(t_j)_{j=1}^J$ of all points $t \in [0,T]$ such that $\Gamma^{\kappa_c e}(\phi^t) = 0$. By smoothness of $\tilde{h}^0$ and by the minimality assumption defining κ_c, the function $f(t) := |\Gamma^{\kappa_c e}(\phi^t)|$ is smooth, hence satisfies for all j,

$$f'(t_j) = 0, \qquad f''(t_j) \geq 0, \tag{8.20}$$

and also

$$0 = \partial_t \Gamma^{\kappa_c e}(\phi^t)|_{t=t_j} = \partial_t \phi^{t_j} \cdot \nabla(\alpha\nabla - \beta\nabla^\perp)\tilde{h}^0(\phi^{t_j}).$$

A direct computation then yields

$$\begin{aligned} f''(t_j) = {} & |\Gamma^{\kappa_c e}(\phi^{t_j})|^{-1} |\partial_t \phi^{t_j} \cdot \nabla(\alpha\nabla - \beta\nabla^\perp)\tilde{h}^0(\phi^{t_j})|^2 - 2|\Gamma^{\kappa_c e}(\phi^{t_j})|^{-1} |\partial_t \phi^{t_j} \cdot \nabla(\alpha\nabla - \beta\nabla^\perp)\tilde{h}^0(\phi^{t_j}) \cdot \partial_t \phi^{t_j}|^2 \\ & + |\Gamma^{\kappa_c e}(\phi^{t_j})|^{-1} \partial_t \phi^{t_j} \cdot \nabla(\alpha\nabla - \beta\nabla^\perp)\tilde{h}^0(\phi^{t_j}) \cdot \nabla(\alpha\nabla - \beta\nabla^\perp)\tilde{h}^0(\phi^{t_j}) \cdot \partial_t \phi^{t_j} \\ & + (\partial_t \phi^{t_j})^{\otimes 3} \odot \nabla^2(\alpha\nabla - \beta\nabla^\perp)\tilde{h}^0(\phi^{t_j}) \\ = {} & (\partial_t \phi^{t_j})^{\otimes 3} \odot \nabla^2(\alpha\nabla - \beta\nabla^\perp)\tilde{h}^0(\phi^{t_j}), \end{aligned}$$

where $\odot$ denotes the complete contraction of 3-tensors. The non-degeneracy assumption now implies $f''(t_j) \neq 0$. Combined with (8.20), this yields

$$2C_j := f''(t_j) = (\partial_t \phi^{t_j})^{\otimes 3} \odot \nabla^2(\alpha\nabla - \beta\nabla^\perp)\tilde{h}^0(\phi^{t_j}) > 0.$$

A Taylor expansion around t_j allows to write

$$|\Gamma^{F_c}(\phi^t_{F_c})| = C_j(t-t_j)^2 + O((t-t_j)^3), \tag{8.21}$$

for $|t - t_j| \ll 1$. Let $\delta > 0$ be small enough such that $|t_j - t_{j+1}| > 2\delta$ is satisfied for all j, and define

$$I_\delta := \biguplus_{j=1}^J [t_j - \delta, t_j + \delta], \qquad c_\delta := \inf_{t \in [0,T] \setminus I_\delta} |\Gamma^{\kappa_c e}(\phi^t)| > 0.$$

For $\kappa > \kappa_c$ sufficiently close to κ_c, we may then compute

$$\int_{[0,T]\setminus I_\delta} |\Gamma^{\kappa e}(\phi^t)|^{-1} dt = (\alpha^2 + \beta^2)^{-1/2} \int_{[0,T]\setminus I_\delta} |\nabla \tilde{h}^0(\phi^t) - \kappa e|^{-1} dt \leq (\alpha^2+\beta^2)^{-1/2} T (c_\delta - |\kappa - \kappa_c|)^{-1},$$

and hence, setting for simplicity $e_{\alpha,\beta} := \alpha e - \beta e^\perp$,

$$\begin{aligned} \int_0^T |\Gamma^{\kappa e}(\phi^t)|^{-1} dt & \leq \int_{I_\delta} |\Gamma^{\kappa_c e}(\phi^t) - (\kappa - \kappa_c) e_{\alpha,\beta}|^{-1} dt + T(\alpha^2+\beta^2)^{-1/2}(c_\delta - |\kappa - \kappa_c|)^{-1} \\ & \leq \sum_{j=1}^J \int_{t_j - \delta}^{t_j+\delta} \left(|C_j (t-t_j)^2 \partial_t \phi^{t_j} - (\kappa - \kappa_c) e_{\alpha,\beta}| - C|t-t_j|^3 \right)^{-1} dt + C(c_\delta - |\kappa-\kappa_c|)^{-1} \\ & = 2\sum_{j=1}^J \int_0^\delta \left(\left(C_j^2 t^4 + (\kappa-\kappa_c)^2 - 2C_j t^2 (\kappa - \kappa_c) \partial_t \phi^{t_j} \cdot e_{\alpha,\beta} \right)^{1/2} - Ct^3 \right)^{-1} dt + C(c_\delta - |\kappa - \kappa_c|)^{-1} \\ & = \frac{2}{C_j^{1/2}(\kappa-\kappa_c)^{1/2}} \sum_{j=1}^J \int_0^{\delta \left(\frac{C_j}{\kappa - \kappa_c}\right)^{1/2}} \left(\left(t^4 - 2t^2 \partial_t \phi^{t_j} \cdot e_{\alpha,\beta} + 1 \right)^{1/2} - CC_j^{-1/2}(\kappa-\kappa_c)^{1/2} t^3 \right)^{-1} dt \\ & \qquad + C(c_\delta - |\kappa - \kappa_c|)^{-1}. \end{aligned}$$

Multiplying by $(\kappa-\kappa_c)^{1/2}$ and letting $\kappa\downarrow\kappa_c$, this yields

$$\limsup_{\kappa\downarrow\kappa_c}(\kappa-\kappa_c)^{1/2}|\Gamma_1^{\kappa e}|^{-1}\le\frac{2}{|\tilde e|C_j^{1/2}}\sum_{j=1}^J\int_0^\infty(t^4-2t^2\partial_t\phi^{t_j}\cdot e_{\alpha,\beta}+1)^{-1/2}dt. \tag{8.22}$$

Symmetrically, we have a similar lower bound

$$\begin{aligned}\int_0^T|\Gamma^{\kappa e}(\phi^t)|^{-1}dt&\ge\sum_{j=1}^J\int_{t_j-\delta}^{t_j+\delta}\left(|C_j(t-t_j)^2\partial_t\phi^{t_j}-(\kappa-\kappa_c)e_{\alpha,\beta}|+C|t-t_j|^3\right)^{-1}dt-C(c_\delta-|\kappa-\kappa_c|)^{-1}\\&=\frac{2}{C_j^{1/2}(\kappa-\kappa_c)^{1/2}}\sum_{j=1}^J\int_0^{\delta\left(\frac{C_j}{\kappa-\kappa_c}\right)^{1/2}}\Big(\big(t^4-2t^2\partial_t\phi^{t_j}\cdot e_{\alpha,\beta}+1\big)^{1/2}+CC_j^{-1/2}(\kappa-\kappa_c)^{1/2}t^3\Big)^{-1}dt\\&\qquad-C(c_\delta-|\kappa-\kappa_c|)^{-1},\end{aligned}$$

so that the equality actually holds in (8.22),

$$\lim_{\kappa\downarrow\kappa_c}(\kappa-\kappa_c)^{1/2}|\Gamma_1^{\kappa e}|^{-1}=\frac{2}{|\tilde e|C_j^{1/2}}\sum_{j=1}^J\int_0^\infty(t^4-2t^2\partial_t\phi^{t_j}\cdot e_{\alpha,\beta}+1)^{-1/2}dt,$$

and the result follows. ☐

8.6 Small applied force implies macroscopic frozenness

Beyond diagonal regimes, we may at least prove the following intuitive result: in the presence of a small applied force $\|F\|_{\mathrm{L}^\infty}\ll\|\nabla h\|_{\mathrm{L}^\infty}$, but with fast oscillating pinning potential, the vortices are pinned in the limit. The proof below is based on energy methods, and is limited to subcritical Ginzburg-Landau regimes (GL_1') and (GL_2').

Proposition 8.12. *We consider the dissipative case $\alpha>0$, $\beta\in\mathbb{R}$, $\alpha^2+\beta^2=1$. Let Assumption A(a) hold, with the initial data $(u_\varepsilon^\circ,\mathrm{v}_\varepsilon^\circ,\mathrm{v}^\circ)$ satisfying the well-preparedness condition (1.14). For all $\varepsilon>0$, let $u_\varepsilon\in\mathrm{L}^\infty(\mathbb{R}^+;H^1_{\mathrm{uloc}}(\mathbb{R}^2;\mathbb{C}))$ denote the unique global solution of (1.5) on $\mathbb{R}^+\times\mathbb{R}^2$ with initial data u_ε°, and with*

$$1\ll N_\varepsilon\ll|\log\varepsilon|,\qquad\frac{N_\varepsilon}{|\log\varepsilon|}\ll\lambda_\varepsilon\lesssim1,\qquad\frac{\varepsilon\lambda_\varepsilon^{-1}}{(N_\varepsilon|\log\varepsilon|)^{1/2}}\ll\eta_\varepsilon\ll\lambda_\varepsilon,$$
$$h(x):=\lambda_\varepsilon\eta_\varepsilon\hat h^0(x,x/\eta_\varepsilon),\qquad\|F\|_{W^{1,\infty}}\ll\lambda_\varepsilon.$$

We consider the regime (GL_1') with $\mathrm{v}_\varepsilon^\circ=\mathrm{v}^\circ$, and the regime (GL_2') with $\operatorname{div}(a\mathrm{v}_\varepsilon^\circ)=0$. Then for all $\gamma>0$ there holds $N_\varepsilon^{-1}\mu_\varepsilon\overset{*}{\rightharpoonup}\operatorname{curl}\mathrm{v}^\circ$ in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;(C_c^{0,\gamma}(\mathbb{R}^2))^*)$.

Proof. We choose $\mathrm{v}_\varepsilon:=\mathrm{v}_\varepsilon^\circ$ in the definition of the modulated energy, thus setting for all $z\in R\mathbb{Z}^2$,

$$\mathcal{E}_{\varepsilon,R}^z:=\int\frac{a\chi_R^z}{2}\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon^\circ|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\Big),\qquad\mathcal{D}_{\varepsilon,R}^z:=\mathcal{E}_{\varepsilon,R}^z-\frac{|\log\varepsilon|}{2}\int a\chi_R^z\mu_\varepsilon,$$

and $\mathcal{E}_{\varepsilon,R}^*:=\sup_z\mathcal{E}_{\varepsilon,R}^z$, $\mathcal{D}_{\varepsilon,R}^*:=\sup_z\mathcal{D}_{\varepsilon,R}^z$. We further consider the following modification of this modulated energy, including suitable lower-order terms,

$$\hat{\mathcal{E}}_{\varepsilon,R}^z:=\int\frac{a\chi_R^z}{2}\Big(|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon^\circ|^2+\frac{a}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2+(1-|u_\varepsilon|^2)(f-N_\varepsilon^2|\mathrm{v}_\varepsilon^\circ|^2-N_\varepsilon|\log\varepsilon|\,\mathrm{v}_\varepsilon^\circ\cdot F^\perp)\Big),$$

and $\hat{\mathcal{E}}^*_{\varepsilon,R} := \sup_z \hat{\mathcal{E}}^z_{\varepsilon,R}$. The lower bound assumption on η_ε allows to choose the cut-off length $R \geq 1$ in such a way that $\lambda_\varepsilon^{-1} \ll R \ll \eta_\varepsilon \varepsilon^{-1}(N_\varepsilon|\log\varepsilon|)^{1/2}$.

By Proposition 5.2, the assumption on the initial data implies $\mathcal{E}^{*,\circ}_{\varepsilon,R} \leq C_0 N_\varepsilon|\log\varepsilon|$ for some $C_0 \simeq 1$. Let $T>0$ be fixed, and define $T_\varepsilon > 0$ as the maximum time $\leq T$ such that the bound $\mathcal{E}^{*,t}_{\varepsilon,R} \leq (C_0+1)N_\varepsilon|\log\varepsilon|$ holds for all $t \leq T_\varepsilon$. Note that, using the bound $\|f\|_{\mathrm{L}^\infty} \lesssim \lambda_\varepsilon\eta_\varepsilon^{-1} + \lambda_\varepsilon^2|\log\varepsilon|^2$ (cf. (1.6)), the choice of η_ε and R, and the assumption $\|\mathrm{v}_\varepsilon^\circ\|_{\mathrm{L}^2\cap\mathrm{L}^\infty(B_{2R})} \lesssim_\theta R^\theta$ for all $\theta>0$, we deduce for all $t \leq T_\varepsilon$,

$$
\begin{aligned}
|\hat{\mathcal{E}}^{z,t}_{\varepsilon,R} - \mathcal{E}^{z,t}_{\varepsilon,R}| &\lesssim \int \chi_R^z|1-|u_\varepsilon^t|^2|(|f| + N_\varepsilon^2|\mathrm{v}_\varepsilon^\circ|^2 + N_\varepsilon|\log\varepsilon||\mathrm{v}_\varepsilon^\circ||F|)\\
&\lesssim \varepsilon R(\lambda_\varepsilon\eta_\varepsilon^{-1} + \lambda_\varepsilon^2|\log\varepsilon|^2)(\mathcal{E}^{z,t}_{\varepsilon,R})^{1/2} + \varepsilon R^\theta o(\lambda_\varepsilon N_\varepsilon|\log\varepsilon|)(\mathcal{E}^{z,t}_{\varepsilon,R})^{1/2} \ll \lambda_\varepsilon N_\varepsilon|\log\varepsilon|, \qquad (8.23)
\end{aligned}
$$

hence in particular $\hat{\mathcal{E}}^{*,t}_{\varepsilon,R} \lesssim N_\varepsilon|\log\varepsilon|$ for all $t \leq T_\varepsilon$. We split the proof into three steps.

Step 1: evolution of the modulated energy. In this step, for all $\varepsilon>0$ small enough, we show that $T_\varepsilon = T$, and that for all $t \leq T$,

$$
\frac{\lambda_\varepsilon\alpha}{4}\int_0^t\int a_\varepsilon\chi_R^z|\partial_t u_\varepsilon|^2 \leq \hat{\mathcal{E}}^{z,\circ}_{\varepsilon,R} - \hat{\mathcal{E}}^{z,t}_{\varepsilon,R} + o_t(\lambda_\varepsilon N_\varepsilon|\log\varepsilon|) \lesssim_t N_\varepsilon|\log\varepsilon|. \qquad (8.24)
$$

By integration by parts, we find

$$
\begin{aligned}
\partial_t\hat{\mathcal{E}}^z_{\varepsilon,R} &= \int a\chi_R^z\Big(\langle\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon^\circ, \nabla\partial_t u_\varepsilon\rangle - N_\varepsilon\mathrm{v}_\varepsilon^\circ\cdot\langle\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon^\circ, i\partial_t u_\varepsilon\rangle\\
&\qquad - \frac{a}{\varepsilon^2}(1-|u_\varepsilon|^2)\langle u_\varepsilon, \partial_t u_\varepsilon\rangle - (f - N_\varepsilon^2|\mathrm{v}_\varepsilon^\circ|^2 - N_\varepsilon|\log\varepsilon|\,\mathrm{v}_\varepsilon^\circ\cdot F^\perp)\langle u_\varepsilon,\partial_t u_\varepsilon\rangle\Big)\\
&= -\int a\chi_R^z\Big\langle\triangle u_\varepsilon + \frac{au_\varepsilon}{\varepsilon^2}(1-|u_\varepsilon|^2) + \nabla h\cdot\nabla u_\varepsilon + i|\log\varepsilon|F^\perp\cdot\nabla u_\varepsilon + fu_\varepsilon, \partial_t u_\varepsilon\Big\rangle\\
&\qquad + N_\varepsilon\int a\chi_R^z(\mathrm{v}_\varepsilon^\circ\cdot\nabla h + \operatorname{div}\,\mathrm{v}_\varepsilon^\circ)\langle\partial_t u_\varepsilon, iu_\varepsilon\rangle - \int a\nabla\chi_R^z\cdot\langle\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon^\circ, \partial_t u_\varepsilon\rangle\\
&\qquad - \int a\chi_R^z(|\log\varepsilon|F^\perp + 2N_\varepsilon\mathrm{v}_\varepsilon^\circ)\cdot\langle\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon^\circ, i\partial_t u_\varepsilon\rangle,
\end{aligned}
$$

hence, inserting equation (1.5) in the first right-hand side term,

$$
\begin{aligned}
\partial_t\hat{\mathcal{E}}^z_{\varepsilon,R} &= -\lambda_\varepsilon\alpha\int a\chi_R^z|\partial_t u_\varepsilon|^2 - \int a\chi_R^z(|\log\varepsilon|F^\perp + 2N_\varepsilon\mathrm{v}_\varepsilon^\circ)\cdot\langle\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon^\circ, i\partial_t u_\varepsilon\rangle\\
&\qquad + N_\varepsilon\int\chi_R^z\operatorname{div}(a\mathrm{v}_\varepsilon^\circ)\langle\partial_t u_\varepsilon, iu_\varepsilon\rangle - \int a\nabla\chi_R^z\cdot\langle\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon^\circ, \partial_t u_\varepsilon\rangle.
\end{aligned}
$$

In particular, using the assumption $\|\nabla h\|_{\mathrm{L}^\infty} \lesssim \lambda_\varepsilon$, and using that $\mathcal{E}^{*,t}_{\varepsilon,R} \lesssim N_\varepsilon|\log\varepsilon|$, we find for all $t \leq T_\varepsilon$,

$$
\begin{aligned}
\partial_t\hat{\mathcal{E}}^z_{\varepsilon,R} &\leq -\frac{\lambda_\varepsilon\alpha}{2}\int a\chi_R^z|\partial_t u_\varepsilon|^2 - \int a\chi_R^z(|\log\varepsilon|F^\perp + 2N_\varepsilon\mathrm{v}_\varepsilon^\circ)\cdot\langle\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon^\circ, i\partial_t u_\varepsilon\rangle\\
&\qquad + C\lambda_\varepsilon^{-1}N_\varepsilon^2\int\chi_R^z|\operatorname{div}(a\mathrm{v}_\varepsilon^\circ)|^2(1+|1-|u_\varepsilon|^2|) + C\lambda_\varepsilon^{-1}R^{-2}\int_{B_{2R}(z)}a|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon^\circ|^2\\
&\leq -\frac{\lambda_\varepsilon\alpha}{2}\int a\chi_R^z|\partial_t u_\varepsilon|^2 - \int a\chi_R^z(|\log\varepsilon|F^\perp + 2N_\varepsilon\mathrm{v}_\varepsilon^\circ)\cdot\langle\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon^\circ, i\partial_t u_\varepsilon\rangle\\
&\qquad + C\lambda_\varepsilon^{-1}N_\varepsilon^2\|\operatorname{div}(a\mathrm{v}_\varepsilon^\circ)\|^2_{\mathrm{L}^2\cap\mathrm{L}^\infty(B_{2R})} + C\lambda_\varepsilon^{-1}R^{-2}N_\varepsilon|\log\varepsilon|,
\end{aligned}
$$

so that the assumptions on $\operatorname{div}(a\mathrm{v}_\varepsilon^\circ)$ and the choice of the cut-off length R yield

$$\partial_t\hat{\mathcal{E}}_{\varepsilon,R}^z \le -\frac{\lambda_\varepsilon\alpha}{2}\int a\chi_R^z|\partial_t u_\varepsilon|^2 - \int a\chi_R^z(|\log\varepsilon|F^\perp + 2N_\varepsilon\mathrm{v}_\varepsilon^\circ)\cdot\langle\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon^\circ, i\partial_t u_\varepsilon\rangle + o(\lambda_\varepsilon N_\varepsilon|\log\varepsilon|). \tag{8.25}$$

Simply using the Cauchy-Schwarz inequality to estimate the second term, with $\|(F,\mathrm{v}_\varepsilon^\circ)\|_{\mathrm{L}^\infty}\lesssim 1$, we find the following rough a priori estimate,

$$\begin{aligned}\partial_t\hat{\mathcal{E}}_{\varepsilon,R}^z &\le -\frac{\lambda_\varepsilon\alpha}{4}\int a\chi_R^z|\partial_t u_\varepsilon|^2 + C\lambda_\varepsilon^{-1}|\log\varepsilon|^2\int a\chi_R^z|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon^\circ|^2 + o(\lambda_\varepsilon N_\varepsilon|\log\varepsilon|)\\ &\le -\frac{\lambda_\varepsilon\alpha}{4}\int a\chi_R^z|\partial_t u_\varepsilon|^2 + O(\lambda_\varepsilon^{-1}N_\varepsilon|\log\varepsilon|^3),\end{aligned}$$

and thus, integrating in time, with $\lambda_\varepsilon \gg N_\varepsilon/|\log\varepsilon|$, we find for all $t\le T_\varepsilon$,

$$\frac{\lambda_\varepsilon\alpha}{4}\int_0^t\int a\chi_R^z|\partial_t u_\varepsilon|^2 \le \hat{\mathcal{E}}_{\varepsilon,R}^{z,\circ} - \hat{\mathcal{E}}_{\varepsilon,R}^{z,t} + o_t(|\log\varepsilon|^4) \lesssim_t |\log\varepsilon|^4.$$

This rough estimate now allows us to apply the product estimate in Lemma 5.4 (with $\mathrm{v}_\varepsilon = \mathrm{v}_\varepsilon^\circ$ and $\mathrm{p}_\varepsilon = 0$), using $|\log\varepsilon|\|F\|_{\mathrm{L}^\infty} + N_\varepsilon \ll \lambda_\varepsilon|\log\varepsilon|$, to the effect of

$$\begin{aligned}&\Big|\int_0^t\int a\chi_R^z(|\log\varepsilon|F^\perp + 2N_\varepsilon\mathrm{v}_\varepsilon^\circ)\cdot\langle\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon^\circ, i\partial_t u_\varepsilon\rangle\Big|\\ &\lesssim \frac{|\log\varepsilon|\|F\|_{\mathrm{L}^\infty} + N_\varepsilon}{|\log\varepsilon|}\Big(\int_0^t\int a\chi_R^z|\partial_t u_\varepsilon|^2 + \int_0^t\int a\chi_R^z|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}_\varepsilon^\circ|^2\Big) + o_t(1)\\ &\lesssim o(\lambda_\varepsilon)\int_0^t\int a\chi_R^z|\partial_t u_\varepsilon|^2 + o_t(\lambda_\varepsilon N_\varepsilon|\log\varepsilon|).\end{aligned}$$

Inserting this into (8.25), and integrating in time, we find for all $t\le T_\varepsilon$,

$$\hat{\mathcal{E}}_{\varepsilon,R}^{z,t} - \hat{\mathcal{E}}_{\varepsilon,R}^{z,\circ} \le -\Big(\frac{\lambda_\varepsilon\alpha}{2} - o(\lambda_\varepsilon)\Big)\int_0^t\int a\chi_R^z|\partial_t u_\varepsilon|^2 + o_t(\lambda_\varepsilon N_\varepsilon|\log\varepsilon|),$$

and the result (8.24) follows for all $t\le T_\varepsilon$. In particular, combined with (8.23), this yields for all $t\le T_\varepsilon$,

$$\mathcal{E}_{\varepsilon,R}^{z,t} \le \hat{\mathcal{E}}_{\varepsilon,R}^{z,t} + o(\lambda_\varepsilon N_\varepsilon|\log\varepsilon|) \le \hat{\mathcal{E}}_{\varepsilon,R}^{z,\circ} + o_t(\lambda_\varepsilon N_\varepsilon|\log\varepsilon|) \le \mathcal{E}_{\varepsilon,R}^{z,\circ} + o_t(\lambda_\varepsilon N_\varepsilon|\log\varepsilon|) \le (C_0 + o_t(1))N_\varepsilon|\log\varepsilon|,$$

and thus, taking the supremum in z, the conclusion $T_\varepsilon = T$ follows for $\varepsilon > 0$ small enough.

Step 2: lower bound on the modulated energy. In this step, we prove that for all $t\le T$,

$$\mathcal{E}_{\varepsilon,R}^{z,t} \ge \frac{|\log\varepsilon|}{2}\int a\chi_R^z\mu_\varepsilon^t - o(\lambda_\varepsilon N_\varepsilon|\log\varepsilon|),$$

hence, by (8.23) and by the assumption that $\mathcal{E}_{\varepsilon,R}^{z,\circ} = \frac12|\log\varepsilon|\int a_\varepsilon\chi_R^z\mu_\varepsilon^\circ + o(N_\varepsilon^2)$,

$$\hat{\mathcal{E}}_{\varepsilon,R}^{z,\circ} - \hat{\mathcal{E}}_{\varepsilon,R}^{z,t} \le \mathcal{E}_{\varepsilon,R}^{z,\circ} - \mathcal{E}_{\varepsilon,R}^{z,t} + o(\lambda_\varepsilon N_\varepsilon|\log\varepsilon|) \le \frac{|\log\varepsilon|}{2}\int a\chi_R^z(\mu_\varepsilon^\circ - \mu_\varepsilon^t) + o(\lambda_\varepsilon N_\varepsilon|\log\varepsilon|).$$

As we show, this is a simple consequence of Lemma 5.1. (However note that we may not directly apply Proposition 5.2(i)–(iii), since in the present situation the assumption $R\gtrsim|\log\varepsilon|$ does not hold.) Noting that $\|\nabla(a\chi_R^z)\|_{\mathrm{L}^\infty} \lesssim \lambda_\varepsilon + R^{-1} \lesssim \lambda_\varepsilon$, we deduce from Lemma 5.1(i) with $\phi = a\chi_R^z$, $\mathcal{E}^*_{\varepsilon,R} \lesssim N_\varepsilon|\log\varepsilon|$, and $e^{-N_\varepsilon} \lesssim r \ll 1$,

$$\begin{aligned}\mathcal{E}^z_{\varepsilon,R} &\geq \frac{\log(r/\varepsilon)}{2}\int a\chi_R^z|\nu^r_{\varepsilon,R}| - O(\lambda_\varepsilon r N_\varepsilon|\log\varepsilon|) - O(r^2N_\varepsilon^2) - O(N_\varepsilon\log N_\varepsilon)\\ &\geq \frac{|\log\varepsilon|}{2}\int a\chi_R^z|\nu^r_{\varepsilon,R}| - O(|\log r|)\int \chi_R^z|\nu^r_{\varepsilon,R}| - o(\lambda_\varepsilon N_\varepsilon|\log\varepsilon|),\end{aligned}$$

hence by item (ii) of Lemma 5.1, with the choice of the radius $r \gtrsim e^{-N_\varepsilon}$,

$$\mathcal{E}^z_{\varepsilon,R} \geq \frac{|\log\varepsilon|}{2}\int a\chi_R^z|\nu^r_{\varepsilon,R}| - O(N_\varepsilon|\log r|) - o(\lambda_\varepsilon N_\varepsilon|\log\varepsilon|) \geq \frac{|\log\varepsilon|}{2}\int a\chi_R^z\nu^r_{\varepsilon,R} - o(\lambda_\varepsilon N_\varepsilon|\log\varepsilon|).$$

By Lemma 5.1(iii) in the form (5.7) and by (5.12) with $\gamma = 1$, using again that $\|\nabla(a\chi_R^z)\|_{\mathrm{L}^\infty} \lesssim \lambda_\varepsilon$, we may now replace $\nu^r_{\varepsilon,R}$ by μ_ε in the right-hand side,

$$\mathcal{E}^z_{\varepsilon,R} \geq \frac{|\log\varepsilon|}{2}\int a\chi_R^z\mu_\varepsilon - O(\lambda_\varepsilon r N_\varepsilon|\log\varepsilon|) - o(\lambda_\varepsilon N_\varepsilon|\log\varepsilon|).$$

and the result follows.

Step 3: estimate on the total vorticity. In this step, we prove that for all $t \leq T$,

$$\Big|\int a\chi_R^z(\mu_\varepsilon^t - \mu_\varepsilon^\circ)\Big| \ll_t \lambda_\varepsilon N_\varepsilon.$$

We first prove (a weaker version of) the result with a replaced by 1, and the conclusion then follows by noting that $a = \exp(\lambda_\varepsilon\eta_\varepsilon\hat h^0)$ indeed converges to 1 as $\varepsilon \downarrow 0$. Using identity (4.8), we write

$$\begin{aligned}\int \chi_R^z(\mu_\varepsilon^t - \mu_\varepsilon^\circ) &= \int_0^t\int \chi_R^z\partial_t\mu_\varepsilon^t = \int_0^t\int \chi_R^z\operatorname{curl} V_\varepsilon^t = -\int_0^t\int \nabla^\perp\chi_R^z\cdot V_\varepsilon^t\\ &= -2\int_0^t\int \nabla^\perp\chi_R^z\cdot\langle\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon^\circ, i\partial_t u_\varepsilon\rangle + N_\varepsilon\int_0^t\int \nabla^\perp\chi_R^z\cdot \mathrm{v}_\varepsilon^\circ\,\partial_t(1-|u_\varepsilon|^2).\end{aligned}$$

Applying the product estimate of Lemma 5.4 as in Step 1, with $|\nabla\chi_R| \lesssim R^{-1}\chi_R^{1/2}$, we find for all $|\log\varepsilon|^{-2} \lesssim K \lesssim |\log\varepsilon|^2$ and for all $t \leq T$,

$$\begin{aligned}\Big|\int \chi_R^z(\mu_\varepsilon^t - \mu_\varepsilon^\circ)\Big| &\lesssim \frac{1}{|\log\varepsilon|}\Big(K^{-2}\int_0^t\int \chi_R^z|\partial_t u_\varepsilon|^2 + K^2R^{-2}\int_0^t\int_{B_{2R}}|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon^\circ|^2\Big) + o_t(|\log\varepsilon|^{-1})\\ &\qquad + N_\varepsilon\int|1-|u_\varepsilon^t|^2||\nabla^\perp\chi_R^z| + N_\varepsilon\int|1-|u_\varepsilon^\circ|^2||\nabla^\perp\chi_R^z|\\ &\lesssim_t \frac{K^{-2}}{|\log\varepsilon|}\int_0^t\int \chi_R^z|\partial_t u_\varepsilon|^2 + K^2R^{-2}N_\varepsilon + \varepsilon N_\varepsilon|\log\varepsilon| + o(|\log\varepsilon|^{-1}).\end{aligned}$$

Using (8.24) to estimate the first right-hand side term, and choosing $\lambda_\varepsilon^{-1} \ll K^2 \ll \lambda_\varepsilon R^2$, we obtain

$$\begin{aligned}\Big|\int \chi_R^z(\mu_\varepsilon^t - \mu_\varepsilon^\circ)\Big| &\lesssim_t \frac{K^{-2}}{\lambda_\varepsilon|\log\varepsilon|}(\hat{\mathcal{E}}^{z,\circ}_{\varepsilon,R} - \hat{\mathcal{E}}^{z,t}_{\varepsilon,R})^+ + o(K^{-2}N_\varepsilon) + K^2R^{-2}N_\varepsilon + o(|\log\varepsilon|^{-1})\\ &\lesssim_t o(|\log\varepsilon|^{-1})(\hat{\mathcal{E}}^{z,\circ}_{\varepsilon,R} - \hat{\mathcal{E}}^{z,t}_{\varepsilon,R})^+ + o(\lambda_\varepsilon N_\varepsilon).\end{aligned}\tag{8.26}$$

It remains to smuggle the weight a into the left-hand side. For all $t \le T$, Lemma 5.1(iii) together with (5.12) yields for $\varepsilon^{1/2} < r \ll \lambda_\varepsilon$,

$$\Big|\int (1-a)\chi_R^z(\mu_\varepsilon^t - \nu_{\varepsilon,R}^{r,t})\Big| \ll \lambda_\varepsilon N_\varepsilon,$$

and hence, by Lemma 5.1(ii), with $\|1-a\|_{\mathrm{L}^\infty} \lesssim \eta_\varepsilon \ll \lambda_\varepsilon$,

$$\int (1-a)\chi_R^z\mu_\varepsilon^t \lesssim \|1-a_\varepsilon\|_{\mathrm{L}^\infty}\int \chi_R^z|\nu_{\varepsilon,R}^{r,t}| + o(\lambda_\varepsilon N_\varepsilon) \ll \lambda_\varepsilon N_\varepsilon.$$

Combining this with (8.26) and with the result of Step 2, we deduce

$$\Big|\int a\chi_R^z(\mu_\varepsilon^t - \mu_\varepsilon^\circ)\Big| \lesssim_t o(|\log\varepsilon|^{-1})(\hat{\mathcal{E}}_{\varepsilon,R}^{z,\circ} - \hat{\mathcal{E}}_{\varepsilon,R}^{z,t})^+ + o(\lambda_\varepsilon N_\varepsilon) \lesssim o(1)\Big|\int a\chi_R^z(\mu_\varepsilon^t - \mu_\varepsilon^\circ)\Big| + o(\lambda_\varepsilon N_\varepsilon),$$

and the result follows.

Step 4: conclusion. Combining the results of Steps 1 and 2 with the assumption $\hat{\mathcal{E}}_{\varepsilon,R}^{z,\circ} = \frac{1}{2}|\log\varepsilon|\int a\chi_R^z\mu_\varepsilon^\circ + o(N_\varepsilon^2)$, we find

$$\frac{\lambda_\varepsilon\alpha}{2}\int_0^T\int a\chi_R^z|\partial_t u_\varepsilon|^2 \le |\log\varepsilon|\int a\chi_R^z(\mu_\varepsilon^\circ - \mu_\varepsilon^T) + o_T(\lambda_\varepsilon N_\varepsilon|\log\varepsilon|),$$

and hence by the result of Step 3,

$$\int_0^T\int a\chi_R^z|\partial_t u_\varepsilon|^2 \ll_T N_\varepsilon|\log\varepsilon|.$$

The product estimate of [82, Appendix A] (see also Lemma 5.4) then yields for all $X \in W^{1,\infty}([0,T]\times\mathbb{R}^2)^2$ and all $|\log\varepsilon|^{-1} \lesssim K \lesssim |\log\varepsilon|$,

$$\begin{aligned}\Big|\int_0^T\int \chi_R^z X\cdot V_\varepsilon\Big| &\lesssim \frac{1}{|\log\varepsilon|}\Big(\frac{1}{K}\int_0^T\int\chi_R^z|\partial_t u_\varepsilon|^2 + K\int_0^T\int\chi_R^z|X\cdot(\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_\varepsilon^\circ)|^2\Big)\\ &\qquad + o(1)\big(1+\|X\|^2_{W^{1,\infty}([0,T]\times\mathbb{R}^2)}\big)\\ &\lesssim_T \big(K^{-1}o(N_\varepsilon) + KN_\varepsilon + o(1)\big)\big(1+\|X\|^2_{W^{1,\infty}([0,T]\times\mathbb{R}^2)}\big),\end{aligned}$$

hence, for a suitable choice of K,

$$\sup_z\Big|\int_0^T\int\chi_R^z X\cdot V_\varepsilon\Big| \ll_T N_\varepsilon\big(1+\|X\|^2_{W^{1,\infty}([0,T]\times\mathbb{R}^2)}\big).$$

This proves $N_\varepsilon^{-1}V_\varepsilon \xrightarrow{*} 0$ in $(C_c^1([0,T]\times\mathbb{R}^2))^*$, so that identity (4.8) yields $\partial_t(N_\varepsilon^{-1}\mu_\varepsilon) = N_\varepsilon^{-1}\mathrm{curl}\,V_\varepsilon \xrightarrow{*} 0$ in $(C^1([0,T];C_c^2(\mathbb{R}^2)))^*$. Arguing as in Step 5 of the proof of Proposition 6.1, the well-preparedness assumption on the initial data implies $N_\varepsilon^{-1}j_\varepsilon^\circ \to \mathrm{v}^\circ$ in $\mathrm{L}^1_{\mathrm{uloc}}(\mathbb{R}^2)^2$, hence in particular $N_\varepsilon^{-1}\mu_\varepsilon^\circ \xrightarrow{*} \mathrm{curl}\,\mathrm{v}^\circ$ in $(C_c^1(\mathbb{R}^2))^*$. We easily conclude $N_\varepsilon^{-1}\mu_\varepsilon \xrightarrow{*} \mathrm{curl}\,\mathrm{v}^\circ$ in $(C([0,T];C_c^2(\mathbb{R}^2)))^*$. The conclusion then follows, noting that by Lemma 5.1(iii) and by (5.12) the sequence $(N_\varepsilon^{-1}\mu_\varepsilon)_\varepsilon$ is bounded in $\mathrm{L}^\infty([0,T];(C_c^{0,\gamma}(\mathbb{R}^2))^*)$ for all $\gamma > 0$, and using interpolation (as e.g. in [53]). ☐

A Appendix: Well-posedness for the modified Ginzburg-Landau equation

In this appendix, we address global well-posedness for equation (1.5), proving Proposition 2.2 as well as additional regularity. We begin with the decaying setting, that is the case when $\nabla h, F, f$ are assumed to have some decay at infinity. Note that in this setting no transport is expected to occur at infinity. As is classical since the work of Bethuel and Smets [9] (see also [63]), we consider the existence of solutions u_ε of (1.5) in the affine space $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; U + H^1(\mathbb{R}^2;\mathbb{C}))$ for some "reference map" U, which is typically chosen smooth and equal (in polar coordinates) to $e^{iD_\varepsilon\theta}$ outside a ball at the origin, for some given $D_\varepsilon \in \mathbb{Z}$. Such a choice $U = U_{D_\varepsilon}$ imposes a total degree D_ε at infinity. More generally, we consider here the following spaces of "admissible" reference maps, for all $k \geq 0$,

$$E_k(\mathbb{R}^2) := \{U \in \mathrm{L}^\infty(\mathbb{R}^2;\mathbb{C}) : \nabla^2 U \in H^k(\mathbb{R}^2;\mathbb{C}), \nabla|U| \in \mathrm{L}^2(\mathbb{R}^2), 1-|U|^2 \in \mathrm{L}^2(\mathbb{R}^2), \nabla U \in \mathrm{L}^p(\mathbb{R}^2;\mathbb{C})\ \forall p > 2\}.$$

(Note that this definition slightly differs from the usual one in [9], but it is more suitable in this form in the presence of pinning and forcing.) The map U_{D_ε} above clearly belongs to the space $E_\infty(\mathbb{R}^2)$. Global well-posedness and regularity in this framework are provided by the following proposition. Note that a stronger decay of the coefficients $\nabla h, F, f$ is required in the Gross-Pitaevskii case, although we do not know whether it is necessary.

Proposition A.1 (Well-posedness for (1.5) — decaying setting)**.** *Set $a := e^h$, with $h : \mathbb{R}^2 \to \mathbb{R}$.*

(i) Dissipative case $\alpha > 0$, $\beta \in \mathbb{R}$:
Given $h \in W^{1,\infty}(\mathbb{R}^2)$, $F \in \mathrm{L}^\infty(\mathbb{R}^2)^2$, $f \in \mathrm{L}^2 \cap \mathrm{L}^\infty(\mathbb{R}^2)$, with $\nabla h, F \in \mathrm{L}^p(\mathbb{R}^2)^2$ for some $p < \infty$, and $u_\varepsilon^\circ \in U + H^1(\mathbb{R}^2;\mathbb{C})$ for some $U \in E_0(\mathbb{R}^2)$, there exists a unique global solution $u_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; U + H^1(\mathbb{R}^2;\mathbb{C}))$ of (1.5) on $\mathbb{R}^+ \times \mathbb{R}^2$ with initial data u_ε°.
Moreover, if for some $k \geq 0$ we have $h \in W^{k+1,\infty}(\mathbb{R}^2)$, $F \in W^{k,\infty}(\mathbb{R}^2)^2$, $f \in H^k \cap W^{k,\infty}(\mathbb{R}^2)$, with $\nabla h, F \in W^{k,p}(\mathbb{R}^2)^2$ for some $p < \infty$, and $U \in E_k(\mathbb{R}^2)$, then $u_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}([\delta,\infty); U + H^{k+1}(\mathbb{R}^2;\mathbb{C}))$ for all $\delta > 0$. In particular, if in addition $u_\varepsilon^\circ \in U + H^{k+1}(\mathbb{R}^2;\mathbb{C})$, then $u_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; U + H^{k+1}(\mathbb{R}^2;\mathbb{C}))$.

(ii) Gross-Pitaevskii case $\alpha = 0$, $\beta \in \mathbb{R}$:
Given $h \in W^{2,\infty}(\mathbb{R}^2)$, $\nabla h \in H^1(\mathbb{R}^2)^2$, $F \in H^2 \cap W^{2,\infty}(\mathbb{R}^2)^2$ with div *$F = 0$, $f \in \mathrm{L}^2 \cap \mathrm{L}^\infty(\mathbb{R}^2)$, and $u_\varepsilon^\circ \in U + H^1(\mathbb{R}^2;\mathbb{C})$ for some $U \in E_0(\mathbb{R}^2)$, there exists a unique global solution $u_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; U + H^1(\mathbb{R}^2;\mathbb{C}))$ of (1.5) on $\mathbb{R}^+ \times \mathbb{R}^2$ with initial data u_ε°.*
Moreover, if for some $k \geq 0$ we have $h \in W^{k+2,\infty}(\mathbb{R}^2)$, $\nabla h \in H^{k+1}(\mathbb{R}^2)^2$, $F \in H^{k+2} \cap W^{k+2,\infty}(\mathbb{R}^2)^2$ with div *$F = 0$, $f \in H^{k+1} \cap W^{k+1,\infty}(\mathbb{R}^2)$, and $u_\varepsilon^\circ \in U + H^{k+1}(\mathbb{R}^2;\mathbb{C})$ for some $U \in E_{k+1}(\mathbb{R}^2)$, then $u_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; U + H^{k+1}(\mathbb{R}^2;\mathbb{C}))$.*

The proof below is based on arguments by [9, 63], which need to be adapted in the present context with both pinning and forcing. The conservative case $\alpha = 0$ is however more delicate, and we then use the structure of the equation to make a crucial change of variables that transforms the first-order terms into zeroth-order ones. As shown in the proof, in the dissipative regime, the decay assumption $\nabla h, F \in \mathrm{L}^p(\mathbb{R}^2)^2$ (for some $p < \infty$) can be simply replaced by $(|\nabla h| + |F|)\nabla U \in \mathrm{L}^2(\mathbb{R}^2;\mathbb{C})^2$.

Proof. We split the proof into seven steps. We begin with the (easiest) case $\alpha > 0$, and then turn to the conservative case $\alpha = 0$ in Steps 4–7.

Step 1: local existence in $U + H^{k+1}(\mathbb{R}^2;\mathbb{C})$ for $\alpha > 0$. In this step, given $k \geq 0$, we assume $h \in W^{k+1,\infty}(\mathbb{R}^2)$, $F \in W^{k,\infty}(\mathbb{R}^2)^2$, $f \in H^k \cap W^{k,\infty}(\mathbb{R}^2)$, $\nabla h, F \in W^{k,p}(\mathbb{R}^2)$ for some $p < \infty$, and $u_\varepsilon^\circ \in U + H^{k+1}(\mathbb{R}^2;\mathbb{C})$ for some $U \in E_k(\mathbb{R}^2)$, and we prove that there exists some $T > 0$ and a unique solution $u_\varepsilon \in \mathrm{L}^\infty([0,T); U + H^{k+1}(\mathbb{R}^2;\mathbb{C}))$ of (1.5) on $[0,T) \times \mathbb{R}^2$. To simplify notation, we replace equation (1.5) by its rescaled version

$$(\alpha + i\beta)\partial_t u = \triangle u + au(1-|u|^2) + \nabla h \cdot \nabla u + iF^\perp \cdot \nabla u + fu, \qquad u|_{t=0} = u^\circ. \tag{A.1}$$

We begin with the case $k = 0$, and briefly comment afterwards on the adaptations needed for $k \geq 1$. We argue by a fixed-point argument in the set $E_{U,u^\circ}(C_0,T) := \{u : \|u - U\|_{\mathrm{L}^\infty_T H^1} \leq C_0, u|_{t=0} = u^\circ\}$, for some

$C_0, T > 0$ to be suitably chosen. We denote by $C \geq 1$ any constant that only depends on an upper bound on α, α^{-1}, $|\beta|$, $\|h\|_{W^{1,\infty}}$, $\|(F,f,U)\|_{\mathrm{L}^\infty}$, $\|1-|U|^2\|_{\mathrm{L}^2}$, $\|\triangle U\|_{\mathrm{L}^2}$, $\|f\|_{\mathrm{L}^2}$, and $\|(|F|+|\nabla h|)\nabla U\|_{\mathrm{L}^2}$, and we add a subscript to indicate dependence on further parameters.

The kernel of the semigroup operator $e^{(\alpha+i\beta)^{-1}t\triangle}$ is given explicitly by $S^t(x) := (\alpha+i\beta)(4\pi t)^{-1}e^{-(\alpha+i\beta)|x|^2/(4t)}$. Since $\alpha > 0$, this kernel decays just like the standard heat kernel,

$$|S^t(x)| \leq Ct^{-1}e^{-\alpha|x|^2/(4t)}, \tag{A.2}$$

and we have the following obvious estimates, for all $1 \leq r \leq \infty$, $k \geq 1$,

$$\|S^t\|_{\mathrm{L}^r} \leq Ct^{1/r-1}, \qquad \|\nabla^k S^t\|_{\mathrm{L}^r} \leq C_k t^{1/r-1-k/2}. \tag{A.3}$$

Setting $\hat{u} := u - U$, we may rewrite equation (A.1) as follows:

$$\begin{aligned}(\alpha+i\beta)\partial_t\hat{u} = \triangle\hat{u} + \triangle U + a(\hat{u}+U)(1-|U|^2) - 2a(\hat{u}+U)\langle U,\hat{u}\rangle - a(\hat{u}+U)|\hat{u}|^2 \\ + \nabla h\cdot\nabla\hat{u} + \nabla h\cdot\nabla U + iF^\perp\cdot\nabla\hat{u} + iF^\perp\cdot\nabla U + f\hat{u} + fU,\end{aligned} \tag{A.4}$$

with initial data $\hat{u}|_{t=0} = \hat{u}^\circ := u^\circ - U$. Any solution $\hat{u} \in \mathrm{L}^\infty([0,T);H^1(\mathbb{R}^2;\mathbb{C}))$ satisfies the Duhamel formula $\hat{u} = \Xi_{U,\hat{u}^\circ}(\hat{u})$, where we have set

$$\begin{aligned}\Xi_{U,\hat{u}^\circ}(\hat{u})^t &:= S^t * \hat{u}^\circ + (\alpha+i\beta)^{-1}\int_0^t S^{t-s} * Z_{U,\hat{u}^\circ}(\hat{u}^s)ds, \\ Z_{U,\hat{u}^\circ}(\hat{u}^s) &:= \triangle U + a(\hat{u}^s+U)(1-|U|^2) - 2a(\hat{u}^s+U)\langle U,\hat{u}^s\rangle - a(\hat{u}^s+U)|\hat{u}^s|^2 \\ &\qquad + \nabla h\cdot\nabla\hat{u}^s + \nabla h\cdot\nabla U + iF^\perp\cdot\nabla\hat{u}^s + iF^\perp\cdot\nabla U + f\hat{u}^s + fU.\end{aligned}$$

Let us examine the map $\Xi_{U,\hat{u}^\circ}$ more closely. Using (A.3) in the forms $\|S^t\|_{\mathrm{L}^1} \leq C$ and $\|\nabla S^t\|_{\mathrm{L}^1} \leq Ct^{-1/2}$, we obtain by the triangle inequality

$$\|\Xi_{U,\hat{u}^\circ}(\hat{u})^t\|_{H^1} \leq \|S^t\|_{\mathrm{L}^1}\|\hat{u}^\circ\|_{H^1} + C\int_0^t (1+(t-s)^{-1/2})\Big(1 + \|\hat{u}^s\|_{\mathrm{L}^2} + \|\hat{u}^s\|_{\mathrm{L}^6}^3 + \|\nabla\hat{u}^s\|_{\mathrm{L}^2}\Big)ds,$$

and hence, by the Sobolev embedding in the form $\|\hat{u}^s\|_{\mathrm{L}^6} \leq C\|\hat{u}^s\|_{H^1}$, for all $\hat{u} \in -U + E_{U,u^\circ}(C_0,T)$,

$$\|\Xi_{U,\hat{u}^\circ}(\hat{u})\|_{\mathrm{L}^\infty_T H^1} \leq C\|\hat{u}^\circ\|_{H^1} + C(T+T^{1/2})(1+C_0^3).$$

Similarly, again using the Sobolev embedding, we easily find for all $\hat{u},\hat{v} \in -U + E_{U,u^\circ}(C_0,T)$

$$\begin{aligned}\|\Xi_{U,\hat{u}^\circ}(\hat{u}) - \Xi_{U,\hat{u}^\circ}(\hat{v})\|_{\mathrm{L}^\infty_T H^1} &\leq C\int_0^t (1+(t-s)^{-1/2})(1+\|\hat{u}^s\|_{H^1}^2 + \|\hat{v}^s\|_{H^1}^2)\|\hat{u}^s-\hat{v}^s\|_{H^1}ds \\ &\leq C(T+T^{1/2})(1+C_0^2)\|\hat{u}-\hat{v}\|_{\mathrm{L}^\infty_T H^1}.\end{aligned}$$

Choosing $C_0 := 1 + C\|\hat{u}^\circ\|_{H^1}$ and $T := 1 \wedge (4C(1+C_0^3))^{-2}$, we deduce that $\Xi_{U,\hat{u}^\circ}$ maps the set $-U + E_{U,u^\circ}(C_0,T)$ into itself, and is contracting on that set. The conclusion follows from a fixed-point argument.

Let us now briefly comment on the case $k \geq 1$ and explain how to adapt the argument above. We again proceed by a fixed point argument, but estimating this time $\Xi_{U,\hat{u}^\circ}(w)$ in $H^{k+1}(\mathbb{R}^2;\mathbb{C})$ as follows

$$\|\Xi_{U,\hat{u}^\circ}(\hat{u})^t\|_{H^{k+1}} \leq \|S^t\|_{\mathrm{L}^1}\|\hat{u}^\circ\|_{H^{k+1}} + C\int_0^t (\|S^{t-s}\|_{\mathrm{L}^1} + \|\nabla S^{t-s}\|_{\mathrm{L}^1})\|Z_{U,\hat{u}^\circ}(\hat{u}^s)\|_{H^k},$$

where we easily check with the Sobolev embedding that

$$\|Z_{U,\hat{u}^\circ}(\hat{u}^s)\|_{H^k} \le C_k(1+\|\hat{u}^s\|^3_{H^{k+1}}), \tag{A.5}$$

for some constant $C_k \ge 1$ that only depends on an upper bound on α, α^{-1}, $|\beta|$, k, $\|h\|_{W^{k+1,\infty}}$, $\|F\|_{W^{k,\infty}}$, $\|f\|_{H^k\cap W^{k,\infty}}$, $\|U\|_{\mathrm{L}^\infty}$, $\|\nabla|U|\|_{\mathrm{L}^2}$, $\|\nabla^2 U\|_{H^k}$, $\|1-|U|^2\|_{\mathrm{L}^2}$, and $\sum_{j\le k}\|(|\nabla^j F|+|\nabla^j\nabla h|)\nabla U\|_{\mathrm{L}^2}$. Similarly estimating the H^{k+1}-norm of the difference $\Xi_{U,\hat{u}^\circ}(\hat{u})-\Xi_{U,\hat{u}^\circ}(\hat{v})$, the result follows.

Step 2: regularizing effect for $\alpha>0$. In this step, given $k\ge 0$, we assume $h\in W^{k+1,\infty}(\mathbb{R}^2)$, $F\in W^{k,\infty}(\mathbb{R}^2)^2$, $f\in H^k\cap W^{k,\infty}(\mathbb{R}^2)$, $\nabla h, F\in W^{k,p}(\mathbb{R}^2)^2$ for some $p<\infty$, and $U\in E_k(\mathbb{R}^2)$, and we prove that any solution $u\in \mathrm{L}^\infty([0,T);U+H^1(\mathbb{R}^2;\mathbb{C}))$ of (A.1) satisfies $u\in\mathrm{L}^\infty([\delta,T);U+H^{k+1}(\mathbb{R}^2;\mathbb{C}))$ for all $\delta>0$. We denote by $C_k\ge 1$ any constant that only depends on an upper bound on α, α^{-1}, $|\beta|$, k, $\|h\|_{W^{k+1,\infty}}$, $\|F\|_{W^{k,\infty}}$, $\|f\|_{H^k\cap W^{k,\infty}}$, $\|U\|_{\mathrm{L}^\infty}$, $\|1-|U|^2\|_{\mathrm{L}^2}$, $\|\nabla|U|\|_{\mathrm{L}^2}$, $\|\nabla^2U\|_{H^k}$, $\sum_{j\le k}\|(|\nabla^jF|+|\nabla^j\nabla h|)\nabla U\|_{\mathrm{L}^2}$, and $\|u^\circ-U\|_{H^1}$. We write C for such a constant in the case $k=1$. We denote by $C_{k,t}\ge 1$ any constant that additionally depends on an upper bound on t, t^{-1}, and $\|u-U\|_{\mathrm{L}^\infty_t H^1}$. We add a subscript to indicate dependence on further parameters.

Let $u\in\mathrm{L}^\infty([0,T);U+H^1(\mathbb{R}^2;\mathbb{C}))$ be a solution of (A.1), and let $\hat{u}:=u-U$. We prove by induction that $\|\hat{u}^t\|_{H^{k+1}}\le C_{k,t}$ for all $t\in(0,T)$ and $k\ge 0$. As it is obvious for $k=0$, we assume that it holds for some $k\ge0$ and we then deduce that it also holds for k replaced by $k+1$. Using the Duhamel formula $\hat{u}=\Xi_{U,\hat{u}^\circ}(\hat{u})$ as in Step 1, we find

$$\begin{aligned}\|\nabla^{k+1}\hat{u}^t\|_{\mathrm{L}^2}\le \|\nabla^kS^t\|_{\mathrm{L}^1}\|\nabla\hat{u}^\circ\|_{\mathrm{L}^2}+C\int_{t/2}^t\|\nabla S^{t-s}*\nabla^kZ_{U,\hat{u}^\circ}(\hat{u}^s)\|_{\mathrm{L}^2}ds\\ +C\int_0^{t/2}\|\nabla^{k+1}S^{t-s}*Z_{U,\hat{u}^\circ}(\hat{u}^s)\|_{\mathrm{L}^2}ds.\end{aligned} \tag{A.6}$$

A finer estimate than (A.5) is now needed. Arguing as in [9, Lemma 2] by means of various Sobolev embeddings, we have for all $1<r<2$

$$\|\nabla Z_{U,\hat{u}^\circ}(\hat{u}^t)\|_{\mathrm{L}^2+\mathrm{L}^r}\le C_r(1+\|\hat{u}^t\|^3_{H^1}+\|\hat{u}^t\|_{H^2}). \tag{A.7}$$

(Note that we cannot choose $r=2$ above because of terms of the form $\|\hat{u}^s\nabla|\hat{u}^s|^2\|_{\mathrm{L}^r}$, and that the term $\|\hat{u}^t\|_{H^2}$ in the right-hand side simply comes from the forcing terms $(\nabla h+iF^\perp)\cdot\nabla\hat{u}^t$ in the expression for $Z_{U,\hat{u}^\circ}(\hat{u}^t)$.) By a similar argument (see e.g. [63, Step 1 of the proof of Proposition A.8]), we find for all $k\ge0$ and $1<r<2$

$$\|\nabla^kZ_{U,\hat{u}^\circ}(\hat{u}^t)\|_{\mathrm{L}^2+\mathrm{L}^r}\le C_{k,r}(1+\|\hat{u}^t\|^3_{H^k}+\|\hat{u}^t\|_{H^{k+1}}). \tag{A.8}$$

We may then deduce from (A.6) together with Young's convolution inequality and with (A.3), for all $1<r<2$,

$$\begin{aligned}\|\nabla^{k+1}\hat{u}^t\|_{\mathrm{L}^2}&\le\|\nabla^kS^t\|_{\mathrm{L}^1}\|\nabla\hat{u}^\circ\|_{\mathrm{L}^2}+C\int_{t/2}^t\|\nabla S^{t-s}\|_{\mathrm{L}^1\cap\mathrm{L}^{2r/(3r-2)}}\|\nabla^kZ_{U,\hat{u}^\circ}(\hat{u}^s)\|_{\mathrm{L}^2+\mathrm{L}^r}ds\\&\qquad+C\int_0^{t/2}\|\nabla^{k+1}S^{t-s}\|_{\mathrm{L}^1}\|Z_{U,\hat{u}^\circ}(\hat{u}^s)\|_{\mathrm{L}^2}ds\\&\le Ct^{-k/2}+C_{k,r}\int_{t/2}^t((t-s)^{-1/2}+(t-s)^{-1/r})(1+\|\hat{u}^s\|^3_{H^k}+\|\hat{u}^s\|_{H^{k+1}})ds\\&\qquad+C\int_0^{t/2}(t-s)^{-(k+1)/2}(1+\|\hat{u}^s\|^3_{H^1})ds\\&\le C_{k,t}+C_{k,t}\sup_{t/2\le s\le t}\|\hat{u}^s\|^3_{H^k}+C_{k,t}\Big(\int_0^t\|\nabla^{k+1}\hat{u}^s\|^3_{\mathrm{L}^2}ds\Big)^{1/3}.\end{aligned}$$

By induction hypothesis, this yields $\|\nabla^{k+1}\hat{u}^t\|_{L^2}^3 \le C_{k,t} + C_{k,t}\int_0^t \|\nabla^{k+1}\hat{u}^s\|_{L^2}^3 ds$, so the result follows from the Grönwall inequality (combined with a simple approximation argument based on the local existence result of Step 1 in the space $U + H^{k+1}(\mathbb{R}^2;\mathbb{C})$).

Step 3: global existence for $\alpha > 0$. In this step, we assume $h \in L^\infty(\mathbb{R}^2)$, $f \in L^2 \cap L^\infty(\mathbb{R}^2)$, $\nabla h, F \in L^p \cap L^\infty(\mathbb{R}^2)$ for some $p < \infty$, $u^\circ \in U + H^1(\mathbb{R}^2;\mathbb{C})$, and $U \in E_0(\mathbb{R}^2)$, and we prove that (A.1) admits a unique global solution $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+; U + H^1(\mathbb{R}^2;\mathbb{C}))$. We denote by $C > 0$ any constant that only depends on an upper bound on α, α^{-1}, $|\beta|$, $\|h\|_{W^{1,\infty}}$, $\|(F,U)\|_{L^\infty}$, $\|1-|U|^2\|_{L^2}$, $\|\triangle U\|_{L^2}$, $\|f\|_{L^2\cap L^\infty}$, and $\|(|F|+|\nabla h|)\nabla U\|_{L^2}$.

Given a solution $u \in L^\infty([0,T); U + H^1(\mathbb{R}^2;\mathbb{C}))$ of (A.1), we claim that the following a priori estimate holds for all $t \in [0,T)$

$$\frac{\alpha}{2}\int_0^t\int |\partial_t u|^2 + \frac{1}{2}\int \Big(|\nabla(u^t - U)|^2 + \frac{a}{2}(1-|u^t|^2)^2 + |u^t - U|^2\Big) \le Ce^{Ct}(1 + \|u^\circ - U\|_{H^1}^2). \tag{A.9}$$

Combining this with the local existence result of Step 1 in the space $U + H^1(\mathbb{R}^2;\mathbb{C})$, we deduce that local solutions can be extended globally in that space, and the result follows. It thus remains to prove the claim (A.9). For simplicity, we assume in the computations below that $u \in L^\infty([0,T); U + H^2(\mathbb{R}^2;\mathbb{C}))$, which in particular implies $\partial_t u \in L^\infty([0,T); L^2(\mathbb{R}^2;\mathbb{C}))$ by (A.1). The general result then follows from a simple approximation argument based on the local existence result of Step 1 in the space $U + H^2(\mathbb{R}^2;\mathbb{C})$.

We set for simplicity $(\alpha + i\beta)^{-1} = \alpha' + i\beta'$, $\alpha' > 0$. Using equation (A.1), we compute the following time-derivative, suitably regrouping the terms and integrating by parts,

$$\begin{aligned}
\frac{1}{2}\partial_t \int |u-U|^2 &= \int \langle u-U, (\alpha'+i\beta')(\triangle u + au(1-|u|^2) + \nabla h\cdot\nabla u + iF^\perp\cdot\nabla u + fu)\rangle \\
&= -\alpha'\int |\nabla(u-U)|^2 + \alpha'\int a|u-U|^2(1-|u|^2) \\
&\qquad + \int \langle u-U, (\alpha'+i\beta')(\nabla h\cdot\nabla(u-U) + iF^\perp\cdot\nabla(u-U) + f(u-U))\rangle \\
&\qquad + \int \langle u-U, (\alpha'+i\beta')(\triangle U + aU(1-|u|^2) + \nabla h\cdot\nabla U + iF^\perp\cdot\nabla U + fU)\rangle,
\end{aligned}$$

which we may now estimate as follows

$$\begin{aligned}
\frac{1}{2}\partial_t \int |u-U|^2 &\le -\alpha'\int |\nabla(u-U)|^2 + C\int |u-U|^2 + C\int |u-U||\nabla(u-U)| \\
&\qquad + \int |u-U|(|\triangle U| + |1-|u|^2| + (|\nabla h| + |F|)|\nabla U| + |f|) \\
&\le -\frac{\alpha'}{2}\int |\nabla(u-U)|^2 + C + C\int |u-U|^2 + C\int (1-|u|^2)^2.
\end{aligned}$$

On the other hand, again using the equation, and integrating by parts, we compute

$$\begin{aligned}
\frac{1}{2}\partial_t \int |\nabla(u-U)|^2 &= \int \langle \nabla(u-U), \nabla\partial_t u\rangle = -\int \langle \triangle(u-U), \partial_t u\rangle \\
= \; & -\int \langle (\alpha+i\beta)\partial_t u - \triangle U - au(1-|u|^2) - \nabla h\cdot\nabla u - iF^\perp\cdot\nabla u - fu, \partial_t u\rangle \\
= \; & -\alpha\int |\partial_t u|^2 - \frac{1}{4}\partial_t \int a(1-|u|^2)^2 + \int \langle \nabla h\cdot\nabla(u-U) + iF^\perp\cdot\nabla(u-U) + f(u-U), \partial_t u\rangle \\
& + \int \langle \triangle U + \nabla h\cdot\nabla U + iF^\perp\cdot\nabla U + fU, \partial_t u\rangle
\end{aligned}$$

and hence

$$\frac{1}{2}\partial_t \int |\nabla(u-U)|^2 + \frac{1}{4}\partial_t \int a(1-|u|^2)^2$$
$$\le -\alpha \int |\partial_t u|^2 + C\int |\partial_t u|(|u-U| + |\nabla(u-U)|) + C\int |\partial_t u|(|\triangle U| + (|\nabla h| + |F|)|\nabla U| + |f|)$$
$$\le -\frac{\alpha}{2}\int |\partial_t u|^2 + C + C\int |u-U|^2 + C\int |\nabla(u-U)|^2.$$

We may thus conclude

$$\frac{\alpha}{2}\int |\partial_t u|^2 + \partial_t \int \Big(\frac{1}{2}|\nabla(u-U)|^2 + \frac{a}{4}(1-|u|^2)^2 + \frac{1}{2}|u-U|^2\Big)$$
$$\le C + C\int \Big(\frac{1}{2}|\nabla(u-U)|^2 + \frac{a}{4}(1-|u|^2)^2 + \frac{1}{2}|u-U|^2\Big),$$

and the claim (A.9) follows from the Grönwall inequality.

Step 4: a useful change of variable. We now turn to the conservative case $\alpha = 0$. The first-order terms (that are forcing terms) in the right-hand side of (1.5) can then no longer be treated as errors, since the lost derivative is not retrieved by the Schrödinger operator. The proof of local existence in Step 1 can thus not be adapted to this case. The global estimates in Step 3 similarly fail, as there is no dissipation to absorb the first-order terms. To remedy this, we begin by performing a useful change of variables transforming first-order terms into zeroth-order ones, which are much easier to deal with. Since by assumption $\operatorname{div} F = 0$ with $F \in \mathrm{L}^\infty(\mathbb{R}^2)^2$, we deduce from a Hodge decomposition that there exists $\psi \in H^1_{\mathrm{loc}}(\mathbb{R}^2)$ such that $F = -2\nabla^\perp\psi$. Using the relation $a = e^h$, and setting $w_\varepsilon := \sqrt{a}u_\varepsilon e^{i|\log\varepsilon|\psi}$, a straightforward computation yields that the equation (1.5) for u_ε is equivalent to

$$\begin{cases} \lambda_\varepsilon(\alpha + i|\log\varepsilon|\beta)\partial_t w_\varepsilon = \triangle w_\varepsilon + \frac{w_\varepsilon}{\varepsilon^2}(a - |w_\varepsilon|^2) + (f_0 + ig_0)w_\varepsilon, & \text{in } \mathbb{R}^+\times\mathbb{R}^2, \\ w_\varepsilon|_{t=0} = w_\varepsilon^\circ := \sqrt{a}e^{i|\log\varepsilon|\psi}u_\varepsilon^\circ. \end{cases} \tag{A.10}$$

where we have set

$$f_0 := f - \frac{\triangle\sqrt{a}}{\sqrt{a}} + \frac{1}{4}|\log\varepsilon|^2|F|^2, \qquad g_0 := \frac{1}{2}|\log\varepsilon|a^{-1}\operatorname{curl}(aF).$$

We look for solutions w_ε of the above in the class $W + H^1(\mathbb{R}^2;\mathbb{C})$, for a "weighted reference map" W, that is an element of

$$E_k^a(\mathbb{R}^2) := \{W \in \mathrm{L}^\infty(\mathbb{R}^2;\mathbb{C}) : \nabla^2 W \in H^k(\mathbb{R}^2;\mathbb{C}), \nabla|W| \in \mathrm{L}^2(\mathbb{R}^2), a - |W|^2 \in \mathrm{L}^2(\mathbb{R}^2), \nabla W \in \mathrm{L}^p(\mathbb{R}^2;\mathbb{C})\ \forall p > 2\}.$$

For $k \ge 0$, and $\nabla h, \nabla\psi \in H^{k+1}(\mathbb{R}^2)^2$, we indeed observe that w_ε is a solution of (A.10) in $\mathrm{L}^\infty([0,T); W + H^{k+1}(\mathbb{R}^2;\mathbb{C}))$ for some $W \in E_k^a$ if and only if u_ε is a solution of (1.5) in $\mathrm{L}^\infty([0,T); U + H^{k+1}(\mathbb{R}^2;\mathbb{C}))$ for some $U \in E_k$.

Step 5: local existence for $\alpha = 0$. In this step, given $k \ge 0$, we assume $h \in W^{k+1,\infty}(\mathbb{R}^2)$, $\nabla h \in H^k(\mathbb{R}^2)^2$, $f_0, g_0 \in H^{k+1} \cap W^{k+1,\infty}(\mathbb{R}^2)$, and $w^\circ \in W + H^{k+1}(\mathbb{R}^2;\mathbb{C})$ for some $W \in E^a_{k+1}(\mathbb{R}^2)$, and we prove that there exists some $T > 0$ and a unique solution $w_\varepsilon \in \mathrm{L}^\infty([0,T); W + H^{k+1}(\mathbb{R}^2;\mathbb{C}))$ of (A.10) on $[0,T)\times\mathbb{R}^2$. To simplify notation, we replace equation (A.10) (with $\alpha = 0$) by its rescaled version

$$i\partial_t w = \triangle w + w(a - |w|^2) + (f_0 + ig_0)w, \qquad w|_{t=0} = w^\circ. \tag{A.11}$$

We begin with the case $k = 0$, and comment afterwards on the adaptations needed for $k \geq 1$. We argue by a fixed-point argument in the set $E_{W,w^\circ}(C_0, T) := \{w : \|w - W\|_{\mathrm{L}^\infty_T H^1} \leq C_0, w|_{t=0} = w^\circ\}$, for some $C_0, T > 0$ to be suitably chosen. We denote by $C \geq 1$ any constant that only depends on an upper bound on $\|\nabla h\|_{\mathrm{L}^2 \cap \mathrm{L}^\infty}$, $\|(f_0, g_0)\|_{H^1 \cap W^{1,\infty}}$, $\|(h, W)\|_{\mathrm{L}^\infty}$, $\|a - |W|^2\|_{\mathrm{L}^2}$, $\|\nabla |W|\|_{\mathrm{L}^2}$, and $\|\triangle W\|_{H^1}$, and we add a subscript to indicate dependence on further parameters.

Let S^t denote the kernel of the semigroup operator $e^{-it\triangle}$. Setting $\hat{w} := w - W$, we may rewrite equation (A.11) as follows:

$$i\partial_t \hat{w} = \triangle \hat{w} + \triangle W + (\hat{w} + W)(a - |W|^2) - 2(\hat{w} + W)\langle W, \hat{w}\rangle - (\hat{w} + W)|\hat{w}|^2 + (f_0 + ig_0)\hat{w} + (f_0 + ig_0)W,$$

with initial data $\hat{w}|_{t=0} = \hat{w}^\circ := w^\circ - W$. Any solution $\hat{w} \in \mathrm{L}^\infty([0,T); H^1(\mathbb{R}^2; \mathbb{C}))$ satisfies the Duhamel formula $\hat{w} = \Xi_{W,\hat{w}^\circ}(\hat{w})$, where we have set

$$\Xi_{W,\hat{w}^\circ}(\hat{w})^t := S^t * \hat{w}^\circ - i\int_0^t S^{t-s} * Z_{W,\hat{w}^\circ}(w^s)ds,$$

$$Z_{W,\hat{w}^\circ}(\hat{w}^s) := \triangle W + (\hat{w}^s + W)(a - |W|^2) - 2(\hat{w}^s + W)\langle W, \hat{w}^s\rangle - (\hat{w}^s + W)|\hat{w}^s|^2 + (f_0 + ig_0)\hat{w}^s + (f_0 + ig_0)W.$$

Similarly as in Step 1, we find $\|Z_{W,\hat{w}^\circ}(\hat{w}^s)\|_{\mathrm{L}^2} \leq C(1 + \|\hat{w}^s\|_{H^1}^3)$. On the other hand, arguing as in [9, Lemma 2] by means of various Sobolev embeddings, we have the following version of (A.7) without forcing: we may decompose $\nabla Z_{W,\hat{w}^\circ}(\hat{w}^s) = Z^1_{W,\hat{w}^\circ}(\hat{w}^s) + Z^2_{W,\hat{w}^\circ}(w^s)$, such that for all $1 < r < 2$

$$\|Z^1_{W,\hat{w}^\circ}(\hat{w}^s)\|_{\mathrm{L}^2} \leq C(1 + \|\hat{w}^s\|_{H^1}^3), \qquad \|Z^2_{W,\hat{w}^\circ}(\hat{w}^s)\|_{\mathrm{L}^r} \leq C_r(1 + \|\hat{w}^s\|_{H^1}^3). \tag{A.12}$$

(Recall that we cannot choose $r = 2$ above because of terms of the form e.g. $\|\hat{w}^s \nabla |\hat{w}^s|^2\|_{\mathrm{L}^r}$.) Let us now examine the map $\Xi_{W,\hat{w}^\circ}$ more closely. We have

$$\|\Xi_{W,\hat{w}^\circ}(\hat{w})^t\|_{H^1} \leq \|S^t * (\hat{w}^\circ, \nabla \hat{w}^\circ)\|_{\mathrm{L}^2} + \left\| \int_0^t e^{-i(t-s)\triangle}(Z_{W,\hat{w}^\circ}(\hat{w}^s), Z^1_{W,\hat{w}^\circ}(\hat{w}^s), Z^2_{W,\hat{w}^\circ}(\hat{w}^s))ds \right\|_{\mathrm{L}^2},$$

and hence by the Strichartz estimates for the Schrödinger operator [56], for any $1 < r \leq 2$,

$$\|\Xi_{W,\hat{w}^\circ}(\hat{w})\|_{\mathrm{L}^\infty_T H^1} \leq C\|\hat{w}^\circ\|_{H^1} + C\|(Z_{W,\hat{w}^\circ}(\hat{w}), Z^1_{W,\hat{w}^\circ}(\hat{w}))\|_{\mathrm{L}^1_T \mathrm{L}^2} + C_r\|Z^2_{W,\hat{w}^\circ}(\hat{w})\|_{\mathrm{L}^{2r/(3r-2)}_T \mathrm{L}^r}.$$

The above estimates for $Z_{W,\hat{w}^\circ}$ then yield for any $1 < r < 2$

$$\|\Xi_{W,\hat{w}^\circ}(\hat{w})\|_{\mathrm{L}^\infty_T H^1} \leq C\|\hat{w}^\circ\|_{H^1} + (CT + C_r T^{\frac{3}{2} - \frac{1}{r}})(1 + \|\hat{w}\|^3_{\mathrm{L}^\infty_T H^1}).$$

Choosing $r = 4/3$, this yields in particular, for all $\hat{w} \in -W + E_{W,\hat{w}^\circ}(C_0, T)$,

$$\|\Xi_{W,\hat{w}^\circ}(\hat{w})\|_{\mathrm{L}^\infty_T H^1} \leq C\|\hat{w}^\circ\|_{H^1} + C(T + T^{3/4})(1 + C_0^3).$$

Similarly, again using Sobolev embeddings and Strichartz estimates, we easily find for all $\hat{v}, \hat{w} \in -W + E_{W,\hat{w}^\circ}(C_0, T)$

$$\|\Xi_{W,\hat{w}^\circ}(\hat{v}) - \Xi_{W,\hat{w}^\circ}(\hat{w})\|_{\mathrm{L}^\infty_T H^1} \leq C(T + T^{3/4})(1 + C_0^2)\|\hat{v} - \hat{w}\|_{\mathrm{L}^\infty_T H^1}.$$

Choosing $C_0 := 1 + C\|\hat{w}^\circ\|_{H^1}$ and $T := 1 \wedge (4C(1 + C_0^3))^{-4/3}$, we may then deduce that $\Xi_{W,\hat{w}^\circ}$ maps the set $-W + E_{W,\hat{w}^\circ}(C_0, T)$ into itself, and is contracting on that set. The conclusion follows from a fixed-point argument.

Let us now briefly comment on the case $k \geq 1$ and explain how to adapt the above argument. We again proceed by a fixed point argument, estimating this time $\Xi_{W,\hat{w}^\circ}(\hat{w})$ hence $Z_{W,\hat{w}^\circ}(\hat{w})$ in $H^{k+1}(\mathbb{R}^2; \mathbb{C})$. Arguing similarly as e.g. in [63, Step 1 of the proof of Proposition A.8] by means of various Sobolev embeddings, we have the following version of (A.8) without forcing: for all $k \geq 1$,

$$\|\nabla^{k+1} Z_{W,\hat{w}^\circ}(\hat{w})\|_{\mathrm{L}^\infty_t(\mathrm{L}^2+\mathrm{L}^r)} \leq C_k(1 + \|\hat{w}\|^3_{\mathrm{L}^\infty_t H^{k+1}}), \tag{A.13}$$

for some constant $C_k \geq 1$ that only depends on an upper bound on k, $\|\nabla h\|_{H^k \cap W^{k,\infty}}$, $\|(f_0, g_0)\|_{H^{k+1}\cap W^{k+1,\infty}}$, $\|(h, W)\|_{\mathrm{L}^\infty}$, $\|a - |W|^2\|_{\mathrm{L}^2}$, $\|\nabla|W|\|_{\mathrm{L}^2}$, and $\|\nabla^2 W\|_{H^{k+1}}$. The result then easily follows as above.

Step 6: global existence for $\alpha = 0$. In this step, we assume $h \in \mathrm{L}^\infty(\mathbb{R}^2)$, $f_0 \in \mathrm{L}^2 \cap \mathrm{L}^\infty(\mathbb{R}^2)$, $g_0 \in H^1 \cap W^{1,\infty}(\mathbb{R}^2)$, and $w^\circ \in W + H^1(\mathbb{R}^2; \mathbb{C})$ for some $W \in E_0^a(\mathbb{R}^2)$, and we prove that (A.11) admits a unique global solution $w \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; W + H^1(\mathbb{R}^2; \mathbb{C}))$. We denote by $C > 0$ any constant that only depends on an upper bound on $\|h\|_{\mathrm{L}^\infty}$, $\|f_0\|_{\mathrm{L}^2\cap\mathrm{L}^\infty}$, $\|g_0\|_{H^1\cap W^{1,\infty}}$, $\|W\|_{\mathrm{L}^\infty}$, $\|1 - |W|^2\|_{\mathrm{L}^2}$, and $\|\triangle W\|_{\mathrm{L}^2}$.

Given a solution $w \in \mathrm{L}^\infty([0,T); W + H^1(\mathbb{R}^2; \mathbb{C}))$ of (A.11), we claim that the following a priori estimate holds for all $t \in [0,T)$

$$\int \Big(|\nabla(w^t - W)|^2 + \frac{1}{2}(a - |w^t|^2)^2 + |w^t - W|^2\Big) \leq Ce^{Ct}(1 + \|w^\circ - W\|^2_{H^1}). \tag{A.14}$$

Combining this with the local existence result of Step 5 in the space $W + H^1(\mathbb{R}^2; \mathbb{C})$, we deduce that local solutions can be extended globally in that space, and the result follows. So it remains to prove the claim (A.14). For simplicity, we assume in the computations below that $w \in \mathrm{L}^\infty([0,T); W + H^2(\mathbb{R}^2; \mathbb{C}))$, which in particular implies $\partial_t w \in \mathrm{L}^\infty([0,T); \mathrm{L}^2(\mathbb{R}^2; \mathbb{C}))$ by (A.11). The general result then follows from a simple approximation argument based on the local existence result of Step 5 in the space $W + H^2(\mathbb{R}^2; \mathbb{C})$.

Using equation (A.11), we compute the following time-derivative, suitably regrouping the terms and integrating by parts,

$$\begin{aligned}
\frac{1}{2}\partial_t \int |w - W|^2 &= \int \langle i(w - W), \triangle w + w(a - |w|^2) + f_0 w + i g_0 w\rangle \\
&= \int \langle i(w - W), \triangle W + W(a - |w|^2) + f_0 W + i g_0 W\rangle + \int g_0 |w - W|^2 \\
&\leq C + C\int |w - W|^2 + C\int (a - |w|^2)^2.
\end{aligned} \tag{A.15}$$

Likewise, we compute

$$\begin{aligned}
\partial_t \int |\nabla(w - W)|^2 &= 2\int \langle \nabla(w - W), \nabla \partial_t w\rangle \\
&= -2\int \langle \triangle(w - W), \partial_t w - g_0 w\rangle \\
&\qquad + 2\int \langle \nabla(w - W), g_0\nabla(w - W) + g_0 \nabla W + (w - W)\nabla g_0 + W\nabla g_0\rangle \\
&\leq -2\int \langle \triangle(w - W), \partial_t w - g_0 w\rangle + C + C\int |\nabla(w - W)|^2 + C\int |w - W|^2,
\end{aligned} \tag{A.16}$$

where we have

$$
\begin{aligned}
& -2\int \langle \triangle(w-W), \partial_t w - g_0 w\rangle \\
= & -2\int \langle i(\partial_t w - g_0 w) - w(a-|w|^2) - f_0 w - \triangle W, \partial_t w - g_0 w\rangle \\
= & \ 2\int \langle w(a-|w|^2) + f_0 w + \triangle W, \partial_t w - g_0 w\rangle \\
= & -\partial_t \int \Big(\frac{1}{2}(a-|w|^2)^2 - f_0|w|^2 - 2\langle \triangle W, w\rangle\Big) + 2\int g_0(a-|w|^2)^2 - 2\int g_0(a-|w|^2) \\
& \qquad\qquad - 2\int f_0 g_0 |w|^2 - 2\int g_0\langle \triangle W, w\rangle \\
\le & -\partial_t \int \Big(\frac{1}{2}(a-|w|^2)^2 - f_0|w-W|^2 - 2\langle w, \triangle W + f_0 W\rangle\Big) + C + C\int (a-|w|^2)^2 + C\int |w-W|^2.
\end{aligned}
$$

Combining this with (A.15) and (A.16), we obtain

$$
\begin{aligned}
\partial_t \int \Big((C-f_0)|w-W|^2 + |\nabla(w-W)|^2 + \frac{1}{2}(a-|w|^2)^2 - 2\langle w, \triangle W + f_0 W\rangle\Big) \\
\le C + C\int \big(|w-W|^2 + |\nabla(w-W)|^2 + (a-|w|^2)^2\big)
\end{aligned}
$$

and the result easily follows from the Grönwall inequality, choosing a large enough constant C in the left-hand side.

Step 7: propagation of regularity for $\alpha = 0$. In this step, given $k \ge 0$, we assume $h \in W^{k+1,\infty}(\mathbb{R}^2)$, $\nabla h \in H^k(\mathbb{R}^2)^2$, $f_0, g_0 \in H^{k+1} \cap W^{k+1,\infty}(\mathbb{R}^2)$, and $w^\circ \in W + H^{k+1}(\mathbb{R}^2;\mathbb{C})$ for some $W \in E^a_{k+1}(\mathbb{R}^2)$, and we prove that the global solution w of Step 6 belongs to $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; W + H^{k+1}(\mathbb{R}^2;\mathbb{C}))$. We denote by $C_k \ge 1$ any constant that only depends on an upper bound on k, $\|\nabla h\|_{H^k \cap W^{k,\infty}}$, $\|(f_0,g_0)\|_{H^{k+1}\cap W^{k+1,\infty}}$, $\|(h,W)\|_{\mathrm{L}^\infty}$, $\|a-|W|^2\|_{\mathrm{L}^2}$, $\|\nabla |W|\|_{\mathrm{L}^2}$, and $\|\nabla^2 W\|_{H^{k+1}}$. We add a subscript to indicate dependence on further parameters.

Let $w \in \mathrm{L}^\infty([0,T]; W + H^1(\mathbb{R}^2;\mathbb{C}))$ be a solution of (A.1), and let $\hat w := w - W$. We argue by induction: as the result is obvious for $k=0$, we assume that it holds for some $k \ge 0$ and we deduce that it then also holds for k replaced by $k+1$. By a similar argument as e.g. in [9, Lemma 4] or in [63, Step 1 of the proof of Proposition A.8], we have the following version of (A.8) without forcing (which generalizes (A.12) to higher derivatives): for all $k \ge 0$ we may decompose $\nabla^{k+1} Z_{W,\hat w^\circ}(\hat w^t) = \nabla^{k+1} Z^1_{W,\hat w^\circ}(\hat w^t) + \nabla^{k+1} Z^2_{W,\hat w^\circ}(w^t)$ such that for all $1 < r < 2$

$$
\|\nabla^{k+1} Z^1_{W,\hat w^\circ}(\hat w^t)\|_{\mathrm{L}^2} + \|\nabla^{k+1} Z^2_{W,\hat w^\circ}(\hat w^t)\|_{\mathrm{L}^r} \le C_{k,r}(1 + \|\hat w^t\|^3_{H^{k+1}}),
$$

or even more precisely,

$$
\|\nabla^{k+1} Z^1_{W,\hat w^\circ}(\hat w^t)\|_{\mathrm{L}^2} + \|\nabla^{k+1} Z^2_{W,\hat w^\circ}(\hat w^t)\|_{\mathrm{L}^r} \le C_{k,r}(1 + \|\hat w^t\|^2_{H^k})(1 + \|\hat w^t\|_{H^{k+1}}). \tag{A.17}
$$

Using Duhamel's formula $\hat w = \Xi_{W,\hat w^\circ}(\hat w)$ and applying the Strichartz estimates for the Schrödinger operator [56] as in Step 5, we find for all $k \ge 0$ and $1 < r \le 2$

$$
\begin{aligned}
\|\nabla^{k+1}\hat w^t\|_{\mathrm{L}^2} &\le \|S^t * \nabla^{k+1}\hat w^\circ\|_{\mathrm{L}^2} + \Big\|\int_0^t S^{t-s} * \nabla^{k+1} Z_{W,\hat w^\circ}(\hat w^s)ds\Big\|_{\mathrm{L}^2} \\
&\le C\|\nabla^{k+1}\hat w^\circ\|_{\mathrm{L}^2} + C\|\nabla^{k+1} Z^1_{W,\hat w^\circ}(\hat w)\|_{\mathrm{L}^1_t\mathrm{L}^2} + C_r\|\nabla^{k+1} Z^2_{W,\hat w^\circ}(\hat w)\|_{\mathrm{L}^{2r/(3r-2)}_t\mathrm{L}^r},
\end{aligned}
$$

and hence, by (A.17), for all $k \geq 0$,

$$\|\hat{w}^t\|_{H^{k+1}} \leq C_k \|\hat{w}^\circ\|_{H^{k+1}} + C_{k,r}(1+t)(1+\|\hat{w}\|^2_{\mathrm{L}^\infty_t H^k})(1+\|\hat{w}\|_{\mathrm{L}^{2r/(3r-2)}_t H^{k+1}}).$$

The result then follows from the induction hypothesis and the Grönwall inequality. $\square$

In the dissipative case, we now prove a well-posedness result for equation (1.5) in the general non-decaying setting, that is without decay assumption on the coefficients $\nabla h, F, f$. Since the forcing does not decay, subtle advection forces may occur at infinity, preventing the solution u_ε from staying in the same affine space $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; U + H^1(\mathbb{R}^2;\mathbb{C}))$ for any stationary reference map U. The well-posedness result below is therefore simply obtained in the space $\mathrm{L}^\infty(\mathbb{R}^+; H^1_{\mathrm{uloc}}(\mathbb{R}^2;\mathbb{C}))$, which yields no information at all on the behavior of the constructed solution at infinity. It is in particular completely unclear whether the total degree of the solution remains well-defined for positive times. In the proof below, the key observation is that the Grönwall argument for the energy in Step 3 of the proof of Proposition A.1 can be localized by means of an exponential cut-off. Note that the same argument does not seem applicable to the Gross-Pitaevskii case.

Proposition A.2 (Well-posedness for (1.5) — non-decaying setting). *Set $a := e^h$, with $h : \mathbb{R}^2 \to \mathbb{R}$. In the dissipative case $\alpha > 0$, $\beta \in \mathbb{R}$, given $h \in W^{1,\infty}(\mathbb{R}^2)$, $F \in \mathrm{L}^\infty(\mathbb{R}^2)^2$, $f \in \mathrm{L}^\infty(\mathbb{R}^2)$, and $u_\varepsilon^\circ \in H^1_{\mathrm{uloc}}(\mathbb{R}^2;\mathbb{C})$, there exists a unique global solution $u_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^1_{\mathrm{uloc}}(\mathbb{R}^2;\mathbb{C}))$ of (1.5) in $\mathbb{R}^+ \times \mathbb{R}^2$ with initial data u_ε°, and this solution satisfies $\partial_t u_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^2_{\mathrm{uloc}}(\mathbb{R}^2;\mathbb{C}))$. Moreover, if for some $k \geq 0$ we have $h \in W^{k+1,\infty}(\mathbb{R}^2)$, $F \in W^{k,\infty}(\mathbb{R}^2)^2$, $f \in W^{k,\infty}(\mathbb{R}^2)$, and $u_\varepsilon^\circ \in H^{k+1}_{\mathrm{uloc}}(\mathbb{R}^2;\mathbb{C})$, then $u_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^{k+1}_{\mathrm{uloc}}(\mathbb{R}^2;\mathbb{C}))$ and $\partial_t u_\varepsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^k_{\mathrm{uloc}}(\mathbb{R}^2;\mathbb{C}))$.*

Proof. We split the proof into four steps. We denote by $\xi^z(x) := e^{-|x-z|}$ the exponential cut-off centered at $z \in R\mathbb{Z}^2$, and $\xi(x) := \xi^0(x) = e^{-|x|}$. To simplify notation, we replace equation (1.5) by its rescaled version

$$(\alpha + i\beta)\partial_t u = \triangle u + au(1-|u|^2) + \nabla h \cdot \nabla u + iF^\perp \cdot \nabla u + fu, \qquad u|_{t=0} = u^\circ. \tag{A.18}$$

Step 1: global existence with $k = 0$. In this step, we assume $h \in W^{1,\infty}(\mathbb{R}^2)$, $F \in \mathrm{L}^\infty(\mathbb{R}^2)^2$, $f \in \mathrm{L}^\infty(\mathbb{R}^2)$, and $u^\circ \in H^1_{\mathrm{uloc}}(\mathbb{R}^2;\mathbb{C})$, and we prove that there exists a global solution $u \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^1_{\mathrm{uloc}}(\mathbb{R}^2;\mathbb{C}))$ of (A.18) on $\mathbb{R}^+ \times \mathbb{R}^2$ with initial data u°. We denote by $C \geq 1$ any constant that only depends on an upper bound on α, α^{-1}, $|\beta|$, $\|(h, \nabla h, F, f)\|_{\mathrm{L}^\infty}$, and $\|u^\circ\|_{H^1_{\mathrm{uloc}}}$.

We argue by approximation: for all $n \geq 1$, we let $\chi_n := \chi(\cdot/n)$ for some fixed cut-off function χ with $\chi|_{B_1} \equiv 1$ and $\chi|_{\mathbb{R}^2 \setminus B_2} \equiv 0$, and we set $h_n := \chi_n h$, $a_n := e^{h_n}$, $F_n := \chi_n F$, and $f_n := \chi_n f$. Note that by construction $\|(h_n, \nabla h_n, F_n, f_n)\|_{\mathrm{L}^\infty} \leq C$. We also need to approximate the initial data $u^\circ \in H^1_{\mathrm{uloc}}(\mathbb{R}^2;\mathbb{C})$: for all $n \geq 1$, we let $\rho_n := n^2\rho(nx)$ for some $\rho \in C^\infty_c(\mathbb{R}^2)$ with $\int \rho = 1$, and we set $u_n^\circ := \chi_n(u^\circ * \rho_n) + 1 - \chi_n$. By definition, we have $u_n^\circ \in E_0$, the sequence $(u_n^\circ)_n$ is bounded in $H^1_{\mathrm{uloc}}(\mathbb{R}^2;\mathbb{C})$, and as $n \uparrow \infty$ we obtain $u_n^\circ \to u^\circ$ in $H^1_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{C})$, and $a_n \to a$, $\nabla h_n \to \nabla h$, and $F_n \to F$ in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^2)^2$. By Proposition A.1, there exists a unique global solution $u_n \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; U + H^1(\mathbb{R}^2;\mathbb{C}))$ of the following truncated equation on $\mathbb{R}^+ \times \mathbb{R}^2$,

$$(\alpha + i\beta)\partial_t u_n = \triangle u_n + a_n u_n(1-|u_n|^2) + \nabla h_n \cdot \nabla u_n + iF_n^\perp \cdot \nabla u_n + f_n u_n, \qquad u_n|_{t=0} = u_n^\circ. \tag{A.19}$$

In order to pass to the limit $n \uparrow \infty$ in (the weak formulation of) this equation, we prove the boundedness of the sequence $(u_n)_n$ in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^1_{\mathrm{uloc}}(\mathbb{R}^2;\mathbb{C}))$, that is, we claim that the following a priori estimate holds for all $t \geq 0$,

$$\|u_n^t\|_{H^1_{\mathrm{uloc}}} \leq \sup_z \|u_n^t\|_{H^1(B(z))} + \alpha^{1/2} \sup_z \|\partial_t u_n\|_{\mathrm{L}^2_t \mathrm{L}^2(B(z))} \leq Ce^{Ct}. \tag{A.20}$$

Before proving this estimate, we show how to conclude from this. Up to a subsequence, the sequence u_n converges weakly-* to some u in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^1_{\mathrm{uloc}}(\mathbb{R}^2;\mathbb{C}))$. Since moreover $\partial_t u_n$ is bounded in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^2(B(z);\mathbb{C}))$, uniformly in z, and as $H^1(B(z);\mathbb{C})$ is compactly embedded into $\mathrm{L}^2(B(z);\mathbb{C})$, we deduce from the Aubin-Simon lemma that $u_n \to u$ strongly in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^1_{\mathrm{uloc}}(\mathbb{R}^2;\mathbb{C}))$. This allows to pass to the limit in the weak formulation of equation (A.19), and deduce that the limit u is a global solution of (A.18) on $\mathbb{R}^+\times\mathbb{R}^2$ with initial data u°.

It remains to prove (A.20). We set for simplicity $(\alpha+i\beta)^{-1}=\alpha'+i\beta'$, $\alpha'>0$. Using equation (A.19), integrating by parts, and using $|\nabla\xi^z|\le\xi^z$, we compute the following time-derivative, for all $z\in R\mathbb{Z}^2$,

$$\begin{aligned}\frac12\partial_t\int\xi^z|u_n|^2&=\int\xi^z\langle u_n,(\alpha'+i\beta')(\triangle u_n+a_nu_n(1-|u_n|^2)+\nabla h_n\cdot\nabla u_n+iF_n^\perp\cdot\nabla u_n+f_nu_n)\rangle\\&\le\int\xi^z\langle u_n,(\alpha'+i\beta')\triangle u_n\rangle+\alpha'\int a_n\xi^z|u_n|^2(1-|u_n|^2)+C\int\xi^z|u_n||\nabla u_n|+C\int\xi^z|u_n|^2\\&\le-\alpha'\int\xi^z|\nabla u_n|^2+C\int\xi^z|u_n||\nabla u_n|+C\int\xi^z|u_n|^2,\end{aligned}$$

and hence

$$\frac12\partial_t\int\xi^z|u_n|^2\le-\frac{\alpha'}{2}\int\xi^z|\nabla u_n|^2+C\int\xi^z|u_n|^2.$$

On the other hand, integration by parts yields

$$\frac12\partial_t\int\xi^z|\nabla u_n|^2=\int\xi^z\langle\nabla u_n,\nabla\partial_tu_n\rangle=-\int\xi^z\langle\triangle u_n,\partial_tu_n\rangle-\int\nabla\xi^z\cdot\langle\nabla u_n,\partial_tu_n\rangle,$$

hence, inserting equation (A.19) in the first right-hand side term,

$$\begin{aligned}&\frac12\partial_t\int\xi^z|\nabla u_n|^2\\&=\ -\int\xi^z\langle(\alpha+i\beta)\partial_tu_n-a_nu_n(1-|u_n|^2)-\nabla h_n\cdot\nabla u_n-iF_n^\perp\cdot\nabla u_n-f_nu_n,\partial_tu_n\rangle-\int\nabla\xi^z\cdot\langle\nabla u_n,\partial_tu_n\rangle\\&\le\ -\alpha\int\xi^z|\partial_tu_n|^2-\frac14\partial_t\int a_n\xi^z(1-|u_n|^2)^2+C\int\xi^z(|u_n|+|\nabla u_n|)|\partial_tu_n|,\end{aligned}$$

and thus

$$\frac12\partial_t\int\xi^z|\nabla u_n|^2+\frac14\partial_t\int a_n\xi^z(1-|u_n|^2)^2\le-\frac{\alpha}{2}\int\xi^z|\partial_tu_n|^2+C\int\xi^z(|u_n|^2+|\nabla u_n|^2).$$

We may then conclude

$$\frac12\partial_t\int\xi^z(|u_n|^2+|\nabla u_n|^2)+\frac14\partial_t\int a_n\xi^z(1-|u_n|^2)^2+\frac{\alpha}{2}\int\xi^z|\partial_tu_n|^2\le C\int\xi^z(|u_n|^2+|\nabla u_n|^2).$$

By the Grönwall inequality, this yields for all $t\ge0$ and $z\in R\mathbb{Z}^2$,

$$\begin{aligned}&\int\xi^z(|u_n^t|^2+|\nabla u_n^t|^2)+\frac12\int a_n\xi^z(1-|u_n^t|^2)^2+\alpha\int_0^t\int\xi^z|\partial_tu_n|^2\\&\qquad\qquad\le e^{Ct}\Big(\int\xi^z(|u_n^\circ|^2+|\nabla u_n^\circ|^2)+\frac12\int a_n\xi^z(1-|u_n^\circ|^2)^2\Big),\end{aligned}$$

and hence, using the Sobolev embedding of $H^1_{\mathrm{uloc}}(\mathbb{R}^2)$ into $\mathrm{L}^4_{\mathrm{uloc}}(\mathbb{R}^2)$ (see e.g. (A.23) below),

$$\int\xi^z(|u_n^t|^2+|\nabla u_n^t|^2)+\frac12\int a_n\xi^z(1-|u_n^t|^2)^2+\alpha\int_0^t\int\xi^z|\partial_tu_n|^2\le e^{Ct}\Big(1+\int\xi^z(|u_n^\circ|^2+|\nabla u_n^\circ|^2)\Big)^2.$$

The claim (A.20) then follows from the boundedness of u_n° in $H^1_{\mathrm{uloc}}(\mathbb{R}^2;\mathbb{C})$, noting that

$$\|\zeta\|^2_{\mathrm{L}^2_{\mathrm{uloc}}} \simeq \sup_{z\in\mathbb{R}^2} \int \xi^z|\zeta|^2. \tag{A.21}$$

Step 2: global existence with $k\geq 0$. In this step, given $k\geq 0$, we assume $h\in W^{k+1,\infty}(\mathbb{R}^2)$, $F\in W^{k,\infty}(\mathbb{R}^2)^2$, $f\in W^{k,\infty}(\mathbb{R}^2)$, and $u^\circ\in H^{k+1}_{\mathrm{uloc}}(\mathbb{R}^2;\mathbb{C})$, and we prove that the global solution u constructed in Step 1 then belongs to $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^{k+1}_{\mathrm{uloc}}(\mathbb{R}^2;\mathbb{C}))$. We denote by $C_k\geq 1$ any constant that only depends on an upper bound on k, α, α^{-1}, $|\beta|$, $\|(h,\nabla h,F,f)\|_{W^{k,\infty}}$, and $\|u^\circ\|_{H^{k+1}_{\mathrm{uloc}}}$, and we write $C_{k,t}$ if it additionally depends on an upper bound on t.

We argue again by approximation. We consider the truncations $h_n,a_n,F_n,f_n,u_n^\circ$ defined in Step 1, as well as the solution u_n to the corresponding equation (A.19). We claim that for all $k\geq 0$, for all $t\geq 0$,

$$\|u_n^t\|_{H^{k+1}_{\mathrm{uloc}}}+\|\partial_t u_n^t\|_{H^k_{\mathrm{uloc}}}\leq C_{k,t}. \tag{A.22}$$

The desired result then follows by passing to the limit $n\uparrow\infty$. This result is proved by induction on k. As for $k=0$ the result already follows from Step 1, we assume that $\|u_n^t\|_{H^k_{\mathrm{uloc}}}\leq C_{k,t}$ holds for some $k\geq 1$, and we deduce that (A.22) also holds for this k. Integrating by parts, we find

$$\frac12\partial_t\int\xi^z|\nabla^{k+1}u_n|^2=\int\xi^z\langle\nabla^{k+1}u_n,\nabla^{k+1}\partial_t u_n\rangle\leq C\int\xi^z|\nabla^{k+1}u_n||\nabla^k\partial_t u_n|-\int\xi^z\langle\nabla^k\triangle u_n,\nabla^k\partial_t u_n\rangle,$$

hence, inserting equation (A.19) in the first right-hand side term, and developing the terms,

$$\begin{aligned}
\frac12\partial_t\int\xi^z|\nabla^{k+1}u_n|^2 &\leq -\alpha\int\xi^z|\nabla^k\partial_t u_n|^2+C\int\xi^z|\nabla^{k+1}u_n||\nabla^k\partial_t u_n|\\
&\qquad+\int\xi^z\big\langle\nabla^k\big(a_nu_n(1-|u_n|^2)+\nabla h_n\cdot\nabla u_n+iF_n^\perp\cdot\nabla u_n+f_nu_n\big),\nabla^k\partial_t u_n\big\rangle\\
&\leq -\alpha\int\xi^z|\nabla^k\partial_t u_n|^2+C_k\sum_{j=0}^{k+1}\int\xi^z|\nabla^j u_n||\nabla^k\partial_t u_n|+C_k\sum_{j=0}^{k-1}\int\xi^z|\nabla^j u_n|^3|\nabla^k\partial_t u_n|\\
&\qquad+C\int\xi^z|u_n|^2|\nabla^k u_n||\nabla^k\partial_t u_n|\\
&\leq -\frac{\alpha}{2}\int\xi^z|\nabla^k\partial_t u_n|^2+C_k\sum_{j=0}^{k+1}\int\xi^z|\nabla^j u_n|^2+C_k\sum_{j=0}^{k-1}\int\xi^z|\nabla^j u_n|^6+C\int\xi^z|u_n|^4|\nabla^k u_n|^2.
\end{aligned}$$

Note that the Sobolev embedding in the balls $B_2(x)$ yields

$$\begin{aligned}
\int\xi^z|\nabla^j u_n|^6 &\lesssim \sum_{x\in\mathbb{Z}^2}\xi^z(x)\int_{B_2(x)}|\nabla^j u_n|^6\\
&\lesssim \sum_{x\in\mathbb{Z}^2}\xi^z(x)\Big(\int_{B_2(x)}(|\nabla^j u_n|^2+|\nabla^{j+1}u_n|^2)\Big)^3\\
&\lesssim \Big(\sum_{x\in\mathbb{Z}^2}\xi^z(x)\int_{B_2(x)}(|\nabla^j u_n|^2+|\nabla^{j+1}u_n|^2)\Big)^3\lesssim\Big(\int\xi^z(|\nabla^j u_n|^2+|\nabla^{j+1}u_n|^2)\Big)^3,
\end{aligned}\tag{A.23}$$

and similarly

$$\int\xi^z|u_n|^4|\nabla^k u_n|^2\leq\Big(\int\xi^z|u_n|^8\Big)^{1/2}\Big(\int\xi^z|\nabla^k u_n|^4\Big)^{1/2}\lesssim\Big(\int\xi^z|\nabla u_n|^2\Big)^2\Big(\int\xi^z(|\nabla^k u_n|^2+|\nabla^{k+1}u_n|^2)\Big).$$

Inserting these estimates in the above, and using (A.21), we obtain

$$\partial_t \int \xi^z |\nabla^{k+1} u_n|^2 + \alpha \int \xi^z |\nabla^k \partial_t u_n|^2 \le C_k \sum_{j=0}^{k} \Big(1 + \int \xi^z |\nabla^j u_n|^2\Big)^3 + C_k \Big(1 + \int \xi^z |\nabla u_n|^2\Big)^2 \int \xi^z |\nabla^{k+1} u_n|^2$$
$$\le C_k \big(1 + \|u_n\|^6_{H^k_{\mathrm{uloc}}}\big) + C_k \big(1 + \|u_n\|^4_{H^1_{\mathrm{uloc}}}\big) \int \xi^z |\nabla^{k+1} u_n|^2.$$

By the induction hypothesis, we deduce for all $t \ge 0$

$$\partial_t \int \xi^z |\nabla^{k+1} u_n^t|^2 + \alpha \int \xi^z |\nabla^k \partial_t u_n^t|^2 \le C_{k,t} + C_{k,t} \int \xi^z |\nabla^{k+1} u_n^t|^2,$$

and the result (A.22) follows from the Grönwall inequality, taking the supremum over z.

Step 3: uniqueness. In this step, we assume $h \in W^{1,\infty}(\mathbb{R}^2)$, $F \in \mathrm{L}^\infty(\mathbb{R}^2)^2$, and $f \in \mathrm{L}^\infty(\mathbb{R}^2)$, and we prove that there exists at most one global solution $u \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^1_{\mathrm{uloc}}(\mathbb{R}^2;\mathbb{C}))$ of (A.18) on $\mathbb{R}^+ \times \mathbb{R}^2$ with given initial data u°. We denote by $C \ge 1$ any constant that only depends on an upper bound on α, α^{-1}, $|\beta|$, and $\|(h, \nabla h, F, f)\|_{\mathrm{L}^\infty}$.

Let $u_1, u_2 \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^1_{\mathrm{uloc}}(\mathbb{R}^2;\mathbb{C}))$ denote two solutions as above. We set for simplicity $(\alpha + i\beta)^{-1} = \alpha' + i\beta'$, $\alpha' > 0$. Using equation (A.18) and integrating by parts, we find

$$\frac{1}{2}\partial_t \int \xi^z |u_1 - u_2|^2 \le -\alpha' \int \xi^z |\nabla(u_1 - u_2)|^2 + C \int \xi^z |u_1 - u_2||\nabla(u_1 - u_2)| + C \int \xi^z |u_1 - u_2|^2$$
$$+ \int a\xi^z \big\langle u_1 - u_2, (\alpha' + i\beta')\big(u_1(1 - |u_1|^2) - u_2(1 - |u_2|^2)\big)\big\rangle$$
$$\le -\frac{\alpha'}{2} \int \xi^z |\nabla(u_1 - u_2)|^2 + C \int \xi^z |u_1 - u_2|^2 (1 + |u_1| + |u_2|)^2. \qquad \text{(A.24)}$$

It remains to estimate the last integral. For that purpose, we decompose

$$\int \xi^z |u_1 - u_2|^2 (|u_1| + |u_2|)^2 \lesssim \sum_{x \in \mathbb{Z}^2} \xi^z(x) \int_{B_2(x)} |u_1 - u_2|^2 (|u_1| + |u_2|)^2$$
$$\lesssim \sum_{x \in \mathbb{Z}^2} \xi^z(x) \Big(\int_{B_2(x)} |u_1 - u_2|^4\Big)^{1/2} \Big(\int_{B_2(x)} (|u_1| + |u_2|)^4\Big)^{1/2},$$

hence, using the Sobolev embedding of $H^{3/4}(B_2(x))$ (and of $H^1(B_2(x))$) into $\mathrm{L}^4(B_2(x))$,

$$\int \xi^z |u_1 - u_2|^2 (|u_1| + |u_2|)^2 \lesssim \|(u_1, u_2)\|^2_{H^1_{\mathrm{uloc}}} \sum_{x \in \mathbb{Z}^2} \xi^z(x) \|u_1 - u_2\|^2_{H^{3/4}(B_2(x))}.$$

Using interpolation and Young's inequality then yields, for any $K \ge 1$,

$$\int \xi^z |u_1 - u_2|^2 (|u_1| + |u_2|)^2 \lesssim \|(u_1, u_2)\|^2_{H^1_{\mathrm{uloc}}} \sum_{x \in \mathbb{Z}^2} \xi^z(x) \|u_1 - u_2\|^{3/2}_{H^1(B_2(x))} \|u_1 - u_2\|^{1/2}_{\mathrm{L}^2(B_2(x))}$$
$$\lesssim K^{-1} \sum_{x \in \mathbb{Z}^2} \xi^z(x) \int_{B_2(x)} |\nabla(u_1 - u_2)|^2 + K^3 (1 + \|(u_1, u_2)\|^8_{H^1_{\mathrm{uloc}}}) \sum_{x \in \mathbb{Z}^2} \xi^z(x) \int_{B_2(x)} |u_1 - u_2|^2$$
$$\lesssim K^{-1} \int \xi^z |\nabla(u_1 - u_2)|^2 + K^3 (1 + \|(u_1, u_2)\|^8_{H^1_{\mathrm{uloc}}}) \int \xi^z |u_1 - u_2|^2.$$

Inserting this into (A.24) with $K \simeq 1$ large enough, we find

$$\frac{1}{2}\partial_t \int \xi^z |u_1 - u_2|^2 \le C\big(1 + \|(u_1, u_2)\|^8_{H^1_{\mathrm{uloc}}}\big) \int \xi^z |u_1 - u_2|^2,$$

and the conclusion $u_1 = u_2$ follows from the Grönwall inequality. $\square$

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