

# A Mean Field Equation as Limit of Nonlinear Diffusions with Fractional Laplacian Operators

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## Abstract

In the limit of a nonlinear diffusion model involving the fractional Laplacian we get a “mean field” equation arising in superconductivity and superfluidity. For this equation, we obtain uniqueness, universal bounds and regularity results. We also show that solutions with finite second moment and radial solutions admit an asymptotic large time limiting profile which is a special self-similar solution: the “elementary vortex patch”.

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## 1 Introduction

We consider the class of evolution models with nonlocal effects given by the system

$$(1.1) \quad u_t = \nabla \cdot (u \nabla p), \quad p = \mathcal{K}u.$$

where  $u$  is a function of the variables  $(x, t)$  to be thought of as a density or concentration, and therefore nonnegative, while  $p$  is the “pressure” and  $\mathbf{v} = -\nabla p$  represent the mass propagation speed. We postulate that  $p$  is related to  $u$  via a linear integral operator  $\mathcal{K}$  which we assume in practice to be the inverse of a fractional Laplacian, that is,  $\mathcal{K} = (-\Delta)^{-s}$ , with  $s \in (0, 1]$ . More general integral operators are also now being considered in the literature.

The problem is posed for  $t > 0$  and  $x \in \mathbb{R}^n$ ,  $n \geq 2$ . (The case  $n = 1$  is easier and a bit different, and a short explanation will be given below). Finally, we give initial conditions

$$(1.2) \quad u(x, 0) = u_0(x), \quad x \in \mathbb{R}^n,$$

where  $u_0$  is a nonnegative integrable function in  $\mathbb{R}^n$ .

NONLOCAL DIFFUSIVE MODELS. These correspond to the equation

$$(1.3) \quad u_t = \nabla \cdot (u \nabla (-\Delta)^{-s} u),$$

in the parameter range  $0 < s < 1$ . This equation can be considered as a nonlocal version of the standard porous medium equation  $u_t = \Delta(u^2)$ . For convenience, we will often refer to it as the FPME (fractional porous medium equation).

The FPME has been derived starting from the continuity equation

$$(1.4) \quad \partial_t u + \nabla \cdot (u \mathbf{v}) = 0,$$

where  $u$  is a density and  $\mathbf{v}$  is a velocity field, which according to Darcy's law derives from a pressure,  $\mathbf{v} = -\nabla p$  (as in the theory of gases in porous media). Physical applications range from macroscopic evolution of particle systems with short- and long-range interactions [33], to phase segregation [31, 32], and dislocation dynamics [35, 15]. The existence of a weak solution, the relevant integral estimates, as well as the property of compact support have been established by Caffarelli and the second author in the paper [20]. The existence is shown by passing to the limit in the sequence of smooth solutions of approximate problems, and using the compactness of the integral operators involved in the weak formulation and its energy estimates for  $s < 1$  (thus the method does not directly apply to  $s = 1$ ). Note that no uniqueness is proven, and examples are constructed where the maximum principle does not hold for  $s > 1/2$ .

HYDRODYNAMIC OR MEAN FIELD EQUATION. The case  $s = 1$  of equation (1.1)–(1.2)

$$(1.5) \quad u_t = \nabla \cdot (u \nabla p), \quad p = (-\Delta)^{-1} u,$$

supplemented with initial data:

$$(1.6) \quad u(x, 0) = u^0(x),$$

is no more a diffusion equation. This equation was first studied in  $\mathbb{R}^2$  by Lin-Zhang [44]. There, existence is proven by a vortex point approximation, and uniqueness by estimates in Zygmund spaces. In dimension 2, the equation is directly related to the Chapman-Rubinstein-Schatzman mean field model of superconductivity [23] and to E's model of superfluidity [26], which would correspond rather to the equation

$$(1.7) \quad u_t = \nabla \cdot (|u| \nabla p)$$

which coincides with (1.5) when  $u \geq 0$ . That model is a mean field model for the motion of vortices in a superconductor in the Ginzburg-Landau theory. There,  $u$  represents the local vortex-density, and  $p$  represents the induced magnetic field in the sample. In this application the equation is really posed in a bounded domain and for signed ‘‘vorticity measures’’  $u$  (which can be measures instead of functions), but the study of (1.5) in  $\mathbb{R}^2$  with  $u \geq 0$ , corresponding to the situation where the vorticity is positive, is a reasonable start. It can also be viewed as the gradient-flow version of the Euler equation in vorticity form i.e.

$$u_t = \nabla \cdot (u \nabla^\perp p) \quad p = (-\Delta)^{-1} u,$$

hence also the term hydrodynamic equation. In [2], Ambrosio and the first author studied the model in a bounded two-dimensional domain (taking into account the possibility of flux of vortices through the boundary), via a gradient flow approach: the equation is the gradient-flow for a natural energy (see below) for the 2-Wasserstein distance. There,  $u$  is denoted  $\mu$ , and  $p$  is denoted  $h$ . Solutions with  $u$  measure (and not  $L^1$ ) are also treated there. The equation (1.7) with changing sign  $u$  has been studied by Masmoudi-Zhang [49] in  $\mathbb{R}^2$ , Ambrosio-Mainini-Serfaty [3] in bounded domains of the plane and in the whole plane, again using a gradient-flow approach.

This equation is also related to nonlocal aggregation models, which have attracted a lot of attention lately, notably in conjunction with the Keller-Segel model [37], see e.g. [10, 11, 12]. They correspond to (1.1) with  $\mathcal{K}$  a general integral operator corresponding to an attractive (instead of repulsive) interaction. Particularly relevant to us is the paper of Bertozzi-Laurent-Léger [12] which deals mostly with aggregation via the Coulomb kernel, but contains a section (Section 3) that deals with the repulsive case, i.e. exactly the equation (1.5)-(1.6). In addition to existence and uniqueness results, it makes a detailed analysis of the property of support propagation and asymptotic behaviour for solutions with compactly supported initial data.

**OUTLINE OF RESULTS.** In the present paper we are interested in obtaining Equation (1.5) as the limit of the nonlocal diffusive equations (1.3) when  $s \rightarrow 1$ ; and in expanding the theory for the limiting equation. We consider the initial-value problem posed in  $\mathbb{R}^n$ , for all dimensions  $n \geq 2$ , with comments on the easier case  $n = 1$ . We develop an existence theory for weak solutions with a nonnegative finite measure as initial data. Such solutions are uniformly bounded for positive times in the form  $|u(x, t)| \leq 1/t$ . We prove a uniqueness result for bounded solutions.

As another important feature of the theory, we derive the support propagation properties (for general solutions), and compare the result with a similar property for the fractional diffusion approximation.

Some insight is obtained with the introduction of the special formulation of the equation under conditions of radial symmetry, leading to a non-homogeneous Burgers' equation for the so-called mass function. With this Burgers' equation approach, we recover the existence of a fundamental solution in the form of a round vortex patch of constant (in space) density, and show that they are attractors for the evolution of all radial solutions. We also find other explicit radial solutions that allow to derive a new non-comparison result.

We also study the question of large time behaviour for another class of initial data where the restriction of compact support is dropped, more precisely integrable data with finite second moment. The method, which is interesting for its own sake, is an energy method that follows the now well-known entropy - entropy dissipation approach, see [21] and references.

**PLAN OF THE PAPER.** The paper is organized as follows: Section 2 is devoted to the definition of solutions for  $s < 1$  and  $s = 1$  and to recalling some of their properties (conserved quantities, etc). In Section 3 we show the existence of solutions to (1.5) by taking the limit  $s \rightarrow 1$  in solutions to (1.1). Section 4 is devoted to the proof of uniqueness for bounded solutions. Section 5 contains the universal decay estimate for the solutions. In Section 6 we investigate the theory in the class of radial solutions via the Burgers' equation approach. In Section 7 we show that the compact support property is preserved during the evolution, see also [12]. Section 8 discusses the question of large time behaviour for two classes of initial data where the restriction of compact

support is dropped: (i) integrable data with finite second moment, (ii) all solutions that are radially symmetric in the space variable.

OPEN PROBLEMS. Here is list of the main problems that remain open: obtaining uniqueness results beyond the class of bounded solutions, analyzing the long-term behaviour of all solutions with integrable data, and finally treating the case of sign changing solutions.

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## 2 Definitions and first properties of the solutions

It will be convenient to write for any  $s \in (0, 1]$  the equation in the form

$$(2.1) \quad \partial_t u = \nabla \cdot (u \nabla \mathcal{K}_s u), \quad \mathcal{K}_s = (-\Delta)^{-s}.$$

$\mathcal{K}_s$  is a positive essentially self-adjoint operator. For convenience, we introduce  $\mathcal{H}_s = \mathcal{K}_s^{1/2} = (-\Delta)^{-s/2}$ . We write  $p = \mathcal{K}_s u$ , so that  $u = (-\Delta)^s p$  and  $\mathcal{H}_s u = (-\Delta)^{s/2} p$ .

We start with the definition of weak solutions. The notation  $\mathcal{M}^+(\mathbb{R}^n)$  denotes the space of positive Radon measures on  $\mathbb{R}^n$  with finite mass.

**Definition 2.1.** *For any  $s \in (0, 1]$ , a weak solution of (2.1) in  $Q_T = \mathbb{R}^n \times (0, T)$  is a nonnegative function  $u(x, t)$  such that  $u \in L^1(Q_T)$ ,  $p = \mathcal{K}_s u \in L^1((\tau, T), W_{loc}^{1,1}(\mathbb{R}^n))$ ,  $u \nabla p \in L^1((\tau, T) \times \mathbb{R}^n)$  for all  $\tau > 0$ , and the identity*

$$(2.2) \quad \iint (u \phi_t - u \nabla p \cdot \nabla \phi) dx dt = 0$$

*holds for all continuously differentiable test functions  $\phi$  compactly supported in  $Q = \mathbb{R}^n \times (0, T)$ . If  $s < 1$  we also assume that  $u$  is continuous in  $Q_T$ .*

**Definition 2.2.** *For any  $s \in (0, 1]$ , a weak solution of the problem formed by equation (2.1) in  $Q_T = \mathbb{R}^n \times (0, T)$  with initial data  $\mu \in \mathcal{M}^+(\mathbb{R}^n)$  is a continuous nonnegative function  $u(x, t)$  with  $u \in L^1(Q_T)$ ,  $p = \mathcal{K}_s u \in L^1((0, T), W_{loc}^{1,1}(\mathbb{R}^n))$ ,  $u \nabla p \in L^1(Q_T)$ , and the identity*

$$(2.3) \quad \iint (u \phi_t - u \nabla p \cdot \nabla \phi) dx dt + \int \phi(x, 0) d\mu(x) = 0$$

*holds for all continuously differentiable test functions  $\phi$  in  $\mathbb{R}^n \times [0, \infty)$  such that  $\phi$  has compact support in the space variable and vanishes for  $t \geq T$ .*

In practice the solutions obtained below have much better properties that can be exploited in the analysis. In particular they will be uniformly bounded for  $t \geq \tau > 0$  and satisfy energy inequalities. Following [44] we may call such improved solutions *dissipative solutions*.

## 2.1 Properties of the solutions in the case $s < 1$

The existence of weak solutions in the sense of Definition 2.2 is proved in [20] under the assumption that the initial data is actually a bounded function  $u_0 \geq 0$  decaying exponentially as  $|x| \rightarrow \infty$ , cf. Theorem 4.1 of [20]. It is extended to general nonnegative  $L^1$  initial data in [19]. Let us list the main properties of those solutions with attention to the way they depend on  $s \in (0, 1)$ . We point out that we do not use all of them in the sequel.

- CONSERVATION OF MASS: The solutions exist globally in time and for every  $t > 0$  we have

$$(2.4) \quad \int u(x, t) dx = \int u_0(x) dx.$$

- CONSERVATION OF POSITIVITY:  $u_0 \geq 0$  implies that  $u(\cdot, t) \geq 0$  for all times.

-  $L^p$  ESTIMATES: the  $L^p$  norm of the constructed solutions does not increase in time, for any  $1 \leq p \leq \infty$ .

- A general COMPARISON THEOREM, i. e., a form of the usual maximum principle, is proven to be false by constructing counterexamples for  $s > 1/2$ , see [20]. However, comparison of solutions for an integrated version of the equation is established in dimension  $n = 1$  by the method of viscosity solutions in Biler-Karch-Monneau [15]. We will return to that issue below.

- FINITE SPEED OF PROPAGATION. If the initial function has compact support so does the solution at any given time  $t > 0$ . Estimates on the growth of the support for very large time are given in [20] and [21].

- LOG-ENTROPY ESTIMATE: We have

$$(2.5) \quad \frac{d}{dt} \int u(x, t) \log u(x, t) dx = - \int |\nabla \mathcal{H}_s u|^2 dx.$$

therefore, solutions with  $u_0 \log u_0 \in L^1(\mathbb{R}^n)$  stay in the same space and moreover satisfy

$$\iint |\nabla \mathcal{H}_s u|^2 dx dt < \infty.$$

- ENERGY ESTIMATE. We first make the observation that if  $p = \mathcal{K}_s u$  we have

$$(2.6) \quad E(u) := \int up dx = \int ((-\Delta)^{s/2} p)^2 dx = \int (\mathcal{H}_s u)^2 dx \geq 0.$$

This formal computation is justified for the class of constructed weak solutions. This is the main energy we are going to use. We also have formally

$$\int |\nabla \mathcal{H}_s u|^2 dx = \int |\nabla (-\Delta)^{s/2} p|^2 dx = \int u(-\Delta p) dx.$$

The estimate on the evolution of the energy according to our flow is then

$$\frac{d}{dt} \int |\mathcal{H}_s u|^2 dx = 2 \int (\mathcal{H}_s u)(\mathcal{H}_s u)_t dx = 2 \int (\mathcal{K}_s u)u_t dx = -2 \int u|\nabla \mathcal{K}_s u|^2 dx.$$

where the last identity is obtained after using the equation and integrating by parts. This is justified for smooth solutions, while in the generality of weak solutions we get

$$(2.7) \quad \int_{\mathbb{R}^n} |\mathcal{H}_s u(t_2)|^2 dx + 2 \int_{t_1}^{t_2} \int u |\nabla \mathcal{K}_s u|^2 dx \leq \int_{\mathbb{R}^n} |\mathcal{H}_s u(t_1)|^2 dx$$

- **BOUNDEDNESS:** The solutions to (2.1) constructed in [20] have the following *boundedness effect* (used in the subsequent paper [19]): For all  $0 < s < 1$ , weak energy solutions in  $L^1$  are bounded for all positive times. More precisely, there is a constant  $C = C(n, s)$  such that

$$(2.8) \quad \|u(t)\|_\infty \leq C(n, s) t^{-\alpha} \|u_0\|_{L^1}^\sigma$$

Dimensional considerations imply that  $\alpha = n/(n + 2 - 2s)$  and  $\sigma = (2 - 2s)/(n + 2 - 2s)$ . This fact is proved in [19]. Notice that as  $s \rightarrow 1$  we have  $\alpha \rightarrow 1$  and  $\sigma \rightarrow 0$ . Once we can control the constant  $C(n, s)$  uniformly in  $s$  we can pass to the limit  $s \rightarrow 1$  to obtain the a priori estimate (already found by Lin and Zhang in their study in dimension  $n = 2$  in [44])

$$(2.9) \quad u(x, t) \leq C/t$$

for the solutions of the limit equation obtained in the limit process, this will be done in the next section in Theorem 3.1. This just requires a closer inspection of the proof of [19].

- **ENERGY SOLUTIONS:** The basis of the boundedness analysis is a property that goes beyond the definition of weak solution. We will review the formulas with attention to the constants that appear since this is not done in [19]. The general energy property is as follows: for any  $F$  smooth and such that  $f = F'$  is bounded and nonnegative, we have for every  $0 \leq t_1 \leq t_2 \leq T$ ,

$$\begin{aligned} \int F(u(t_2)) dx - \int F(u(t_1)) dx &= - \int_{t_1}^{t_2} \int \nabla[f(u)]u \nabla p dx dt = \\ &= - \int_{t_1}^{t_2} \int \nabla h(u) \nabla (-\Delta)^{-s} u dx dt \end{aligned}$$

where  $h$  is a function satisfying  $h'(u) = u f'(u)$ . We can write the last integral as a bilinear form

$$(2.10) \quad \int \nabla h(u) \nabla (-\Delta)^{-s} u dx = \mathcal{B}_s(h(u), u)$$

and this bilinear form  $\mathcal{B}_s$  is defined on the Sobolev space  $W^{1,2}(\mathbb{R}^n)$  by

$$(2.11) \quad \mathcal{B}_s(v, w) = C_{n,s} \iint \nabla v(x) \frac{1}{|x-y|^{n-2s}} \nabla w(y) dx dy = \iint \mathcal{N}_{-s}(x, y) \nabla v(x) \nabla w(y) dx dy$$

where  $\mathcal{N}_{-s}(x, y) = C_{n,s} |x-y|^{-(n-2s)}$  is the kernel of operator  $(-\Delta)^{-s}$ . After some integrations by parts we also have

$$(2.12) \quad \mathcal{B}_s(v, w) = C_{n,1-s} \iint (v(x) - v(y)) \frac{1}{|x-y|^{n+2(1-s)}} (w(x) - w(y)) dx dy$$

since  $-\Delta \mathcal{N}_{-s} = \mathcal{N}_{1-s}$ . It is known [56] that  $\mathcal{B}_s(u, u)$  is an equivalent norm for the fractional Sobolev space  $W^{1-s,2}(\mathbb{R}^n)$  and we will see just below that  $C_{n,1-s} \sim K_n(1-s)$  as  $s \rightarrow 1$ , for some constant  $K_n$  depending only on  $n$ .

**Proposition 2.1.** *The constant  $C(n, s)$  in (2.8) is uniformly bounded as  $s \rightarrow 1$ , i.e. for any solution to (2.1), we have*

$$\|u(t)\|_\infty \leq \frac{C_n \|u_0\|_{L^1}^\sigma}{t^\alpha},$$

with  $\alpha = n/(n+2-2s)$  and  $\sigma = (2-2s)/(n+2-2s)$ .

*Proof.* It is based on a careful scrutiny of the ideas and proof of paper [19], Section 4.

(i) We have  $C_{n,1-s} \sim K_n(1-s)$  as  $s \rightarrow 1$ . Indeed, if  $0 < s < 1$  we can also use the integral representation

$$(2.13) \quad (-\Delta)^s g(x) = C_{n,s} \text{P.V.} \int_{\mathbb{R}^n} \frac{g(x) - g(z)}{|x-z|^{n+2s}} dz = \mathcal{N}_s * g(x)$$

where P.V. stands for principal value and  $C_{n,s} = \frac{4^s s \Gamma((n/2)+s)}{\pi^{n/2} \Gamma(1-s)}$  is a normalization constant, see for example [41, 56]. Note that  $C_{n,s} \sim K_n s$  as  $s \rightarrow 0$  and  $C_{n,s} \sim C'_n(1-s)$  as  $s \rightarrow 1$ .

(ii) Next, we need the following result from [50]

$$\lim_{r \rightarrow 0^+} r \iint \frac{(u(x) - u(y))^2}{|x-y|^{n+2r}} dx dy = |\mathbb{S}^{n-1}| \int u(x)^2 dx.$$

Setting  $r = 1-s$  and combining this with (2.12) and  $C_{n,1-s} \sim K_n(1-s)$  as  $s \rightarrow 1^-$ , we deduce that

$$\lim_{s \rightarrow 1^-} \mathcal{B}_s(u, u) = K_n |\mathbb{S}^{n-1}| \|u\|_2^2.$$

(iii) We can now revisit the proof of boundedness of Theorem 4.1, in Section 4 of [19], and check that the constants in all the arguments are uniform in  $s$  as  $s \rightarrow 1$ .  $\square$

We end this section with some further properties of the solutions.

- **HÖLDER ESTIMATES** are proven for bounded energy solutions in [19] and hold for all  $s \in (0, 1)$ . The Hölder exponent has to deteriorate as  $s \rightarrow 1$  since we do not expect solutions of the limit equation to be continuous, as can be seen in the case of the “vortex-patch solutions” below.

- **MORE GENERAL DATA.** The property of boundedness and the  $C^\alpha$  regularity are used in [19] to obtain solutions with similar properties for all nonnegative initial data in  $L^1(\mathbb{R}^n)$ . We will generalize that result to measures in Section 3 below.

### 3 Existence of solutions as limits of fractional diffusion

In dimension  $n = 2$ , existence and uniqueness of positive  $L^\infty$  solutions has been proven by Lin and Zhang in [44] (they also proved existence of positive measure-valued solutions). Existence with positive initial data of finite energy has also been proven by a gradient flow approach, in bounded domains of the plane in [2], and in all of  $\mathbb{R}^2$  in [3]. In general dimension, existence can be obtained by taking the limit  $s \rightarrow 1$  in the solutions for  $s < 1$  constructed in [20] as we will see below.

We examine here the situation  $s = 1$  as limit of the equations for  $s < 1$ . Since all the estimates above are uniform in  $s$  as  $s \rightarrow 1$ , we may write the formal version for  $s = 1$ , where the equation is (1.5). Here we need to distinguish the case  $n \geq 3$  from the case  $n = 2$  which we will examine separately below. For  $n \geq 3$ , and noting that now  $\mathcal{H}u = (-\Delta)^{1/2}p$ , the energy is

$$(3.1) \quad E(u) = \int_{\mathbb{R}^n} up \, dx = \int_{\mathbb{R}^n} (\mathcal{H}u)^2 \, dx = \int_{\mathbb{R}^n} |\nabla p|^2 \, dx,$$

which is an analogue of the energy of [2] (there, it was set in bounded domains) for which the equation is a Wasserstein gradient flow, and its evolution is given by

$$(3.2) \quad \frac{dE}{dt} = -2 \int_{\mathbb{R}^n} u |\nabla p|^2 \, dx.$$

This is a formal computation that will be justified as a consequence of the passage to the limit  $s \rightarrow 1$  in formula (2.7). This energy would not make sense for  $n = 2$  because, unless  $\int u = 0$ ,  $p = (-\Delta)^{-1}u$  is never of gradient in  $L^2$  in dimension 2 (the Green kernel is a logarithm), however a definition is given below in Section 3.2.

### 3.1 The existence result by $s \rightarrow 1$ limit

We next prove the following general result in all dimensions  $n \geq 2$ .

**Theorem 3.1.** *For every  $\mu \in \mathcal{M}^+(\mathbb{R}^n)$ , there exists a continuous and nonnegative weak solution of equation (1.5), with data  $\mu$  in the sense of Definition 2.2. It can be obtained as the limit  $s \rightarrow 1$  of the solutions of the FPME with  $0 < s < 1$  with data  $u_k^0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$  that approximate  $\mu$  in the sense of measures. In addition these solutions satisfy*

$$(3.3) \quad \|u(t)\|_{L^\infty} \leq \frac{C}{t}$$

for some  $C > 0$  depending only on  $n$ .

*Proof.* Existence is proved by passing to the limit  $s \rightarrow 1$  in the solutions of the FPME (1.1). The outline is similar to the proof of Theorem 13.1 in [19] where  $s$  was considered constant, but we cannot use the continuity in time of the solution, since it may be lost in the limit  $s \rightarrow 1$ .

Let us consider initial data  $u^0 = \mu$  in  $\mathcal{M}^+(\mathbb{R}^n)$ , and a sequence  $u_k^0$  of nonnegative functions approaching  $u^0$  as measures as  $k \rightarrow \infty$ . (If  $u^0 \in L^1$  we may simply take  $u_k^0 = u^0$ .) Let us then consider the family of solutions  $u_k(x, t)$  with parameter  $s_k \rightarrow 1$  of (1.1) given by Theorem 4.1 of [20] or Theorem 13.1 of [19], and let  $p_k(x, t) = (-\Delta)^{-s}u_k(x, t)$  be the corresponding pressures.

(i) In a first step, we assume that  $n \geq 3$ , and prove that the limit exists (along subsequences  $s_{k_j} \rightarrow 1$ ) and is a solution of the equation in the sense of Definition 2.1, without discussing the question of the initial data. By Proposition 2.1, we have that for any  $\tau > 0$ ,

$$(3.4) \quad \|u_k\|_{L^\infty((0,\tau),L^1_x)} \leq C$$



where  $C > 0$  only depends on  $\tau$ . In order to pass to the limit in the weak formulation

$$(3.5) \quad \iint (u_k \phi_t - u_k \nabla p_k \cdot \nabla \phi) dxdt = 0$$

for  $\phi$  compactly supported in  $\mathbb{R}^n \times (0, T)$ , it suffices to have weak convergence of  $u_k$ , together with strong convergence of  $p_k$  and  $\nabla p_k$ . Let us work on  $[\tau, T)$  containing the support of  $\phi$ . From (3.4) we have weak convergence (up to subsequence extraction) of  $u_k$  in  $L^q([\tau, T) \times B_R)$  for any  $q \geq 1$ ,  $R > 0$ .

- For  $p_k$ , we may invoke the embedding theorems associated to the Riesz kernels, which tell us that  $p_k = (-\Delta)^{-s} u_k$  is uniformly bounded in  $L^\infty((\tau, T), L_x^q(B_R))$  for all large  $q$  and  $R > 0$ . More precisely,  $q > n/(n - 2s)$ , which is uniform in  $s \approx 1$  if  $n \geq 3$ .

- Moreover,  $\nabla p_k$  has a similar bound in spaces of the form  $L^\infty((\tau, T), W_{loc}^{s', q})$  with  $s' < 2s - 1$  and convenient  $q > n/(n - s)$ . Since we are interested in  $s \rightarrow 1$ , we can restrict to  $2s - 1 > 0$  so we can take  $s' > 0$  and both  $(p_k)$  and  $(\nabla p_k)$  are a locally compact family with respect to the space variable.

- We still need compactness in time for  $p_k$  and  $\nabla p_k$ , and this is obtained from Aubin-Simon's results, cf. [6], [55]. First, for  $p_k$ , we use the facts that

$$(3.6) \quad \begin{aligned} p_k \text{ is bounded in } & L^\infty((\tau, T), L^p(B_R)) \\ \partial_t p_k = (-\Delta)^{-s_k} \partial_t u_k = & (-\Delta)^{-s_k} \nabla \cdot (u_k p_k) \end{aligned}$$

and since  $u_k p_k$  is bounded in  $L^q(B_R)$  for any  $q > 0$  and uniformly in  $t \in [\tau, +\infty)$ , it follows that

$$(3.7) \quad \partial_t p_k \text{ is bounded in } L^\infty((\tau, T), W^{2s_k-1, q}(B_R)).$$

(3.6) and (3.7) ensure, via Aubin-Simon's theorem, that  $p_k$  is compact in  $C([\tau, T], L^q(B_R))$ , for any  $q \geq 1$  and  $R > 0$ . The proof for  $\nabla p_k$  is entirely similar: first, from the above, we have

$$(3.8) \quad \begin{aligned} \nabla p_k \text{ is bounded in } & L^\infty((\tau, T), W^{s_k', q}(B_R)) \\ \partial_t \nabla p_k = (-\Delta)^{-s_k} \partial_t u_k = & \nabla (-\Delta)^{-s_k} \nabla \cdot (u_k p_k) \end{aligned}$$

and since  $u_k p_k$  is bounded in  $L^q(B_R)$  for any  $q > 0$  and uniformly in  $t \in [\tau, T]$ , it follows that

$$\partial_t \nabla p_k \text{ is bounded in } L^\infty((\tau, T), W^{2s_k-2, q}(B_R)).$$

Again, these suffice to apply Aubin-Simon's result and obtain that  $\nabla p_k$  is compact in say  $C([\tau, T], L^q(B_R))$  for every  $\tau > 0, T > \tau, R > 0, q > 1$ . The limits after extraction of  $u_k$  and  $p_k$  are denoted  $u$  and  $p$ .

Taking  $\phi$  smooth compactly supported in space and vanishing for  $t \geq T$ , it follows from these convergence results that

$$(3.9) \quad \lim_{k \rightarrow \infty} \int_0^\infty \int (u_k \phi_t - u_k \nabla p_k \cdot \nabla \phi) dxdt = \int_0^\infty \int (u \phi_t - u \nabla p \cdot \nabla \phi) dxdt.$$

(ii) When  $n = 2$  we do not have a uniform bound of the pressures  $p_k$  in  $L^q$  spaces valid for all  $s \approx 1$ , since this is a limit case of the Sobolev embeddings. Fortunately, the pressures do not appear directly in the equation, but only through the pressure gradients for which the estimates are uniform in the range  $s \approx 1$ . The rest of the argument follows.

(iii) There remains to control terms corresponding to the time interval  $[0, \tau]$  in order to complete the proof that the solution takes on the initial data, as in Definition 2.2. For that we follow [19], proof of Theorem 13.1. We consider  $\phi \in C^1(\mathbb{R}^n \times [0, \infty))$  compactly supported in space and vanishing for  $t \geq T$ . First, let  $\tau > 0$  be arbitrary. Applying the result of (3.9) we have that

$$(3.10) \quad \lim_{k \rightarrow \infty} \int_{\tau}^{\infty} \int (u_k \phi_t - u_k \nabla p_k \cdot \nabla \phi) dx dt = \int_{\tau}^{\infty} \int (u \phi_t - u \nabla p \cdot \nabla \phi) dx dt.$$

In addition, by weak convergence of  $u_k^0$  to  $u^0$ , we have

$$(3.11) \quad \lim_{k \rightarrow \infty} \int u_k^0(x) \phi(x, 0) dx = \int u^0(x) \phi(x, 0) dx.$$

There remains to show that the contributions on the interval  $[0, \tau]$  can be made small.

Starting again from Proposition 2.1 we have

$$\|u_k(t)\|_{L^\infty} \leq C(u_0)t^{-\alpha} \quad \alpha = n/(n+2-2s_k),$$

which yields by conservation of mass

$$\|u_k(t)\|_{L^p}^p \leq \|u_k(t)\|_{L^1} \|u_k(t)\|_{L^\infty}^{p-1} \leq C(u_0)t^{-\alpha(p-1)}.$$

By Riesz embedding we deduce

$$\int |\nabla \mathcal{H}_{s_k} u_k|^2 dx \leq C \|u_k(t)\|_{L^p}^2 \leq C t^{-\alpha(p-1)\frac{2}{p}}, \quad \frac{1}{2} = \frac{1}{p} - \frac{s_k}{n}$$

where we recall  $\mathcal{H}_s u = (-\Delta)^{-s/2} u$  for any  $u$ , and combining the definitions of  $\alpha$  and  $p$ , this yields

$$\int |\nabla \mathcal{H}_{s_k} u_k|^2 \leq C t^{\frac{2}{n+2-2s_k}-1}.$$

Combining with the energy inequality (2.7), we deduce that we also have

$$(3.12) \quad \int_t^\infty \int u_k |\nabla p_k|^2 \leq C t^{\frac{2}{n+2-2s_k}-1}.$$

We may now bound the remaining contribution  $\int_0^\tau \int (u_k \phi_t - u_k \nabla p_k \cdot \nabla \phi) dx dt$  as follows: for  $t_m = 2^{-m}$ , we first write

$$\left| \int_{t_m}^{t_{m-1}} \int u_k \phi_t \right| \leq t_{m-1} \|\phi\|_{C^1} \|u_k\|_{L^\infty((0, \infty), L^1)} \leq C t_{m-1}$$

and second

$$\begin{aligned} \left| \int_{t_m}^{t_{m-1}} \int u_k \nabla p_k \cdot \nabla \phi \right| &\leq \|\phi\|_{C^1} \left( \int_{t_m}^{t_{m-1}} \int u_k \right)^{\frac{1}{2}} \left( \int_{t_m}^{t_{m-1}} \int u_k |\nabla p_k|^2 \right)^{\frac{1}{2}} \\ &\leq C t_{m-1}^{\frac{1}{2}} t_m^{\frac{1}{n+2-2s_k} - \frac{1}{2}} \leq 2C t_m^{\frac{1}{2} - \frac{1}{2} + \frac{1}{n+2-2s_k}} = C t_m^{\frac{1}{n+2-2s_k}}. \end{aligned}$$

Summing over  $m$  we find that the contributions on  $[0, \tau]$  tend to 0 as  $\tau$  tends to 0 and combining with (3.10)–(3.11) we obtain

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_0^\infty \int (u_k \phi_t - u_k \nabla p_k \cdot \nabla \phi) dx dt + \int u_k^0(x) \phi(x, 0) dx \\ = \int_0^\infty \int (u \phi_t - u \nabla p \cdot \nabla \phi) dx dt + \int u^0(x) \phi(x, 0) dx. \end{aligned}$$

as desired.

(iii) Finally, by passing to the limit  $s_k \rightarrow 1$  in (2.8) and using the fact that  $C(n, s)$  does not blow up in this limit, we obtain the universal estimate (3.3).  $\square$

**Remark 3.1.** *The handling of the initial data in the proof above could be replaced by showing that the initial data is taken via a result of continuity of the solution in Wasserstein metric, see Lemma 5.2 below.*

When  $s_k$  is taken fixed this is the proof of existence of solutions with measure data that improves Theorem 10.1 of [19].

**Theorem 3.2.** *For every  $\mu \in \mathcal{M}^+(\mathbb{R}^n)$ , there exists a continuous and nonnegative weak solution of FPME,  $0 < s < 1$ , with data  $\mu$  in the sense of Definition 2.2. It is a bounded energy solution for all  $t \geq \tau > 0$ .*

**Remark 3.2.** *We will provide in Proposition 5.1 below another approach to the existence of solutions for  $s = 1$  via the vanishing viscosity method.*

## 3.2 Definition of the energy in two space dimensions

In dimension 2, even though  $p = (-\Delta)^{-1}u$  is not in general of finite energy  $\int |\nabla p|^2 < \infty$ , we may still define an energy by a “renormalization” procedure. Here we reproduce the definition of [3], which itself follows a similar construction for Ginzburg-Landau in [51] and [13].

The first idea is to define the energy in a “renormalized way”, over functions  $u$  such that  $\int u = k$  with  $k$  a fixed constant (we may do this without loss of generality since the integral is preserved by the evolution) by

$$(3.13) \quad \mathcal{E}(u) = \lim_{R \rightarrow \infty} \left( \int_{B(0, R)} |\nabla p|^2 - \frac{k^2}{2\pi} \log R \right).$$

Contrarily to  $\int |\nabla p|^2$ , it is possible to make this energy finite even if  $k \neq 0$ .

Another definition is the following: consider any smooth positive compactly supported  $u_0$  with  $\int_{\mathbb{R}^2} u_0 = k$ , and let  $p_0 = (-\Delta)^{-1}u_0$ . For any  $u$  with  $\int u = k$  such that  $w := p - p_0$  is in  $H^1(\mathbb{R}^2)$ , we define the energy by

$$(3.14) \quad E(u) = \int_{\mathbb{R}^2} |\nabla w|^2 + 2 \int_{\mathbb{R}^2} u_0 w$$

In fact, expanding  $w$  as  $p - p_0$  and noting that  $\int_{B(0,R)} |\nabla p_0|^2 \sim \frac{k^2}{2\pi} \log R + cst$  as  $R \rightarrow \infty$ , and using an integration by parts we see that if  $w \in H^1(\mathbb{R}^2)$ , the two definitions differ only by the constant (depending only on the choice of  $u_0$ ). So, for energy we may take  $E$ .

Formally, along the flow, since  $w_t = p_t$ , we have

$$\begin{aligned} \frac{d}{dt} E(u(t)) &= 2 \int_{\mathbb{R}^2} \nabla w \cdot \nabla w_t + 2 \int_{\mathbb{R}^2} u_0 w_t \\ &= -2 \int_{\mathbb{R}^2} w \Delta p_t + 2 \int_{\mathbb{R}^2} u_0 p_t \\ &= 2 \int_{\mathbb{R}^2} (w + p_0) u_t = 2 \int_{\mathbb{R}^2} p u_t \\ &= - \int_{\mathbb{R}^2} |\nabla p|^2 u. \end{aligned}$$

An alternate approach would have been to view the  $(-\Delta)^{-1}$  operator as a limit of  $(-\Delta + hI)^{-1}$  as  $h \rightarrow 0$ .

## 4 Uniqueness of bounded solutions for $s = 1$

In this section we focus on obtaining a uniqueness result for  $s = 1$  for bounded solutions. Uniqueness is not expected in all situations, see [1]. We also recall that there is no maximum principle so uniqueness cannot follow from comparison arguments. The proof works only for  $s = 1$ ; no uniqueness proof is known for the FPME in dimensions  $n \geq 2$ .

A natural space in which to work for such continuity equations is the space of probability measures endowed with the Kantorovich-Rubinstein-Wasserstein distance (commonly called Wasserstein), indeed the PDE is a gradient flow for the 2-Wasserstein distance (point of view which was exploited in [2, 3]). For completeness, we recall (see e.g. [1]) that the  $p$ -Wasserstein distance between two probability measures  $\mu$  and  $\nu$  on  $\mathbb{R}^n$  (and by extension between two positive measures with same mass) is defined by

$$(4.1) \quad W_p(\mu, \nu)^p = \inf_{\gamma \in \Gamma(\mu, \nu)} \iint_{\mathbb{R}^n \times \mathbb{R}^n} |x - y|^p d\gamma(x, y)$$

where  $\Gamma(\mu, \nu)$  is the set of probability measures on  $\mathbb{R}^n \times \mathbb{R}^n$  whose first marginal is  $\mu$  and whose second marginal is  $\nu$ . This inf is achieved by an ‘‘optimal transport plan’’, and when  $1 \leq p < \infty$ ,  $W_p$  defines a distance.

**Theorem 4.1.** *There exists at most a unique solution of the initial-value problem for Equation (1.5) in  $L^\infty((0, T) : L^\infty(\mathbb{R}^n))$ .*

When  $n = 2$ , this theorem recovers with a different method the result on existence and uniqueness of bounded solutions of [44] but without requiring initial compact support.

*Proof.* The proof is inspired by the method of Loeper [47] (see also [2]). We will need the following result, adapted from [48, Lemma 8.1]. The result there is only stated in dimension  $n = 2$ , we include here a proof in any dimension  $n$ .

**Lemma 4.1.** *Let  $p = (-\Delta)^{-1}u$ . There exists a constant  $C > 0$  depending only on the dimension  $n$  such that*

$$\forall x, y \in \mathbb{R}^n, \quad |\nabla p(x) - \nabla p(y)| \leq C(\|u_0\|_{L^1} + \|u_0\|_\infty)|x - y| (|\log|x - y||\mathbf{1}_{|x-y| \leq 1/e} + 1)$$

*Proof.* Using the integral expression for  $\Delta^{-1}$ , we have, for any  $n \geq 1$ ,

$$(4.2) \quad \nabla p(x) = C_n \int_{\mathbb{R}^n} K(x - z)u(z) dz \quad K(x) = \frac{x}{|x|^n}.$$

Let us denote  $d = |x - y|$ . First, we treat the case  $d \leq 1/e$ . We may write  $\nabla p(x) - \nabla p(y) = C_n(I_1 + I_2 + I_3)$  with

$$I_1 := \int_{\mathbb{R}^n \setminus B(x, 4)} (K(x - z) - K(y - z))u(z) dz, \quad I_2 := \int_{B(x, 4) \setminus B(x, 2d)} (K(x - z) - K(y - z))u(z) dz$$

$$I_3 := \int_{B(x, 2d)} (K(x - z) - K(y - z))u(z) dz.$$

For  $z \in \mathbb{R}^n \setminus B(x, 4)$ , and since  $d = |x - y| \leq 1/e$  we have  $|K(x - z) - K(y - z)| \leq C|x - y|$  for some constant  $C$ . Hence we may write  $|I_1| \leq C|x - y|\|u\|_{L^1}$ .

For  $z \in B(x, 4) \setminus B(x, 2d)$ , by a mean value argument, and since  $d = |x - y|$ , we have  $|K(x - z) - K(y - z)| \leq C \frac{|x - y|}{|x - z|^n}$ . We may thus write

$$|I_2| \leq C|x - y|\|u\|_{L^\infty} \int_{B(x, 4) \setminus B(x, 2d)} \frac{dz}{|z - x|^n} = C|x - y|\|u\|_{L^\infty} \log d = C\|u\|_{L^\infty}|x - y| \log|x - y|.$$

For  $z \in B(x, 2d)$  we have  $|K(x - z) - K(y - z)| \leq |K(x - z)| + |K(y - z)| = \frac{1}{|x - z|^{n-1}} + \frac{1}{|y - z|^{n-1}}$  and we conclude

$$|I_3| \leq \|u\|_{L^\infty} \int_{B(x, 2d)} \frac{dz}{|x - z|^{n-1}} + \frac{dz}{|y - z|^{n-1}} \leq C\|u\|_{L^\infty}d = C|x - y|\|u\|_{L^\infty}.$$

Combining these estimates on  $I_1, I_2$  and  $I_3$  we obtain the result in this case.

If  $d \geq 1/e$ , we split the integral into two parts:

$$I_1 := \int_{\mathbb{R}^n \setminus B(x, 2d)} (K(x - z) - K(y - z))u(z) dz, \quad I_2 := \int_{B(x, 2d)} (K(x - z) - K(y - z))u(z) dz.$$

For the first part, we use again the fact that for  $z \notin B(x, 2d)$  we have  $|K(x - z) - K(y - z)| \leq C \frac{|x - y|}{|x - z|^n} \leq C|x - y|$  since  $d \geq 1/e$ . We thus find  $|I_1| \leq C|x - y|\|u\|_{L^1}$ . For  $z \in B(x, 2d)$  we argue just as above for  $I_3$  and find  $|I_2| \leq C|x - y|\|u\|_{L^\infty}$ . This concludes the proof.  $\square$

We now give the proof of the theorem. Let us consider two bounded solutions  $u_1, u_2$  on  $[0, T]$  with initial data  $u^0$ , and let  $p_1, p_2$  be the corresponding pressures.

We consider the characteristics equation (for  $i = 1, 2$ )

$$(4.3) \quad \frac{dX_i}{dt}(t, x) = -\nabla p_i(t, X_i(t, x)) \quad X_i(0, x) = x.$$

According to [1] (see also Corollary 3.3 in [47]), if  $u_i$  solve (1.5), then we have

$$(4.4) \quad u_i = (X_i)_\# u^0.$$

We then define

$$Q(t) = \int u^0(x) |X_1(t, x) - X_2(t, x)|^2 dx.$$

We have that  $Q(0) = 0$ , and from (4.4),  $W_2^2(u_1(t), u_2(t)) \leq Q(t)$ .

Let us study the evolution of  $Q$ . Differentiating  $Q$  in time yields

$$Q'(t) = -2 \int u^0(x) (X_1(t, x) - X_2(t, x)) \cdot (\nabla p_1(t, X_1(t, x)) - \nabla p_2(t, X_2(t, x))) dx$$

hence using the Cauchy-Schwarz inequality

$$(4.5) \quad \frac{1}{2} |Q'(t)| \leq \left( \int u^0 |X_1 - X_2|^2 dx \right)^{1/2} \left( \int u^0 |\nabla p_1(t, X_1) - \nabla p_2(t, X_2)|^2 dx \right)^{1/2} \\ = Q^{1/2} \left( \int u^0 |\nabla p_1(t, X_1) - \nabla p_2(t, X_2)|^2 dx \right)^{1/2}.$$

On the other hand,  $(\int u^0 |\nabla p_1(t, X_1) - \nabla p_2(t, X_2)|^2 dx)^{1/2} \leq T_1^{1/2} + T_2^{1/2}$  where

$$T_1 = \int u^0 |\nabla p_1(t, X_1) - \nabla p_1(t, X_2)|^2 dx$$

and

$$T_2 = \int u^0 |\nabla p_1(t, X_2) - \nabla p_2(t, X_2)|^2 dx.$$

Note first that from (4.4)

$$(4.6) \quad T_2 = \int u_2(x, t) |\nabla p_1(t, x) - \nabla p_2(t, x)|^2 dx.$$

But Theorem 4.4. in [47] asserts that, if  $\omega_1$  and  $\omega_2$  are two positive measures of same finite total mass and with  $L^\infty$  densities, then  $\Psi_i = -(\Delta)^{-1} \omega_i$  satisfy

$$(4.7) \quad \|\nabla \Psi_1 - \nabla \Psi_2\|_{L^2(\mathbb{R}^n)} \leq (\max(\|\omega_1\|_{L^\infty}, \|\omega_2\|_{L^\infty}))^{1/2} W_2(\omega_1, \omega_2)$$

Applying this to  $u_1(t)$  and  $u_2(t)$ , using the uniform  $L^\infty$  bound on these solutions, and inserting into (4.6), we find

$$|T_2| \leq \max(\|u_1(t)\|_\infty^2, \|u_2(t)\|_\infty^2) W_2^2(u_1(t), u_2(t)) \leq C W_2^2(u_1(t), u_2(t)).$$

There remains to bound  $T_1$ . Using Lemma 4.1, and denoting  $M = \|u^0\|_{L^1}$ , we may write

$$|T_1| \leq C(M^2 + \|u_1(t)\|_\infty^2) \left( \int u^0(x) |X_1 - X_2|^2 \log^2 |X_1 - X_2| \mathbf{1}_{|X_1 - X_2| \leq 1/e} dx + \int u^0(x) |X_1 - X_2|^2 dx \right).$$

Let us define  $f(x) = x \log^2(x)$  for  $0 \leq x \leq 1/e$  and  $f(x) = 1/e$  for  $x \geq 1/e$ . It is easy to check that  $f$  is concave,  $f(x) \leq x \log^2 x$ , and moreover from the above (and the boundedness of  $u_1$ )

$$|T_1| \leq C \left( \int u^0(x) f(|X_1 - X_2|^2) dx + \int u^0(x) |X_1 - X_2|^2 dx \right)$$

Using the concavity of  $f$ , by Jensen's inequality, we have

$$\int u^0(x) f(|X_1 - X_2|^2) dx \leq M f \left( \int \frac{u^0(x)}{M} |X_1 - X_2|^2 dx \right) = M f \left( \frac{Q(t)}{M} \right) \leq C_M Q(t) \log^2 Q(t).$$

We conclude that  $|T_1| \leq C Q(t) (\log^2 Q(t) + 1)$ . Combining with the bound for  $T_2$ , recalling that  $W_2^2(u_1, u_2) \leq Q$ , and inserting all these bounds into (4.5), we find

$$Q'(t) \leq |Q'(t)| \leq C Q(t) \sqrt{1 + \log^2 \frac{1}{Q(t)}},$$

where  $C$  depends on  $M$  and the uniform bounds on  $u_1$  and  $u_2$ .

We may now conclude by a standard integration argument, and since  $\int \frac{dv}{v \sqrt{1 + \log^2 v}} = \log(\log v + \sqrt{1 + \log^2 v})$  (using a change of variables  $u = \log v$ ), that  $Q(0) = 0$  implies  $Q(t) = 0$  for  $t \in [0, T]$ , and since  $W_2^2(u_1(t), u_2(t)) \leq Q(t)$  and  $W_2$  is a distance, the conclusion follows.  $\square$

## 5 The universal bound for $s = 1$

We now show that the  $L^\infty$  norm of solutions is decreasing in time. Indeed, the equation can be written as

$$u_t = -u^2 - \nabla u \cdot \nabla p.$$

which is the sum of absorption and transport with speed  $\mathbf{v} = -\nabla p$ . This expression suggests a universal decay in time of the form

$$(5.1) \quad \|u(\cdot, t)\|_\infty \leq \frac{1}{t}$$

which we will establish below. This agrees with the limit  $s \rightarrow 1$  of the smoothing effect of Proposition 2.1 for  $s < 1$ , cf. Theorem 3.1. What is surprising is that the bound is universal in the sense that it does not depend on any norm of the data. Note that in dimension 2, this bound appears in [44]. We have a slightly more precise result in terms of the  $L^\infty$  norm of the data. The subject was taken up in the recent paper [12], which is very close to the ideas we explain next in some detail for the reader's convenience.

**Theorem 5.1.** *For every weak solution  $u$  with bounded initial data  $u_0$  we have the estimate*

$$(5.2) \quad u(x, t) \leq \frac{1}{t + \tau}$$

where  $\tau = 1/\sup_x u_0(x)$ . Therefore these solutions coincide (via the uniqueness result for bounded solutions, Theorem 4.1) with those of Theorem 3.1.

**Remark 5.1.** *A similar result holds for the lower bound of the negative values when dealing with signed solutions.*

We start by studying a viscous approximation of the equation (which, by the way, is like the repulsive version of the Keller-Segel equation for chemotaxis [37, 36]).

**Proposition 5.1.** *Let  $u^0 \in L^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$  with  $u^0 \geq 0$ . There exists a unique solution to*

$$(5.3) \quad \begin{cases} \partial_t u_\varepsilon = \varepsilon \Delta u_\varepsilon + \nabla \cdot (u_\varepsilon \nabla p_\varepsilon) & p_\varepsilon = (-\Delta)^{-1} u_\varepsilon \\ u_\varepsilon(\cdot, 0) = u^0 \end{cases}$$

Moreover  $u_\varepsilon$  is smooth, satisfies

$$(5.4) \quad 0 \leq u_\varepsilon(x, t) \leq \|u^0\|_{L^\infty}$$

and converges locally in  $L^q(\mathbb{R}^n \times [0, +\infty))$  for any  $1 \leq q < \infty$ , to  $u$ , the unique solution to (1.5)–(1.6).

We note that this “vanishing viscosity approach” is an alternative way of obtaining existence of solutions to (1.5)–(1.6).

*Proof.* This natural “viscosity approximation” was previously considered in Schätzle-Styles [53], for a very similar equation, but in a bounded domain of  $\mathbb{R}^2$ . Because the proof is tailored for a bounded domain, it is not immediate to adapt it to our case. In order to prove the existence of solutions to (5.3), we set up a fixed point argument.

Take  $T > 0$  and let

$$K = \left\{ v(x, t) \in L^\infty(\mathbb{R}^n \times (0, T)), v \geq 0, \text{ and } \forall t \in [0, T], \int_{\mathbb{R}^n} v(x, t) dx \leq \int_{\mathbb{R}^n} u^0(x) dx \right\}.$$

$K$  is clearly a closed convex subset of the Banach space  $L^\infty(\mathbb{R}^n \times [0, T])$ .



**Lemma 5.1.** *Let  $u^0 \in L^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ . For any  $v \in K$ , letting  $p_v = (-\Delta)^{-1}v$ . Then, we have  $\nabla p_v \in L^\infty(\mathbb{R}^n \times [0, T])$  and there exists a unique solution  $u := \mathcal{T}(v)$  over  $[0, T]$  of the problem*

$$(5.5) \quad \begin{cases} \partial_t u = \varepsilon \Delta u + \nabla \cdot (u \nabla p_v), \\ u(\cdot, 0) = u^0. \end{cases}$$

*That solution satisfies  $\mathcal{T}(v) \in K$  and  $\|\mathcal{T}(v)\|_{C^{0,\alpha}} \leq C$ .  $\mathcal{T}$  is continuous and compact from  $K$  to itself.*

*Proof.* First we note that if  $v \in K$ , then for all  $t \in [0, T]$ ,  $v(\cdot, t) \in L^1 \cap L^\infty$ , which implies that  $\nabla p_v \in L^\infty(\mathbb{R}^n \times [0, T])$  by Nash-Sobolev embedding, or more precisely, by observing

$$\begin{aligned} |\nabla p_v|(x, t) &\leq \int_{\mathbb{R}^n} \frac{v(y, t)}{|x - y|^{n-1}} dy \leq \|v\|_{L^\infty(\mathbb{R}^n \times [0, T])} \int_{B(x, 1)} \frac{dy}{|x - y|^{n-1}} + \int_{\mathbb{R}^n} v(y) dy \\ &\leq C (\|v\|_{L^\infty(\mathbb{R}^n \times [0, T])} + \|v(\cdot, t)\|_{L^1(\mathbb{R}^n)}), \end{aligned}$$

where  $C$  depends only on the dimension. In fact, direct inspection allows to prove in addition that  $\nabla p_v(\cdot, t) \in C^{0,\alpha}(\mathbb{R}^n)$  for some  $\alpha > 0$ .

We are thus led to solving a linear equation of the form

$$(5.6) \quad \begin{cases} \partial_t u = \varepsilon \Delta u + \nabla \cdot (uf) \\ u(\cdot, 0) = u^0 \end{cases}$$

with  $f \in L^\infty(\mathbb{R}^n \times [0, T], \mathbb{R}^n)$ . This can be solved by a standard convolution argument, see for example Lemma 12 in [16] which applies exactly (the fact that the dimension is  $n$  and not 2 does not affect at all the argument), and it provides us with a unique solution  $\mathcal{T}(v) \geq 0$  defined for all positive times, hence in  $[0, T]$ . The fact that for all  $t \in [0, T]$ ,  $\int_{\mathbb{R}^n} \mathcal{T}(v) dx \leq \int_{\mathbb{R}^n} u^0$  follows by integrating in time the equation. On order to show that  $\mathcal{T}(v) \in K$ , there remains to show that  $\mathcal{T}(v) \in L^\infty(\mathbb{R}^n \times [0, T])$ .

Let us consider  $u = \mathcal{T}(v)$  solving (5.6) and let  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  be a smooth function. Using (5.6), integration by parts and  $\Delta p_v = -v$ , we have

$$\begin{aligned} (5.7) \quad \frac{d}{dt} \int_{\mathbb{R}^n} \phi(u) &= \int \phi'(u) (\varepsilon \Delta u + \nabla \cdot (u \nabla p_v)) \\ &= -\varepsilon \int \phi''(u) |\nabla u|^2 - \int u \phi''(u) \nabla p_v \cdot \nabla u \\ &= -\varepsilon \int \phi''(u) |\nabla u|^2 - \int \nabla p_v \cdot \nabla [H(u)] \\ &= -\varepsilon \int \phi''(u) |\nabla u|^2 - \int v H(u), \end{aligned}$$

where we define  $H(s) = \int_0^s t \phi''(t) dt$ . If in addition  $\phi$  is convex it follows that  $H \geq 0$  and, since  $v \geq 0$ , we find

$$\frac{d}{dt} \int_{\mathbb{R}^n} \phi(u) \leq 0.$$

Taking  $\phi$  nonnegative convex and supported over  $[\|u^0\|_{L^\infty}, +\infty)$  we conclude that  $0 \leq u(x, t) \leq \|u^0\|_{L^\infty}$  for every  $(x, t) \in \mathbb{R}^n \times [0, T]$ . This proves that  $\mathcal{T}(v) \in L^\infty$  and it belongs to  $K$ .  $\mathcal{T}$  thus maps  $K$  into itself. It is also easy to see that it is continuous.

Now, standard regularity theory (cf. [39]) gives us Schauder estimates

$$\|\mathcal{T}(v)\|_{C^{0,\alpha}(\mathbb{R}^n \times [0, T])} \leq C (\|v\|_{L^\infty(\mathbb{R}^n \times [0, T])} + \|v\|_{L^1(\mathbb{R}^n \times [0, T])}) \leq CT (\|u^0\|_{L^\infty} + \|u^0\|_{L^1}).$$

Since  $C^{0,\alpha}$  embeds compactly into  $L^\infty$ ,  $\mathcal{T}(K)$  is thus precompact.  $\square$

We may now apply Schauder's fixed point theorem [43] which gives us the existence of a fixed point for  $\mathcal{T}$  hence a solution to (5.3), satisfying  $u_\varepsilon \geq 0$  and  $\|u_\varepsilon\|_{L^\infty(\mathbb{R}^n \times [0, T])} \leq \|u^0\|_{L^\infty}$ .

Finally once we have established (5.4), convergence of  $u_\varepsilon$  to a solution of (1.5) can be proven following the same steps as in the proof of Theorem 3.1 (except replacing  $s_k$  by 1). The solution is unique by Theorem 4.1, which implies convergence of the whole sequence  $u_\varepsilon$ .  $\square$

*Proof of Theorem 5.1.* We first prove the desired estimate for  $u_\varepsilon$  solution to (5.3), which is smooth by standard parabolic regularity theory. Let us consider  $\phi$  smooth such that  $\phi(u) = 0$  for  $u \leq 0$ ,  $\phi'' \geq 0$ ,  $\phi' \geq 0$ , and let us compute  $\frac{d}{dt} \int \phi(u_\varepsilon - M(t)) dx$ , with  $M$  some smooth nonnegative function of time. This yields

$$(5.8) \quad \frac{d}{dt} \int \phi(u_\varepsilon - M(t)) dx = \int \phi'(u_\varepsilon - M(t)) \partial_t u_\varepsilon dx - M'(t) \int \phi'(u_\varepsilon - M(t)) dx.$$

Using the equation, we find, after several integrations by parts

$$\begin{aligned} \int \phi'(u_\varepsilon - M(t)) \partial_t u_\varepsilon &= \varepsilon \int \phi'(u_\varepsilon - M(t)) \Delta u_\varepsilon - \int \phi''(u_\varepsilon - M(t)) \nabla u_\varepsilon \cdot u_\varepsilon \nabla p_\varepsilon \\ &= -\varepsilon \int \phi''(u_\varepsilon - M(t)) |\nabla u_\varepsilon|^2 - \int \phi''(u_\varepsilon - M(t)) (u_\varepsilon - M(t)) \nabla u_\varepsilon \cdot \nabla p_\varepsilon \\ &\quad - M(t) \int \phi''(u_\varepsilon - M(t)) \nabla u_\varepsilon \cdot \nabla p_\varepsilon \\ &\leq - \int \nabla [H(u_\varepsilon - M(t))] \cdot \nabla p_\varepsilon - M(t) \int \phi''(u_\varepsilon - M(t)) \nabla u_\varepsilon \cdot \nabla p_\varepsilon. \end{aligned}$$

Here, we have defined again  $H(s) = \int_0^s t \phi''(t) dt$ . Using  $-\Delta p_\varepsilon = u_\varepsilon$  and another integration by parts, we find that the first term on the right-hand side is  $-\int H(u_\varepsilon - M(t)) u_\varepsilon \leq 0$ . There remains to estimate in a similar way

$$\begin{aligned} -M(t) \int \phi''(u_\varepsilon - M(t)) \nabla u_\varepsilon \cdot \nabla p_\varepsilon &= -M(t) \int \nabla [\phi'(u_\varepsilon - M(t))] \cdot \nabla p_\varepsilon \\ &= -M(t) \int \phi'(u_\varepsilon - M(t)) u_\varepsilon \leq -M(t)^2 \int \phi'(u_\varepsilon - M(t)), \end{aligned}$$

because  $\phi'$  is supported on the positive part of the real line and  $M(t) \geq 0$ . We conclude that

$$\int \phi'(u_\varepsilon - M(t)) \partial_t u_\varepsilon dx \leq -M(t)^2 \int \phi'(u_\varepsilon - M(t)) dx$$

and combining with (5.8)

$$(5.9) \quad \frac{d}{dt} \int \phi(u_\varepsilon - M(t)) dx \leq -(M^2(t) + M'(t)) \int \phi'(u_\varepsilon - M(t)) dx.$$

Choosing  $M(t) = \frac{1}{t+\tau}$  with  $\tau = (\sup u_0(x))^{-1}$ , we have  $M'(t) = -M^2(t)$ , and it follows that  $\frac{d}{dt} \int \phi(u_\varepsilon - M(t)) dx \leq 0$ . On the other hand, since  $u^0 \leq M(0)$  we have  $\int \phi(u^0 - M(0)) dx \leq 0$ , and thus we have obtained that  $\int \phi(u_\varepsilon - M(t)) dx \leq 0$  for all  $t \geq 0$ .

Taking now  $\phi = \phi_n$  to be a sequence that approximates the function  $\phi_0(x) = \text{sgn}_+(x)$ . we obtain  $u_\varepsilon(t) \leq M(t)$  a.e. for  $t \geq 0$ . Passing to the limit  $\varepsilon \rightarrow 0$ , and using Proposition 5.1, we arrive at the result.  $\square$

**Remark 5.2.** *The estimate  $|u(x, t)| \leq \frac{1}{t + \|u_0\|_\infty^{-1}}$  also applies to solutions of every sign which generalizes the case of “dissipative weak solutions” studied in [49].*

We conclude this section with a result of continuity of the solutions in Wasserstein distance, which we do not use however.

**Lemma 5.2.** *The solutions to (1.5) with  $u^0 \in \mathcal{M}^+(\mathbb{R}^n)$  satisfying (3.3) are “absolutely continuous” (hence continuous) for the  $W_q$ -Wasserstein distance for any  $q \in [1, \infty)$ : more precisely  $W_q(u(t_2), u(t_1)) \leq \int_{t_1}^{t_2} m(s) ds$  with  $m \in L^1([0, T])$  for any  $T > 0$ .*

*Proof.* By conservation of mass, for every  $t > 0$  we have the two a priori estimates  $\|u(t)\|_1 \leq M$ ,  $\|u(t)\|_\infty \leq 1/t$ . By the Nash-Sobolev embeddings we then get  $\nabla p(\cdot, t) \in L^\infty(\mathbb{R}^n)$ . A simple scaling argument then shows that

$$(5.10) \quad \|\nabla p(\cdot, t)\|_\infty \leq C \|\Delta p(\cdot, t)\|_1^{1/n} \|\Delta p(\cdot, t)\|_\infty^{(n-1)/n} \leq C_1 M^{1/n} t^{-(n-1)/n}$$

The claimed continuity is a direct consequence of the nature of the equation (1.5) as a continuity equation, i.e. of the form  $\partial_t u + \nabla \cdot (u \mathbf{v}) = 0$ . Theorem 8.3.1 in [1] asserts that for such an equation, if  $\int_0^T \|\mathbf{v}(t)\|_{L^q(u(t))} dt < +\infty$  then the solution  $u(t)$  is absolutely continuous in  $[0, T]$  with values in the space of measures equipped with the distance  $W_q$ . The precise definition of this (cf. [1]) is that  $W_q(u(t_1), u(t_2)) \leq \int_{t_1}^{t_2} m(s) ds$  with  $m \in L^1([0, T])$ , and it implies continuity in the usual sense. In the present case,  $\mathbf{v} = \nabla p$  and, from (5.10) we have

$$\begin{aligned} \int_0^T \|\mathbf{v}(t)\|_{L^q(u(t))} dt &= \int_0^T \left( \int_{\mathbb{R}^n} |\nabla p(t)|^q u(t) \right)^{1/q} dx dt \\ &\leq \int_0^T C t^{1/n-1} \left( \int_{\mathbb{R}^n} u(t) dx \right)^{1/q} dt \leq C' M^{1/q} T^{1/n}. \end{aligned}$$

It follows that the result of [1] applies and  $u(t)$  is continuous for the Wasserstein distance  $W_q$ .  $\square$

## 6 Radial solutions. Burgers' equation

In this section we analyze radially symmetric solutions and exhibit the special class of vortex-patch solutions. We use the Burgers' equation approach via the mass transform, which already appeared in [11] and has precedents in chemotaxis problems, [36]. The main purpose of most of this section is to fix the ideas and notation that we will use in the last two subsections, and in the asymptotic analysis of Section 8.2.

**Lemma 6.1.** *Let  $u$  be a solution to (1.5) with radially symmetric bounded initial data  $u_0(r)$ ,  $r = |x|$ . Then  $u$  is radial, and the “mass function” defined by*

$$(6.1) \quad M(r, t) = \int_0^r s^{n-1} u(s, t) ds$$

*solves the Burgers' equation*

$$(6.2) \quad M_t + \frac{1}{r^{n-1}} M M_r = 0,$$

*which after the change of variables  $s = r^n/n$  can be written in the standard form  $M_t + M M_s = 0$ .*

*Proof.* This follows from elementary computations, or can be found in [11].  $\square$

Note that (6.2) is now a local equation, contrary to (1.5) and that it looks very much like the equation for chemotaxis, but with the opposite sign. The speed of movement is  $M/r^{n-1} = v \geq 0$ . Notice that

$$(6.3) \quad M_t \leq 0.$$

A common problem with Burgers' equations and other first order conservation laws is that they may develop shocks in finite time and the uniqueness of solutions needs the concept of entropy solutions, cf. [42, 38]. This is not the case here, due to the fact that  $M(r, t)$  is a monotone nondecreasing function of  $r$  and that the velocity  $v$  is positive, so that the characteristic lines spread. Conversely, it is easy to see that weak solutions  $M(s, t)$  of the Burgers equation with  $M_s \geq 0$  bounded, give rise to nonnegative, bounded, radial, weak solutions of the hydrodynamic equation  $u_t = \nabla(u \nabla(-\Delta)^{-1} u)$ .

Burgers' Equation (6.2) has the property of comparison (maximum principle) in the class of nonnegative and monotone solutions  $M$ . Since the equation is also 2-homogeneous in  $M$ , Bénilan-Crandall's result [8] applies to give the pointwise inequality

$$(6.4) \quad M_t \geq -\frac{M}{t}.$$

Using the equation this implies the further inequality (that we already know)

$$(6.5) \quad M_r \leq \frac{r^{n-1}}{t}, \quad \text{i. e.} \quad u(x, t) \leq \frac{1}{t}.$$

These inequalities are optimal as we will show by means of the examples of the next section.

The approach via the mass function  $M(r, t)$  will be used below to construct interesting classes of explicit solutions, that serve as prototypes or as counter-examples of the theory.

## 6.1 Solutions in dimension $n = 1$

The Burgers' transformation, i. e., the formulation in terms of the mass function, applies easily when we work in dimension  $n = 1$  and we do not need the condition of radial symmetry to get in a direct way the standard Burgers' equation

$$(6.6) \quad M_t + MM_r = 0,$$

where we are still assuming that we deal with nonnegative solutions  $u \geq 0$ .

## 6.2 The elementary vortex patch

We next recover the existence of the vortex patch self-similar solution.

**Proposition 6.1.** *The equation (1.5) admits the family of radial “vortex patch solutions” of the form*

$$(6.7) \quad u(x, t) = \frac{1}{t + \tau} \chi_{|x| \leq R(t+\tau)^{1/n}}$$

(as well as all their translates) where  $R > 0, \tau \geq 0$  are free parameters.

We skip the proof of this known result, see for instance [12]. This configuration is called in the terminology of transport equations a *rarefaction wave*. The simplest form of the solution is

$$(6.8) \quad u(r, t) = \frac{1}{t} \chi_{r \leq Rt^{1/n}}$$

which formally corresponds to  $\tau = 0$ . We call this solution the *elementary vortex patch*. Notice that it is a solution in the sense of Definition 2.2. It has a Dirac mass as initial data for the density, hence it qualifies as the *fundamental solution* of the problem.

All of the above solutions can also be described as *expanding mesas*. The use of the term “mesa” appears in some limit cases to nonlinear diffusions, see e.g. [27] and [28] or [59].

**Largest solution.** Note that in the limit when the initial mass goes to infinity we obtain the solution

$$(6.9) \quad u(x, t) = \frac{1}{t}, \quad M(r, t) = r^n nt$$

that does not have an interface. This solution is an absolute a priori upper bound for all solutions of our problem with  $s = 1$ . It satisfies equations (6.4) and (6.5) with equality.

## 6.3 Self-similar solutions

The vortex patch solution can be obtained as the limit of interesting self-similar solutions of the FPME with  $0 < s < 1$ . Such solutions are constructed in the paper [21] as self-similar solutions of the form

$$(6.10) \quad u(x, t) = t^{-\alpha} U(x/t^\beta)$$

In order to cancel the factors including  $t$  explicitly, we get the condition on the exponents

$$(6.11) \quad \alpha + (2 - 2s)\beta = 1$$

If we also impose conservation of (finite) mass, which amounts to the condition  $\alpha = n\beta$ , we arrive at the precise value for the exponents:

$$\beta = 1/(n + 2 - 2s), \quad \alpha = n/(n + 2 - 2s).$$

We also arrive at the *nonlinear, nonlocal elliptic equation* for the self-similar profiles

$$(6.12) \quad \nabla_y \cdot (U \nabla_y (P + a|y|^2)) = 0, \quad \text{with } P = \mathcal{K}_s U.$$

where  $a = \beta/2$ , and  $\beta$  are defined just above. The solution of equation (6.12) is obtained in [21] by reducing it to an obstacle problem for the  $s$ -Laplacian operator. On the other hand, Biler, Imbert and Karch [14] have obtained the explicit formula for a self-similar solution in the form

$$(6.13) \quad u(x, t; C_1) = t^{-\alpha} (C_1 - k_1 x^2 t^{-2\alpha/n})_+^{1-s}$$

with  $k_1, C_1 > 0$  and  $\alpha = n/(n + 2 - 2s)$  as before. These solutions bear a resemblance to those of the standard porous medium equation [59, 15, 60]. This is why they have been termed *Barenblatt solutions for the FPME*.

**Selfsimilar solution for  $s = 1$ :** This approach works for  $s = 1$  and gives  $\alpha = 1$ ,  $\beta = 1/n$ , where we replace  $\mathcal{K}_s$  by  $\mathcal{K}_1 = (-\Delta)^{-1}$ . We then recover the vortex patch solution (6.8). It is also easy to see that the family of self-similar solutions (6.13) pass to the limit  $s \rightarrow 1$  to produce the vortex patch solution (6.8) (after careful selection of the constants). This is an explicit example of application of Theorem 3.1.

**Comment.** For the reader's convenience, we give more details in the case  $s = 1$ . The solution of the obstacle problem has to satisfy  $U = -\Delta P = 2na = n\beta = 1$  in the coincidence set  $|y| \leq R$ , while  $\Delta P = 0$  outside (for  $|y| \geq R$ ). Therefore, in the outer region

$$P = D |y|^{2-n}, \quad D > 0.$$

$C^1$  agreement of the two expressions at  $y = r$  gives  $D = R^n/n(n - 2)$  and  $R^2 = 2(n - 2)C$ , and we recover the solution (6.8).

## 6.4 Other examples of vortex patches

We may next construct many solutions of the radial problem as rarefaction waves, which are not very different from the elementary vortex patch. We take an initial configuration formed by a ball patch at the origin plus an annular patch around, i.e.,

$$u_0(r) = c_1 \quad \text{for } 0 \leq r \leq R_1, \quad u_0(r) = c_2 \quad \text{for } R_2 \leq r \leq R_3,$$

with  $c_1, c_2 > 0$  and  $0 < R_1 < R_2 < R_3$ ,  $u_0(r) = 0$  otherwise. Then

$$\begin{aligned} M(r, 0) &= (c_1/n)r^n && \text{for } 0 \leq r \leq R_1 \\ &= (c_1/n)R_1^n && \text{for } R_1 \leq r \leq R_2 \\ &= (c_2/n)r^n + (c_1/n)R_1^n - (c_2/n)R_2^n && \text{for } R_2 \leq r \leq R_3 \\ &= (c_2/n)(R_3^n - R_2^n) + (c_1/n)R_1^n && \text{for } r \geq R_3 \end{aligned}$$

Note that the continuity of  $M$  requires that we choose  $c_2 R_2^n = c_1 R_1^n$ . At later times the solution keeps the same shape, i.e. the support of  $u(r, t)$  is still the union of a ball and an annulus (both expanding).  $u(r, t)$  and  $M(r, t)$  are explicit, in regions that are separated by interfaces:

(i) The first region is an inner ball where  $u$  and  $M$  have the form

$$u_1(r, t) = \frac{c_1}{c_1 t + 1} = \frac{1}{t + \tau_1}; \quad M_1(r, t) = \frac{c_1 r^n}{n(c_1 t + 1)} = \frac{r^n}{n(t + \tau_1)}$$

( $\tau_1 = 1/c_1$ ). The bounding circle (centered at the origin) has radius  $S_1(t)$  satisfying  $S_1'(t) = v(S_1(t), t) = M(S_1(t), t)/S_1(t)^{n-1}$  with  $S_1(0) = R_1$ , hence

$$S_1(t) = R_1(c_1 t + 1)^{1/n} = R_1 c_1^{1/n} (t + \tau_1)^{1/n}.$$

(ii) There is then an annular region with  $M(r, t) = c_1 R_1^n/n$  and  $u(r, t) = 0$ , bounded by this interface and the circle of radius

$$S_2(t) = R_1(c_1 t + (R_2/R_1)^n)^{1/n} = R_2(c_2 t + 1)^{1/n} = R_2 c_2^{1/n} (t + \tau_2)^{1/n}, \quad \tau_2 = (R_2/R_1)^n \tau_1.$$

(iii) There is then an annular region of positive density (initially  $R_2 < r < R_3$ ) given by

$$u_2(r, t) = \frac{c_2}{c_2 t + 1} = \frac{1}{t + \tau_2}; \quad M_2 = \frac{c_2 r^n}{n(c_2 t + 1)} = \frac{r^n}{n(t + \tau_2)}$$

where  $\tau_2 = 1/c_2 > \tau_1$ . The outer interface of this annulus is the circle of radius  $S_3(t) = R_3(c_2 t + 1)^{1/n} = R_3 c_2^{1/n} (t + \tau_2)^{1/n}$ .

(iv) Finally, in the exterior region, complement of the ball of radius  $S_3(t)$ ,  $M(r, t)$  is constant  $M$  and  $u$  is zero.

**Conclusion.** In this example there are two connected components for the density that expand out with time. The inner patch is unperturbed by the outer mass, while the outer annular patch is spreading out in a self-similar way. The density in both patches is constant in space but the density outside depends on the density inside, time, and the ratio  $R_2/R_1$ . Both densities approach  $1/t$  in first approximation as  $t \rightarrow \infty$ . Moreover, the gap between the two patches is

$$d(t) = C [(t + \tau_2)^{1/n} - (t + \tau_1)^{1/n}]$$

which relatively goes to zero like  $O(t^{-(n-1)/n})$  as  $t \rightarrow \infty$ .

## 6.5 Lack of comparison principle

This example allows to construct a counterexample to comparison of densities. Let us take for first solution  $u_1(r, t)$  the above example with parameters  $0 < R_1 < r_2 < R_3$ . For second solution  $u_2(x, t)$  let us take the elementary vortex patch with center  $x_0$  such that  $|x_0| = (R_2 + R_3)/2$ , with height equal or less than  $c_2$  and initial radius  $R_4$  much less than  $(R_3 - R_2)/2$ . In this way we have  $u_1(x, 0) \geq u_2(x, 0)$ . However, the support of  $u_2$  expands from the initial ball  $B_{R_4}(x_0)$  while the support of  $u_1$  has a “hole” that eventually will “cover” that ball at some time. Hence comparison of densities does not persist in time.

Note that the lack of a comparison of densities principle was shown for the solutions of the FPME with  $s > 1/2$  in [20]. The argument there was less explicit due to the lack of explicit solutions.

Recall that for radial solutions on the other hand we have comparison of the mass functions  $M(r, t)$ . A similar result for FPME is known in dimension  $n = 1$  where the approach of integrating in space is viable, [15].

## 7 Solutions with compact support

We now return to non necessarily radial solutions to (1.5) but impose the condition of compact support on the data. This short section is essentially contained in [12], where more details can be found. We have kept the proof since some of the arguments we explain here are used in what follows.

It is known [20] that the solutions of the FPME for  $0 < s < 1$  have the property of finite propagation so that initial data with compact support produces solutions with the same property for every fixed  $t > 0$ . We can check that the calculation of the supersolution used to prove the property for  $s < 1$  can be extended to  $s = 1$ . However, a comparison argument with explicit solutions is easier in this case, by using the selfsimilar solutions constructed in Section 6.2. It works as follows.

**Theorem 7.1.** *Given bounded initial data  $u_0$  with compact support in the ball of radius  $R_0$ , then for every  $t > 0$  the solution  $u(\cdot, t)$  is supported in the ball of radius  $R_0(1 + \|u_0\|_\infty t)^{1/n}$ .*

*Proof.* We consider the characteristics that according to [1] transport the solution. Using the equation we see that the speed of such lines is given by

$$(7.1) \quad v(x_0, t) = \nabla p(x_0, t) = (\nabla(-\Delta)^{-1}u)(x_0, t) = C_1 \int \frac{u(y, t)(x_0 - y)}{|x_0 - y|^n} dy$$

where  $C_1 = 1/((n - 2)\omega_n)$ , with  $\omega_n$  the volume of the unit ball in  $\mathbb{R}^n$ . So it is bounded at all  $x$  and  $t$  and we conclude that the initial support propagates only to a finite distance in any finite time interval.

Take now  $t > 0$  fixed and assume that we are at  $x_0$ , a point on the free boundary or boundary of the support  $\Omega(t) = \{x : u(x, t) > 0\}$ . We consider an extreme point, i. e., such that  $|x_0(t)| = \sup\{|x| : x \in \partial\Omega(t)\}$ . We may assume after rotation that  $x_0 = (R(t), 0, \dots, 0)$ . We calculate



the velocity along the radius direction (which is the velocity with which the boundary separates from the origin). Then,

$$(7.2) \quad v_1(x_0, t) = C_1 \int \frac{u(y, t)(R(t) - y_1)}{|x_0 - y|^n} dy,$$

The integrand has a positive sign for every  $y \in \Omega(t)$ . Therefore, the worst case is when we replace  $u$  by its maximum estimate given by Theorem 5.1, to get

$$|v_1(x_0, t)| \leq C_1 \frac{1}{t + \tau} \int_{B_{R_0}(0)} \frac{(R - y_1)}{|x_0 - y|^n} dy,$$

where  $\tau = 1/\|u_0\|_\infty$ . Using scaling we get

$$|v_1(R(x_0, t))| \leq C_1 \frac{R(t)}{t + \tau} \int_{B_1(0)} \frac{(1 - y_1)}{|\mathbf{e}_1 - y|^n} dy,$$

with  $\mathbf{e}_1 = (1, 0, \dots, 0)$ . Next, we notice that this is the integral that would be obtained for the self-similar solution of Proposition 6.1. After a calculation in radial coordinates we have

$$R'(t) = |v_1(x_0, t)| \leq \frac{R(t)}{n(t + \tau)}$$

which, integrated in time, yields the estimate  $R(t) \leq R_0 \left(\frac{t+\tau}{\tau}\right)^{1/n}$ , i.e. the result.  $\square$

**Remark 7.1.** *We have really proved the following: Let  $u$  be a bounded nonnegative weak solution. Assume the support of the initial data is contained in a ball of radius  $R_0$ . Let  $U$  be the vortex-patch solution with same initial  $L^\infty$  norm and initial support  $B_{R_0}(0)$ . Then the support of  $u$  at time  $t$  is contained in the support of  $U$  at time  $t$ .*

We conclude the section with a similar statement for solutions which are not necessarily initially bounded.

**Proposition 7.1.** *For every initial measure  $\mu \in \mathcal{M}^+(\mathbb{R}^n)$  with compact support, there exists a weak energy solution with compact support at all times and satisfying a bound of the type (3.3).*

*Proof.* The existence of a solution satisfying (3.3) is proven in Theorem 3.1. To check it has compact support, let us approximate first the initial data by smooth initial data with same compact support and same bound  $M$  for their total mass  $\int u_0 dx$ . Then, by conservation of mass, for every  $t > 0$  we have the two a priori estimates  $\|u(t)\|_{L^1} \leq M$ ,  $\|u(t)\|_\infty \leq 1/t$ . By the Nash-Sobolev embeddings we then get  $\nabla p(\cdot, t) \in L^\infty(\mathbb{R}^n)$ . A simple scaling argument then shows that

$$(7.3) \quad \|\nabla p(\cdot, t)\|_\infty \leq C \|\Delta p(\cdot, t)\|_1^{1/n} \|\Delta p(\cdot, t)\|_\infty^{(n-1)/n} \leq C_1 M^{1/n} t^{-(n-1)/n}$$

Using the description by characteristics (valid for the approximating solutions, as we saw in the previous theorem) and integrating, it follows that the supports of the approximating solutions (compact by the previous theorem) are included in a ball of radius

$$(7.4) \quad R(t) \leq R_0 + C_2 M^{1/n} t^{1/n},$$

where  $C_2$  is independent of the approximation (and neither does  $R_0$ ). The conclusion follows by passing to the limit. Note that we estimate the growth of the support like  $O(t^{1/n})$  as expected in the best case.  $\square$

## 8 Asymptotic behaviour

The class of nonnegative solutions with finite mass has a property of simple asymptotic behaviour of self-similar form that is typical of diffusion processes, and also of some conservation laws. We want to prove that the self-similar vortex patch solutions of Section 6.2 are attractors of the finite-mass solutions. This has been proven in [12] in the case of compactly supported initial data. We extend it here to solutions with finite second moment and to all radial solutions.

### 8.1 Asymptotics by the entropy method

By using the entropy method we can prove asymptotic behaviour for solutions with general data, with the only extra requirement that the second moment of the initial mass distribution has to be finite. The proof follows the outline of the one done for the diffusive process with  $s < 1$  in [21].

In order to consider the question of large-time behaviour it is better to switch to the renormalized flow. More specifically, we introduce a change of the space, time and density variables corresponding to the expected large time behaviour, as follows:

$$(8.1) \quad y = x/(t+1)^{1/n}, \quad \tau = \log(1+t), \quad U(y, \tau) = (t+1)u(x, t),$$

as well as the corresponding change for pressure and velocity:

$$P(y, \tau) = (t+1)^{(n-2)/n}p(x, t), \quad V(y, \tau) = (t+1)^{(n-1)/n}v(x, t) = -\nabla P(y, \tau).$$

As a model for the asymptotic behaviour we consider the elementary vortex of Subsection 6.2 with initial height 1, i. e. ,

$$u_*(x, t) = \frac{1}{t+1} \quad \text{for } x \in B_R(t), \quad R(r) = R_0(t+1)^{1/n}.$$

and  $u_*(x, t)$  zero otherwise. We further choose  $R_0$  so that the total mass  $M$  is the same:

$$\int u_*(x, t) dx = \int u_0(x, t) dx,$$

(i. e.,  $\omega_n R_0^n = \|u_0\|_1 = M$ ). The corresponding renormalized density is

$$(8.2) \quad U_*(y, \tau) = \chi_{|y| \leq R_0},$$

and the corresponding pressure, defined by  $P_*(\tau, y) = (-\Delta)^{-1}U_*(\tau, y)$  satisfies

$$P_*(\tau, y) = (C - (y^2/2n)) \quad \text{and} \quad -\nabla P_*(\tau, y) = \frac{y}{n} \quad \text{in } B_{R_0}(0).$$

In the proofs below we will consider the solution  $U(y, \tau)$  of the renormalized flow associated to a solution  $u(x, t)$  via (8.1). Some direct calculations show that  $U$  solves

$$(8.3) \quad U_\tau = \nabla \cdot [U(\nabla P + y/n)], \quad \Delta P + U = 0,$$

where  $\nabla$  and  $\Delta$  indicate now differentiation w.r.t.  $y$ . Note that this is a mass conservation law

$$U_\tau + \nabla \cdot (U\vec{v}) = 0, \quad \vec{v} = -\nabla P - y/n,$$

where the velocity is the sum of the renormalized particle velocity and the confining velocity  $-y/n$  produced by the change of variables.

We next state the asymptotic behaviour result.

**Theorem 8.1.** *Let  $n \geq 2$  and assume that  $u_0 \in L^1(\mathbb{R}^n)$ ,  $u_0 \geq 0$ , and  $u_0$  has a finite second moment,  $\int_{\mathbb{R}^n} y^2 u_0(y) dy < \infty$ . Then, the speed and density converge to equilibrium after normalization, as follows:*

(i)  *$U$  stabilizes towards  $U_*$ , the elementary patch with the same mass, in the sense that*

$$(8.4) \quad \lim_{\tau \rightarrow \infty} \|U(y, \tau) - U_*(y)\|_1 = \lim_{t \rightarrow \infty} \|u(\cdot, t) - u_*(\cdot, t)\|_1 = 0.$$

*We also have strong convergence  $U(y, \tau) \rightarrow U_*(y)$  in all the  $L^p(\mathbb{R}^n)$  spaces,  $1 \leq p < \infty$ , and weak-\* convergence in  $L^\infty(\mathbb{R}^n)$ .*

(ii) *The corresponding velocities converge*

$$(8.5) \quad \lim_{\tau \rightarrow \infty} \left\| V(y, \tau) - \frac{y}{n} \right\|_{L^2(B_{R_0})} = 0.$$

*Proof.* (i) By displacing the origin of time by one unit, we may assume that our solution satisfies the a priori bound  $u(x, t) \leq \frac{1}{t+1}$ , hence  $U(y, \tau) \leq 1$ . The mass of the renormalized solution will be  $M$  at all times  $\tau \geq 0$ . This time displacement also allows to assume that the initial energy is finite, i. e.,  $\nabla p_0(y) \in L^2(\mathbb{R}^n)$ .

The boundedness of the orbit  $U(\cdot, \tau)$  in  $L^1 \cap L^\infty$  implies that as  $\tau \rightarrow \infty$  there is a sequence  $\tau_k$  such that  $U(\cdot, \tau_k)$  converges weakly to some  $U_\infty(y)$ , and also  $0 \leq U_\infty(y) \leq 1$ .

(ii) If  $n \geq 3$ , we consider the ‘‘entropy’’

$$(8.6) \quad \text{Ent}(U) := \frac{1}{2} \int_{\mathbb{R}^n} (|\nabla P|^2 + \frac{1}{n} U y^2) dy$$

which is the version for the renormalized flow of the energy (3.1). Recall that  $\nabla P = -V$  and  $\nabla \cdot V = U$ . Not also that under the stated assumptions  $\int |\nabla P|^2 dy = \int U P dy$ . It is easily calculated that the entropy evolves according to the rule

$$\frac{d}{d\tau} \text{Ent}(U(\tau)) = -D(U(\tau)), \quad D(U(\tau)) := \int U \left| V - \frac{y}{n} \right|^2 dy.$$

(See [21] for a similar entropy calculation for  $s < 1$ .)

(ii') If  $n = 2$ , this entropy is not finite and its definition needs to be modified according to Section 3.2: we set  $U_0$  to be any smooth compactly supported nonnegative function such that  $\int U_0 = \int U$ , and let  $P_0 = (-\Delta)^{-1} U_0$ . We then define

$$\text{Ent}(U) := \int_{\mathbb{R}^2} \left( \frac{1}{2} |\nabla(P - P_0)|^2 + U_0(P - P_0) + \frac{1}{2n} U y^2 \right) dy$$

which is finite. Computing as in Section 3.2 we find that

$$\frac{d}{d\tau} \text{Ent}(U(\tau)) = -D(U(\tau))$$

still holds, and we can continue the proof in the same way as for  $n \geq 3$ .

(iii) The fact that  $\text{Ent}(U(\tau))$  is nonincreasing along the orbit of the renormalized flow implies that there exists the limit

$$(8.7) \quad E_*(U) = \lim_{\tau \rightarrow \infty} \text{Ent}(U(\tau)).$$

Moreover, the integral  $\int_1^\infty \int U |V - (y/n)|^2 dy d\tau$  is convergent as an integral in an infinite time interval. Another consequence is that the second moment  $\int U(y, \tau) y^2 dy$  is bounded for all  $\tau$  (by  $2n \text{Ent}(U(0))$ , to be precise), and this control of the behaviour as  $y \rightarrow \infty$  allows to make sure that in the limit  $\int U_\infty(y) dy = M$ .

(iv) We now take limits as  $\tau_k \rightarrow \infty$  not only in the family  $\{U(y, \tau_k)\}_k$ ,  $k \rightarrow \infty$ , but in the family of orbits  $\{U(y, \tau + \tau_k)\}_k$ , for  $y \in \mathbb{R}^n$  and  $0 < \tau \leq T$  for fixed  $T > 0$ ;  $k \rightarrow \infty$ . The compactness (which can be simply reproduced from Theorem 3.1) allows to show that along a further subsequence that we also call  $\tau_k \rightarrow \infty$  we get

$$U(y, \tau + \tau_k) \rightarrow \tilde{U}(y, \tau),$$

in the weak sense, as well as  $V(y, \tau + \tau_k) \rightarrow \tilde{V}(y, \tau)$  strongly. Notice that  $\tilde{U}(y, 0) = U_\infty(y)$ . Since (8.7) holds, we also have

$$\text{Ent}(\tilde{U}(\tau)) = E_*(U), \quad \text{a constant; and } D(\tilde{U}(\tau)) = 0.$$

The last expression means that for a.e.  $\tau > 0$  and a.e.  $y$  we have

$$\tilde{U}(\tilde{V} - (y/n)) = 0,$$

which according to the renormalized evolution equation (8.3) and  $V = -\nabla P$  means that  $\partial_\tau \tilde{U} = 0$ . Therefore, the limit orbit is a stationary solution of the renormalized equation.

(v) We claim that our candidate  $U_*(y)$  of (8.2) is the only stationary solution satisfying the conditions  $\int \tilde{U}(y) dy = M$  and  $0 \leq \tilde{U} \leq 1$  and  $\tilde{V}(y) = y/n$  on the support of  $\tilde{U}$ .

First of all the a priori bound for  $\tilde{V}$  in  $L^\infty$  (due to integral kernel estimates on the representation formula performed in Formula (7.3)) implies that  $\tilde{V} = y/n$  cannot be true for large  $y$ , therefore  $\tilde{U}$  must be compactly supported. Once we know the compact support property, it is already proven in [12] that the asymptotic limit is  $\tilde{U} = U_*$  (this also follows easily from our analysis of Theorem 7.1).

(vi) The convergence of  $U(\cdot, \tau)$  towards  $U_*$  is at this point weak in  $L^1$ . But we also have  $\sup_y U(y, \tau) \leq 1$ , this unilateral bound allows to improve the weak convergence to strong convergence in the ball  $B_{R_0}(0)$ , i. e.,  $\|U(y, \tau) - U_*(y)\|_1 \rightarrow 0$  in that ball. But the mass of  $U_*$

contained in that ball is the whole mass, so that  $U(y, \tau)$  must converge to zero in the complement of the ball. This proves the main asymptotic formula (8.4). Since the solutions are uniformly bounded, the convergence in all the  $L^p$  spaces also follows for  $1 \leq p < \infty$ . However, the convergence in  $L^\infty$  is impossible due to the discontinuous form of the limit function.

(vii) The convergence of the velocities now follows from properties of the convolution formula for  $V$  in terms of  $U$ .

(viii) We still have to check that in displacing the origin of times from  $t = 0$  to  $t = 1$  we have not lost the property of finite second moment. Indeed, using the equation and several integrations by parts, we have

$$\begin{aligned} \frac{d}{dt} \int u x^2 dx &= \int \nabla \cdot (u \nabla p) x^2 dx = -2 \int x \cdot (u \nabla p) dx = 2 \int x \cdot \nabla p \Delta p dx \\ &= -2 \int \nabla p \cdot \nabla (x \cdot \nabla p) = -2 \int |\nabla p|^2 - \int x \cdot \nabla |\nabla p|^2 \\ &= (n-2) \int |\nabla p|^2 dx = (n-2) \int u p dx. \end{aligned}$$

If  $n = 2$  then we are done. If  $n \geq 3$ , we have found that the time derivative of the second moment is a constant times the energy (3.1) which is itself nonincreasing in time, hence bounded independently of time. It follows that the second moment remains bounded for all times.  $\square$

## 8.2 Asymptotic behaviour for radial solutions

We will perform next the proof that the large-time behaviour of finite-mass solutions is given by the self-similar vortex patch solutions of Section 6.3 in the case of radially symmetric data, where the Burgers' reformulation allows to use familiar tools. We do not need any further restriction on the class of initial data. A reference for asymptotic convergence of conservation laws is [46]. Section 3 of that paper applies directly in the case  $n = 1$  and it also does when  $n \geq 2$  after replacing the space variable  $r$  by  $s$ . Since we also want to estimate  $u(r, t)$ , which is a weighted derivative of the mass function  $M(r, t)$ , we give the details of the argument.

**Theorem 8.2.** *For any radial solution to (1.5) with integrable initial data, letting  $M(x, t)$  be its "mass function" as in (6.1) and assuming  $\max_{r \geq 0} M(r, 0) = M_0$ , we have  $0 \leq M(rt^{1/n}, t) \leq r^n/n$  and*

$$(8.8) \quad |r^n/n - M(rt^{1/n}, t)| \rightarrow 0$$

*uniformly in  $r \in \mathbb{R}$  and  $t \rightarrow \infty$ . Moreover, if  $u_0$  is compactly supported, the set of  $x$  such that  $M(|x|, t) < M_0$ , i.e. the support of  $u(x, t)$ , is included in a disk centered at the origin of radius  $S(t)$  such that  $S(t) \geq (nM_0t)^{1/n}$  and*

$$(8.9) \quad S(t) - (nM_0t)^{1/n} \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

*Proof.* (i) We first check the upper bound. First, since  $M(r, t)$  is increasing in  $r$ , it has a limit  $\lim_{r \rightarrow \infty} M(r, t) = \sup_r M(r, t)$  which is also equal to  $\int u(r, t) r^{n-1} dr = M_0$  hence is constant in time.

In view of the universal estimate of Section 5 we have  $u(x, t) \leq 1/t$  so that  $M(r, t) \leq r^n/nt$  and since  $\lim_{r \rightarrow \infty} M(r, t) = M_0$ , we deduce in view of the above that  $S(t) \geq (nM_0t)^{1/n}$  (this includes that possibility that  $S(t) \rightarrow \infty$ ). This settles half of the estimates.

(ii) In order to bound the solution from below we consider a vertically displaced function

$$M_1(r, t) = M(r, t) + \varepsilon, \quad \varepsilon > 0.$$

Then,

$$(M_1)_t = -\frac{1}{r^{n-1}}(M_1 - \varepsilon)(M_1)_r \geq -\frac{1}{r^{n-1}}M_1(M_1)_r$$

i.e.  $M_1$  is a supersolution of the original Burgers' equation. If  $R_0$  be determined by the condition  $M(R_0, 0) \geq M_0 - \varepsilon$ .  $R_0$  depends on  $\varepsilon$  and may be large but is finite. Then

$$M_1(r, 0) \geq \varepsilon \quad \text{for } 0 \leq r \leq R_0, \quad M_1(r, 0) = M_0 + \varepsilon \quad \text{for } r \geq R_0$$

We now consider the vortex patch solution  $u_2(r, t)$  with initial data  $\frac{1}{\tau}\chi_{B_R}$  and total mass  $M_0$ . For that we must have  $R^n/(n\tau) = M_0$ . We let  $M_2(r, t)$  be its mass function. Then

$$M_2(r, t) = \frac{r^n}{n(t + \tau)} \chi_{\{r \leq R(1+t/\tau)^{1/n}\}}.$$

If we choose  $\tau > 0$  so that  $\varepsilon \geq R_0^n/n\tau$ , then we will have  $M_1(r, 0) \geq M_2(r, 0)$ . By the maximum principle for (6.2), we deduce  $M_1(r, t) \geq M_2(r, t)$  for all  $r \geq 0$  and all  $t > 0$ . This means that

$$M(rt^{1/n}, t) \geq \frac{r^n}{n(1 + (\tau/t))} - \varepsilon \quad \text{for } r \leq [n(M_0 + \varepsilon)(1 + (\tau/t))]^{1/n}.$$

Passing to the limit  $t \rightarrow \infty$ , the term in  $\tau$  disappears, and letting then  $\varepsilon \rightarrow 0$  we easily arrive at both estimates (8.8).

(iii) The estimates on the support for compactly supported data are derived in Section 3 of [12], even with rates of convergence.  $\square$

As a consequence of this result, we have estimates for the radial velocity  $v = M/r^{n-1}$  that we resemble the velocity of the elementary vortex patch  $V(r, t) = r/nt$ . They are as follows: the velocity satisfies the estimates

$$v(r, t) \leq \frac{r}{nt} \quad \text{for } 0 < r < R(t)$$

and

$$(8.10) \quad \frac{r}{n} - t^{(n-1)/n} v(rt^{1/n}, t) \rightarrow 0$$

uniformly in  $r \leq S(t)/t^{1/n}$  as  $t \rightarrow \infty$ . We can also give estimates for the density.

**Corollary 8.1.** *The density satisfies the estimates*

$$(8.11) \quad u(r, t) \leq \frac{1}{t} \quad \text{for } 0 < r < R(t)$$

and  $tu(rt^{1/n}, t) \rightarrow \chi_{B_{R_1}(0)}(r)$  in  $L^\infty$  (with weak-\* convergence) as  $t \rightarrow \infty$ , where  $R_1 = (nM_0)^{1/n}$ .

*Proof.* It is convenient to introduce the rescaled density  $\hat{u}(r, \tau) = tu(rt^{1/n}, t)$ . We know that  $\hat{u}$  is bounded uniformly for large times by Theorem 5.1 and its support is uniformly bounded, by Theorem 8.2. It therefore converges in the weak-\* topology of  $L^\infty(\mathbb{R}^n)$  to a function  $U$  with compact support. Due to the relation between  $u$  and  $M$ , the limit  $U$  must be related to the rescaled mass function of the elementary vortex patch by  $r^{n-1}U(r) = tM'(rt^{1/n}) = M'(r)$ . We conclude that  $U(r)$  is the characteristic function of the ball of radius  $R_1$ . Since the limit is unique the convergence does not depend on the subsequence  $t_k \rightarrow \infty$ .  $\square$

**Remark 8.1.** *The example of the radial solution with an expanding ball together with an expanding annulus shows that the convergence of  $\hat{u}$  toward the equilibrium state  $\hat{U}(r) = \chi_{B_{M_0/n}(0)}(r)$  cannot happen in the uniform topology.*

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