□

8.2 Lower Bound

We now wish to compute a lower bound for $G_\varepsilon(u, A)$ which matches the upper bound of the previous section. In the course of the proof we will see clearly that if (u, A) minimizes G_ε, then its energy is accounted for by the vortex-energy.

In what follows we denote $B_\lambda^x = B(x, \lambda^{-1})$ and we will often omit the subscript ε, where x is the center of the blow-up.

Proposition 8.2. *Assume* $|\log\varepsilon| \ll h_{ex} \ll 1/\varepsilon^2$ *and* $(u_\varepsilon, A_\varepsilon)$ *minimizes* G_ε*. Then for any* $K > 0$*, there exists* $1 \ll \lambda \ll \frac{1}{\varepsilon}$ *such that for every* $x \in \Omega$ *such that* $B_\lambda^x \subset \Omega$*, we have*

$$G_\varepsilon(u_\varepsilon, A_\varepsilon, B_\lambda^x) \geq \frac{\alpha_K |B_\lambda^x|}{2} h_{ex} \log \frac{1}{\varepsilon\sqrt{h_{ex}}} (1 - o(1)), \tag{8.15}$$

where $\lim_{K\to+\infty} \alpha_K = 1$.

Proof. As already mentioned, the proof is achieved by blowing-up at the scale λ.

From Lemma 8.1 (and after translation), dropping the ε subscripts, the left-hand side of (8.15) is equal to

$$\frac{1}{2}\int_{B_1} |\nabla_{A_\lambda} u_\lambda|^2 + \lambda^2 \left(\operatorname{curl} A_\lambda - \frac{h_{\text{ex}}}{\lambda^2}\right)^2 + \frac{\left(1 - |u_\lambda|^2\right)^2}{2(\lambda\varepsilon)^2}$$

thus, letting $u' = u_\lambda$, $A' = A_\lambda$, $\varepsilon' = \lambda\varepsilon$ and $h'_{\text{ex}} = h_{\text{ex}}/\lambda^2$, the inequality (8.15) that we wish to prove is equivalent to

$$\frac{1}{2}\int_{B_1} |\nabla_{A'} u'|^2 + \lambda^2 \left(\operatorname{curl} A' - h'_{\text{ex}}\right)^2 + \frac{\left(1 - |u'|^2\right)^2}{2\varepsilon'^2} \geq \frac{\alpha_K |B_1|}{2} h'_{\text{ex}} \log \frac{1}{\varepsilon\sqrt{h_{\text{ex}}}} (1 - o(1)). \tag{8.16}$$

Now for any $\varepsilon > 0$ we choose λ such that

$$h'_{\text{ex}} = K|\log\varepsilon'|. \tag{8.17}$$

Let us check that this is possible and give the behavior of λ as $\varepsilon \to 0$. Condition (8.17) is equivalent to $\varepsilon^2 h_{\text{ex}} = f(\varepsilon\lambda)$, where $f(x) = Kx^2 \log(1/x)$.

Since $\varepsilon^2 h_{\rm ex} \to 0$ as $\varepsilon \to 0$, it is easy to check that for ε small enough, there is a unique $x_\varepsilon \in (0, 1/2)$ satisfying $f(x_\varepsilon) = \varepsilon^2 h_{\rm ex}$. Moreover from $|\log\varepsilon| \ll h_{\rm ex} \ll 1/\varepsilon^2$ we deduce $\varepsilon \ll x_\varepsilon \ll 1$. Therefore (8.17) can indeed be verified, and the corresponding λ, ε' satisfy

$$1 \ll \lambda \ll \frac{1}{\varepsilon}, \quad \varepsilon' \ll 1, \quad \log\frac{1}{\varepsilon\sqrt{h_{\rm ex}}} \approx |\log\varepsilon'|, \tag{8.18}$$

the last identity being deduced from $\varepsilon^2 h_{\rm ex} = f(\varepsilon\lambda) = f(\varepsilon')$ by taking logarithms. Thus with this choice of λ, (8.16) becomes

$$\frac{1}{2}\int_{B_1} |\nabla_{A'}u'|^2 + \lambda^2\left(\operatorname{curl} A' - h'_{\rm ex}\right)^2 + \frac{\left(1-|u'|^2\right)^2}{2\varepsilon'^2} \geq \frac{\alpha_K|B_1|}{2} h'_{\rm ex}|\log\varepsilon'|\left(1 - o(1)\right). \tag{8.19}$$

Two cases may now occur, depending on the blow-up origin x. Either

$$\frac{1}{2}\int_{B_1} |\nabla_{A'}u'|^2 + \lambda^2\left(\operatorname{curl} A' - h'_{\rm ex}\right)^2 + \frac{\left(1-|u'|^2\right)^2}{2\varepsilon'^2} \gg {h'_{\rm ex}}^2$$

as $\varepsilon \to 0$ and then, from (8.17), (8.19) is clearly satisfied, or

$$\frac{1}{2}\int_{B_1} |\nabla_{A'}u'|^2 + \lambda^2\left(\operatorname{curl} A' - h'_{\rm ex}\right)^2 + \frac{\left(1-|u'|^2\right)^2}{2\varepsilon'^2} \leq C{h'_{\rm ex}}^2.$$

This way, we have reduced to the case of configurations with a relatively small energy, for which all the analysis of previous chapters apply.

In this case, replacing ε by ε' and $h_{\rm ex}$ by $h'_{\rm ex}$, the hypotheses of Theorem 7.1, item 1) are satisfied and we deduce from (7.6), (7.8) that

$$\liminf_{\varepsilon'\to 0} \frac{1}{2{h'_{\rm ex}}^2}\int_{B_1} |\nabla_{A'}u'|^2 + \left(\operatorname{curl} A' - h'_{\rm ex}\right)^2 + \frac{\left(1-|u'|^2\right)^2}{2\varepsilon'^2} \geq \min_\mu E_K(\mu),$$

where E_K is defined in (7.6) (K plays now the role of λ in (7.6)). But, from the description of the minimizer μ_* following Corollary 7.1, we have, using the notations there, that $\mu_* = \left(1 - \frac{1}{2K}\right)\mathbf{1}_{\omega_K}$, and that

$$E_K(\mu_*) \geq \frac{1}{2K}\left(1 - \frac{1}{2K}\right)|\omega_K|,$$

where $|\omega_K| \to |B_1|$ as $K \to +\infty$. Therefore, replacing above, we find

$$\liminf_{\varepsilon' \to 0} \frac{1}{2{h'_{\rm ex}}^2} \int_{B_1} |\nabla_{A'} u'|^2 + \left(\operatorname{curl} A' - h'_{\rm ex}\right)^2 + \frac{\left(1 - |u'|^2\right)^2}{2{\varepsilon'}^2} \geq,$$

and we note now that

$${h'_{\rm ex}}^2 = K \frac{h_{\rm ex}}{\lambda^2} \log \frac{1}{\varepsilon \sqrt{h_{\rm ex}}},$$

therefore

$$\int_{B_1} |\nabla_{A'} u'|^2 + \left(\operatorname{curl} A' - h'_{\rm ex}\right)^2 + \frac{\left(1 - |u'|^2\right)^2}{\geq} \frac{h_{\rm ex}}{2} \log \frac{1}{\varepsilon \sqrt{h_{\rm ex}}} | \left(1 - \frac{1}{2K}\right) \frac{|\omega_K|}{\lambda^2},$$

which proves the desired result with

$$\alpha_K = | \left(1 - \frac{1}{2K}\right) \frac{|\omega_K|}{|B_1|}.$$

Clearly this tends to 1 as K tends to $= \infty$. □

To conclude the proof of Theorem 8.1, we integrate (8.15) with respect to x. Letting U be any open subdomain of Ω, using Fubini's theorem, we have

$$\int_{x \in U} G_\varepsilon(u, A, B_\lambda^x \cap U) = \iint_{\substack{x \in U \\ y \in B_\lambda^x \cap U}} g_\varepsilon(u, A)(y)\, dy\, dx$$

$$= \iint_{\substack{x \in U \\ y \in B_\lambda^x \cap U}} g_\varepsilon(u, A)(y)\, dx\, dy = \int_{y \in U} |B_\lambda^y \cap U| g_\varepsilon(u, A)(y)\, dy \leq \frac{\pi}{\lambda^2} G_\varepsilon(u, A, U)$$

We deduce that

$$\begin{aligned} \liminf_{\varepsilon \to 0} \frac{G_\varepsilon(u, A, U)}{h_{\rm ex} \log \frac{1}{\varepsilon \sqrt{h_{\rm ex}}}} &\geq \liminf_{\varepsilon \to 0} \int_{x \in U} \frac{\lambda^2 G_\varepsilon(u, A, B_\lambda^x \cap U)}{\pi h_{\rm ex} \log \frac{1}{\varepsilon \sqrt{h_{\rm ex}}}} \\ &\geq \liminf_{\varepsilon \to 0} \int_{x \in U, B_\lambda^x \subset U} \frac{\lambda^2 G_\varepsilon(u, A, B_\lambda^x \cap U)}{\pi h_{\rm ex} \log \frac{1}{\varepsilon \sqrt{h_{\rm ex}}}} \\ &\geq \int_{x \in U} \liminf_{\varepsilon \to 0} \left(\mathbf{1}_{B_\lambda^x \subset U} \frac{G_\varepsilon(u, A, B_\lambda^x)}{h_{\rm ex} |B_\lambda^x| \log \frac{1}{\varepsilon \sqrt{h_{\rm ex}}}} \right) \\ &\geq \alpha_K \frac{|U|}{2}, \end{aligned} \tag{8.20}$$

where we have used Fatou's lemma and (8.15). Since this is true for any $K > 0$, we may take the limit $K \to +\infty$ on the right-hand side and find

$$\liminf_{\varepsilon \to 0} \frac{G_\varepsilon(u, A, U)}{h_{\mathrm{ex}} \log \frac{1}{\varepsilon\sqrt{h_{\mathrm{ex}}}}} \geq \frac{|U|}{2}.$$

In view of Proposition 8.1, we know that $\left(h_{\mathrm{ex}} \log \frac{1}{\varepsilon\sqrt{h_{\mathrm{ex}}}}\right)^{-1} g_\varepsilon(u_\varepsilon, A_\varepsilon)$ is bounded in $L^1(\Omega)$, hence has a weak limit g in the sense of measures. Since continuous functions on Ω can be uniformly approximated by characteristic functions, (8.20) allows to say that $g \geq \frac{dx}{2}$. But since (8.5) holds, there must be equality, which proves (8.1), and (8.2) immediately follows.

Bibliographic notes on Chapter 8: The result of this chapter was obtained in [180], but the proof is presented here under a much simpler form. The case of higher h_{ex}, of order b/ε^2 with $b < 1$, was studied in [182].