Limiting Domain Wall Energy for a Problem Related to Micromagnetics

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Abstract

We study the asymptotic limit of a family of functionals related to the theory of micromagnetics in two dimensions. We prove a compactness result for families of uniformly bounded energy. After studying the corresponding one-dimensional profiles, we exhibit the Γ-limit ("wall-energy") which is a variational problem on the folding of solutions of the eikonal equation $|\nabla g| = 1$. We prove that the minimal wall-energy is twice the perimeter.

I Introduction

Let Ω be a magnetic body in $\mathbb{R}^3$ and $u : \Omega \to S^2$ its magnetization per unit volume. Following [HS] (page 148) the energy associated to such a magnetization is given by

$$F(u) = w^2 \int_\Omega |\nabla u|^2 + \int_{\mathbb{R}^3} |H_u|^2 + Q \int_\Omega \Phi(u) - 2 \int_\Omega h_e \cdot u, \tag{I.1}$$

where H_u is the demagnetizing field induced by the magnetization u and given by the following Maxwell equations

$$\begin{cases} \text{div } (\tilde{u} + H_u) = 0 & \text{in } \mathbb{R}^3 \\ \text{curl } H_u = 0 & \text{in } \mathbb{R}^3, \end{cases} \tag{I.2}$$

and $\tilde{u}$ is the extension of u by 0 out of the domain Ω. (In fact, H_u is defined as $-\nabla\Delta^{-1}(\text{div } \tilde{u})$.) h_e is some applied external field that will be zero in the present paper. Finally w^2 and Q are respectively the exchange constant and the quality parameter that only depend on the material. $F(u)$ can be decomposed as follows : $w^2\int_\Omega |\nabla u|^2$ is the exchange energy, $\int_{\mathbb{R}^3}|H_u|^2 = -\int_\Omega u\cdot H_u$ the demagnetizing energy, $Q\int_\Omega \Phi(u)$ the anisotropy energy and lastly $-2\int_\Omega h_e\cdot u$ the interaction energy. The reader is invited to consult the paper by A. DeSimone, R. Kohn, S. Müller and F. Otto [DKMO2] for a general presentation of the model together with related open mathematical questions. In this paper we will restrict to the simplified model in which the body is an infinite cylinder and the magnetization is invariant under translations along the axis of the cylinder so that we can reduce to the study in the cross section still denoted $\Omega\subset\mathbb{R}^2$. Then we assume that there is a very strong planar anisotropy so that u is forced to take values into S^1. Although such materials with strong planar anisotropy do exist (see [HS]), the restriction on u to be strictly in S^1, which is mathematically convenient, induces new topology that will generate perhaps non physically relevant parts in the limiting energy. Nevertheless, we believe that the local part of the limiting energy that we will exhibit in Theorem 4 below should arise in the original problem with strong planar anisotropy but with u free to evolve in all of S^2, and thus has some physical relevance. So we are led to study the following family of energy functionals:

$$E_\varepsilon(u) = \int_\Omega \varepsilon|\nabla u|^2 + \int_{\mathbb{R}^2}\frac{1}{\varepsilon}|H_u|^2, \tag{I.3}$$

depending on the positive parameter ε. $\Omega\subset\mathbb{R}^2$ is a smooth bounded simply connected domain, on which we shall make additional assumptions, and E_ε is defined over $H^1(\Omega, S^1)$, where S^1 is the unit circle in $\mathbb{C}\simeq\mathbb{R}^2$. Any $u:\Omega\to S^1$ is extended to

$$\tilde{u} = \begin{cases} u & \text{in } \Omega \\ 0 & \text{in } \mathbb{R}^2\backslash\Omega, \end{cases}$$

and H_u is a vector-field that is deduced from u by the equations

$$\begin{cases} \text{div } (\tilde{u}+H_u) = 0 & \text{in } \mathbb{R}^2 \\ \text{curl } H_u = 0 & \text{in } \mathbb{R}^2. \end{cases} \tag{I.4}$$

We are interested in the special regime $\varepsilon\to 0$ where the magnetization is allowed to vary on smaller and smaller scales. This corresponds to the limit $w\to 0$, of an exchange energy tending to 0. This exchange energy is often completely neglected and set to zero, allowing real discontinuities of u, which leads to a quite different behavior, only governed by the competition between the demagnetizing and anisotropy energies (see [JaKi]). However, here we will see how the exchange energy, though small, balances the demagnetizing energy in the limiting procedure.

There has been some work on this functional for a fixed ε, notably by Gilles Carbou [Ca], and then by R. Hardt and D. Kinderlehrer [HK], who studied the regularity of minimizers for $\Omega = B_1(0)$. Critical points of E_ε satisfy the following Euler equation

$$-\Delta u - u|\nabla u|^2 = \frac{1}{\varepsilon^2}(H_u \cdot u^\perp)u^\perp, \tag{I.5}$$

where $\cdot$ denotes the scalar product in $\mathbb{R}^2$, and $u^\perp = iu$. Carbou proved that a critical u_ε is continuous in Ω.

Now, let us explain briefly the expected behavior of minimizers. More generally, we can consider families u_ε such that $E_\varepsilon(u_\varepsilon) \leq C$. The term $\varepsilon \to 0$ forces the term $\int_{\mathbb{R}^2} |H_u|^2$ to tend to 0. In fact, we will prove more here, since we will show that $\min E_\varepsilon$ remains bounded above as $\varepsilon \to 0$. On the other hand, since $|u_\varepsilon| = 1$, we can extract a subsequence converging weakly in L^∞ to some u_0. One of the first questions that arises, and that we will answer positively, is to know whether the equality

$$|u_0| = 1 \tag{I.6}$$

holds. Then, since $H_u \to 0$ in $L^2(\mathbb{R}^2)$, in view of equation (I.2), u_0 must satisfy, in the sense of distributions

$$\text{div } \tilde{u}_0 = 0 \quad \text{in } \mathbb{R}^2. \tag{I.7}$$

This is equivalent to

$$\begin{cases} \text{div } u_0 = 0 & \text{in } \Omega \\ u_0 \cdot n = 0 & \text{on } \partial\Omega, \end{cases} \tag{I.8}$$

where n is the exterior unit normal to $\partial\Omega$. Therefore, the possible limiting fields lie among all divergence-free (in a weak sense) unit vector fields, tangent to $\partial\Omega$, which is a very large class of fields. Since (I.8) holds, there exists a $C_0^{0,1}(\Omega)$ real-valued function g such that

$$u_0 = \nabla^\perp g$$

where $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$. Thus, in other words, the limit $\varepsilon \to 0$ leads to solutions of the **eikonal equation**

$$\begin{cases} |\nabla g| = 1 & \text{a.e. in } \Omega \\ g = 0 & \text{on } \partial\Omega. \end{cases} \tag{I.9}$$

The difficulty is to understand which ones among these solutions can be reached through this limiting process. Then, we can examine the similarity between this problem and another problem, first raised by Aviles and Giga (see [AG1]), which has also been studied independently by L. Ambrosio, C. De Lellis and C. Mantegazza in [ADM], by A. DeSimone, R. Kohn, S. Müller and F. Otto in [DKMO1], and more recently by W. Jin and R. Kohn in [JK], consisting in minimizing the energy functionals

$$F_\varepsilon(\psi) = \int_\Omega \varepsilon|\nabla\nabla\psi|^2 + \frac{1}{\varepsilon}(1 - |\nabla\psi|^2)^2, \tag{I.10}$$

or more generally, in studying sequences such that $F_\varepsilon(\psi_\varepsilon) \le C$. The question that is addressed in [ADM, DKMO1], and also in a more recent work by P.E. Jabin and B. Perthame [JP1, JP2] with a kinetic equation approach, is to prove that for such sequences $u_\varepsilon = \nabla\psi_\varepsilon$ is compact in $\cap_{q<\infty} L^q$. Here we get a similar result for (I.3), ensuring that the constraint $|u| = 1$ is true in the limit. This type of problem already appeared in the study of the Ginzburg-Landau energy by F. Bethuel, H. Brezis, and F. Hélein [BBH]. F_ε can be seen as a Ginzburg-Landau-type energy for $u = \nabla\psi$, but the problem is quite different from [BBH] in that it is not elliptic in its essence and the singularities are not of vortex-type because the fields $\nabla\psi$ are constrained to be curl-free. Observe that being curl-free is the same as being divergence-free up to a rotation of $\frac{\pi}{2}$. On the other hand, while our fields are constrained to be of unit norm, the fields $\nabla\psi$ in (I.10) only tend to be of norm 1 in the limit $\varepsilon \to 0$. Hence, in some sense, the limiting processes are reversed. But as we shall see, there are other important differences between both problems. First, F_ε is a local funtional whereas E_ε contains a **nonlocal term** $\int_{\mathbb{R}^2} |H_u|^2$, and the same is then true for the Euler equations. Secondly, the expected Γ-limits for these functionals are quite different.

It is clear by a degree argument that there are no regular divergence-free unit vector fields that are tangent to $\partial\Omega$, for a general simply connected domain. All admissible vector-fields have line-singularities (except in the case of the ball, where the field $\frac{1}{2\pi}\frac{\partial}{\partial\theta}$ only has a point singularity at the origin). In the case of problem (I.10), the limiting fields also have line singularities. In fact, the conjecture made by Aviles and Giga (see also [AG2, JK]) is that the energy-functionals F_ε Γ-converge to

$$\frac{1}{3}\int_{J_{\nabla\psi}} |\nabla^+\psi - \nabla^-\psi|^3 d\mathcal{H}^1, \tag{I.11}$$

where $\mathcal{H}^1$ is the one-dimensional Hausdorff measure, $J_{\nabla\psi}$ is the jump set of $\nabla\psi$, expected to be countably rectifiable, and $\nabla^+\psi, \nabla^-\psi$ are the traces on both sides of the jump set. Thus, the energy E_ε is also expected to concentrate on lines, at a scale ε around the lines (because of the scaling in the functional), and the limit $\varepsilon \to 0$ corresponds to allowing sharper and sharper jumps. These lines would correspond in three dimensions to jumps accross surfaces, called "domain walls" in the theory of micromagnetics.

Let us state our main results. For reasons that will appear clearer later, we need to work on smaller functional spaces involving liftings of u. These definitions can be skipped in a first reading.

Definition I.1 *We call "admissible covering of Ω" a collection $\mathcal{U}$ of $((U_i)_{i\in I}, (k_{ij})_{i,j\in I})$ where*
1) Card $I < \infty$.
2) U_i is open and $\Omega \subset \cup_{i\in I}\overline{U_i}$, $\forall i,j \in I$, $U_i \cap U_j$ is diffeomorphic to $\mathbb{R}^2$.
3) $\forall i,j \in I$, $k_{ij} \in \mathbb{Z}$ and they satisfy the cocycle relations $k_{ij} = -k_{ji}$, $k_{ij} + k_{jl} + k_{li} = 0$.
For any admissible $\mathcal{U}$, we define $\Lambda_{\mathcal{U}}$ to be the set of $u \in H^1(\Omega, S^1)$ such that $\forall i \in I$, there exists $l_i \in [0, 2\pi]$ and $\psi_i \in H^1(U_i, [l_i, l_i + 2\pi])$ with

1) $u = e^{i\psi_i}$ *on* U_i, $\forall i \in I$.
2) $\forall i, j \in I$, $\psi_i - \psi_j = 2\pi k_{ij}$ *in* $U_i \cap U_j$.

We will say that the $u \in \Lambda_{\mathcal{U}}$ satisfy the "locally bounded-phase condition" or LBP condition.

Lemma I.1 *If* $u \in \Lambda_{\mathcal{U}}$ *there exists* $\varphi \in H^1(\Omega, \mathbb{R}) \cap L^\infty(\Omega, \mathbb{R})$ *such that*

$$u = e^{i\varphi} \quad \text{and} \quad ||\varphi||_{L^\infty(\Omega)} \leq C_{\mathcal{U}},$$

where $C_{\mathcal{U}}$ *is a constant that depends only on* $\mathcal{U}$.

This comes from the assumption that Ω is simply connected and that $\mathcal{U}$ is an admissible covering, thus our locally constant presheave is globally constant (see [BT] page 141). For existence of H^1-liftings of $H^1(\Omega, S^1)$, see [Ca2, BZ].

These definitions mean that we restrict to the set of u which have a $H^1(\Omega, \mathbb{R})$-lifting locally taking its values in an interval of length $\leq 2\pi$ of $\mathbb{R}$. This assumption, which is mainly required in the proof of Theorem 2, is not only there to make our life easier but seems to have also a fundamental mathematical relevance in our problem : indeed we will prove later that any solution of the limiting profile equation (I.19) solves the LBP condition : the smooth lifting has to be locally included in an interval of size 2π. The variational problem is well-posed in these spaces, ie, for any admissible $\mathcal{U}$, E_ε admits a minimum in $\Lambda_{\mathcal{U}}$, that we denote $I_\varepsilon^{\mathcal{U}}$.

Theorem 1 *If* $\partial\Omega$ *is a finite union of analytic curves, then*

$$\limsup_{\varepsilon \to 0} \min_{H^1(\Omega, S^1)} E_\varepsilon \leq 2|\partial\Omega|,$$

where $|\partial\Omega|$ *denotes the perimeter of* Ω.
In addition, there exists an admissible covering $\mathcal{U}$ *such that*

$$\limsup_{\varepsilon \to 0} I_\varepsilon^{\mathcal{U}} \leq 2|\partial\Omega|.$$

(We call such a $\mathcal{U}$ *a "suitable covering of* Ω*".)*

The second result is a compactness result, in the spirit of that of [ADM], together with a type of Γ-convergence result. We start by defining the "micromagnetism domain wall configuration space" $\mathcal{C}$, which is the suitable limiting configuration-set.

Definition I.2 $\mathcal{C}$ *is the class of couples* (u, φ) *such that*
1) $u : \Omega \to S^1$
2) $\operatorname{div} \tilde{u} = 0$ *in* $\mathcal{D}'(\mathbb{R}^2)$
3) $\varphi \in L^1(\Omega, \mathbb{R})$ *and* $u = e^{i\varphi}$ *a.e. in* Ω
4) $\mu_{u,\varphi} := \operatorname{div}(\varphi u + u^\perp)$ *is a bounded Radon measure on* Ω.
We denote by $||\mu_{u,\varphi}||$ *the total mass of* $\mu_{u,\varphi}$: $\int_\Omega |\mu_{u,\varphi}|$.

Note that $\mu_{u,\varphi}$ does not change if a constant is added to φ.
We are going to show that the limiting "functional" associated to E_ε is the following "domain-wall energy" :

$$\|\mu_{u,\varphi}\| = \int_\Omega |\text{div } (\varphi u + u^\perp)|,$$

defined over $\mathcal{C}$. This energy is a good candidate for making a selection among the solutions of the eikonal equation (I.9) and this selection should be compared to Conjecture 6.2 of [DKMO2]. We expect solutions of the eikonal equation belonging to $\mathcal{C}$ to satisfy nice properties (see the comments at the end of the introduction). Our second main result is the following compactness property.

Theorem 2 *(Without assumptions on $\partial\Omega$). Let $\mathcal{U}$ be an admissible covering of Ω, $\varepsilon_n \to 0$, and $u_n \in \Lambda_{\mathcal{U}}$ such that $E_{\varepsilon_n}(u_n) \leq C$, for some constant C independent of n, and let φ_n be a lifting of u_n given by Lemma I.1. Then, after extraction of a subsequence, there exists u and φ in $\cap_{1\leq q<\infty} L^q$ such that*

$$u_n \to u \quad \text{in } L^q(\Omega) \quad \forall 1 \leq q < \infty.$$

$$\varphi_n \to \varphi \quad \text{in } L^q(\Omega) \quad \forall 1 \leq q < \infty.$$

In addition, $(u, \varphi) \in \mathcal{C}$ and

$$\liminf_{n\to\infty} E_{\varepsilon_n}(u_n) \geq 2\|\mu_{u,\varphi}\|.$$

Thus, a sequence whose energy E_{ε_n} is uniformly bounded (which, from Theorem 1, is the case for a sequence of minimizers) is compact in $\cap L^q$, and its limits (after extraction) are necessarily of unit norm. $\mathcal{C}$ appears to be the right limiting configuration space. The LBP assumption is crucial in our proof of compactness.

The proof of the connection between E_ε and $\|\mu_{u,\varphi}\|$ at the limit can be sketched in the following way. Consider $\varepsilon_n \to 0$ and u_n such that $E_{\varepsilon_n}(u_n) \leq C$, and that u_n has a $H^1(\Omega, \mathbb{R}^2)$-lifting φ_n in Ω ($u_n = e^{i\varphi_n}$). Denoting by H_n the demagnetizing fields associated to u_n by (I.2), one has

$$\begin{aligned} C \geq E_{\varepsilon_n}(u_n) &= \int_\Omega \varepsilon_n |\nabla u_n|^2 + \int_{\mathbb{R}^2} \frac{1}{\varepsilon_n}|H_n|^2 \\ &\geq \int_\Omega \varepsilon_n |\nabla u_n|^2 + \frac{1}{\varepsilon_n}|H_n|^2 \\ &\geq 2\int_\Omega |\nabla u_n||H_n|. \end{aligned}$$

But $|\nabla u_n| = |\nabla \varphi_n|$ in Ω, hence we can write

$$C \geq E_{\varepsilon_n}(u_n) \geq 2\int_\Omega |\nabla \varphi_n \cdot H_n|. \tag{I.12}$$

We thus find a quantity $\mu_n = \nabla\varphi_n \cdot H_n$, which remains bounded in the sense of measures. This measure has the particular property that it is a divergence : indeed

$$\mu_n = \nabla\varphi_n \cdot H_n = -\nabla\varphi_n \cdot u_n + \nabla\varphi_n \cdot (u_n + H_n). \tag{I.13}$$

But if u_n is regular enough, it is easy to check that

$$\text{curl } u_n = \partial_{x_1}(\sin\varphi_n) - \partial_{x_2}(\cos\varphi_n) = \nabla\varphi_n \cdot u_n. \tag{I.14}$$

On the other hand, from (I.2), div $(u_n + H_n) = 0$ in Ω, hence

$$\nabla\varphi_n \cdot (u_n + H_n) = \text{div } (\varphi_n(u_n + H_n)). \tag{I.15}$$

Combining (I.13), (I.14) and (I.15), we have

$$\mu_n = \text{div } (\varphi_n u_n + u_n^\perp + \varphi_n H_n).$$

Since in addition $H_n \to 0$, $\varphi_n u_n + u_n^\perp + \varphi_n H_n$ is the sum of a local term $\varphi_n u_n + u_n^\perp$ and a term which vanishes and can be neglected. We have thus reduced a **nonlocal** problem to a **local** one, and in first approximation, we can write in view of (I.12) :

$$C \geq E_{\varepsilon_n}(u_n) \geq 2\int_\Omega |\mu_n|. \tag{I.16}$$

Since μ_n is bounded in measures, it has a weak limit which is a measure μ, and we can expect that

$$\liminf_{n\to\infty} E_{\varepsilon_n}(u_n) \geq 2\liminf_{n\to\infty}\int_\Omega |\mu_n| \geq 2\int_\Omega |\mu|$$

with $\mu = \text{div } (\varphi u + u^\perp)$ associated to the limit $u = e^{i\varphi}$ of u_n.

Next, we state a result on minimizers of the limiting problem.

Definition I.3 *$u_\star$ is the unit vector-field $\nabla^\perp(dist(.,\partial\Omega))$.*

We will prove in Section III that there exists $\varphi_\star$ such that $(u_\star, \varphi_\star)$ belongs to $\mathcal{C}$.

Theorem 3 *If $\partial\Omega$ is analytic by parts,*

$$\min_{(u,\varphi)\in\mathcal{C}} \|\mu_{u,\varphi}\| = |\partial\Omega|.$$

The minimum is achieved by $(u_\star, \varphi_\star)$.

We conjecture that the only minimizers are $u_\star$ and $-u_\star$. We give an heuristic justification of this fact in the last part of the introduction. These considerations have to be compared with the recent non-uniqueness result of [JK] for F_ε. As a consequence of Theorems 1, 2 and 3, we get the global understanding of minimizers of E_ε under the LBP condition.

Theorem 4 *Assume $\partial\Omega$ is analytic by parts. Let $\mathcal{U}$ be a suitable covering of Ω (i.e. $\limsup_{\varepsilon\to 0} I_\varepsilon^{\mathcal{U}} \le 2|\partial\Omega|$), then*

$$\lim_{\varepsilon\to 0} I_\varepsilon^{\mathcal{U}} = 2 \min_{(u,\varphi)\in\mathcal{C}} \|\mu_{u,\varphi}\| = 2|\partial\Omega|,$$

and a sequence of minimizers $u_\varepsilon = e^{i\varphi_\varepsilon}$ of E_ε over $\Lambda_{\mathcal{U}}$ converges strongly in $\cap L^q$ (both u_ε and φ_ε converge) to a minimizer (u,φ) of $\|\mu_{u,\varphi}\|$ over $\mathcal{C}$.

To give an idea of the lower bound $\|\mu_{u,\varphi}\| \ge |\partial\Omega|$ for $u \in \mathcal{C}$, recall that $u\cdot n = 0$ on $\partial\Omega$, and assume for instance that $u\cdot\tau = -1$ on $\partial\Omega$ where τ is the unit tangent to $\partial\Omega$ (with positive orientation). Then, integrating by parts, formally we have

$$\int_\Omega |\text{div } (\varphi u + u^\perp)| \ge \int_\Omega \text{div } (\varphi u + u^\perp) = \int_{\partial\Omega} \varphi u\cdot n - u\cdot\tau = |\partial\Omega|.$$

The proof needs to be refined when $u\cdot\tau$ changes sign on $\partial\Omega$, but the result is still true.

If $(u,\varphi)\in\mathcal{C}$, as already mentioned, u has singularities, and they are expected to be line singularities. Since, u is divergence-free, its normal component accross such singularity lines is preserved. Wherever u and φ are smooth, one has

$$\text{div } (\varphi u) = \nabla\varphi\cdot u = \text{curl } u = -\text{div } u^\perp,$$

hence $\mu_{u,\varphi} = 0$. $\mu_{u,\varphi}$ is thus supported on the singularities of u. Unfortunately, the condition that div $(\varphi u + u^\perp)$ is a measure does not guarantee that φ or u is in $BV(\Omega)$ (one can see this from the ideas of the counter-examples of [ADM]) : the class of possible limits $\mathcal{C}$ is quite large. Nevertheless, we may expect the measure $\mu_{u,\varphi}$ to be carried on a countably rectifiable set.

We have a more precise result in the case where $\varphi \in BV(\Omega)$ (where φ is a lifting of u given by the definition of $\mathcal{C}$). Then by definition, $D\varphi$ is a Radon measure, and it is standard that $D\varphi$ can be split into three mutually singular parts

$$D\varphi = \nabla\varphi\mathcal{L}^2 + (\varphi^+ - \varphi^-)\otimes n\mathcal{H}^1\lfloor_S + D_c\varphi \tag{I.17}$$

where $\mathcal{L}^2$ is the Lebesgue measure, $\mathcal{H}^1$ is the one-dimensional Hausdorff measure, $D_c\varphi$ is the Cantor part of $D\varphi$, S is the jump set of φ, n is the normal to S pointing from S into the + half-space, and φ^+ and φ^- are the approximate limits of φ on both "sides", + and − of S (see Section V.2 for more details).

Theorem 5 *1) Let $(u,\varphi)\in\mathcal{C}$ be such that $\varphi\in BV(\Omega)$. Then,*

$$\mu_{u,\varphi} = \text{div } (\varphi u + u^\perp) = \left((\varphi^+ - \varphi^-)(u\cdot n) - (u^+ - u^-)\cdot n^\perp\right)\mathcal{H}^1\lfloor_S.$$

Thus

$$\|\mu_{u,\varphi}\| = \int_S |(\varphi^+ - \varphi^-)(u\cdot n) - (u^+ - u^-)\cdot n^\perp| d\mathcal{H}^1.$$

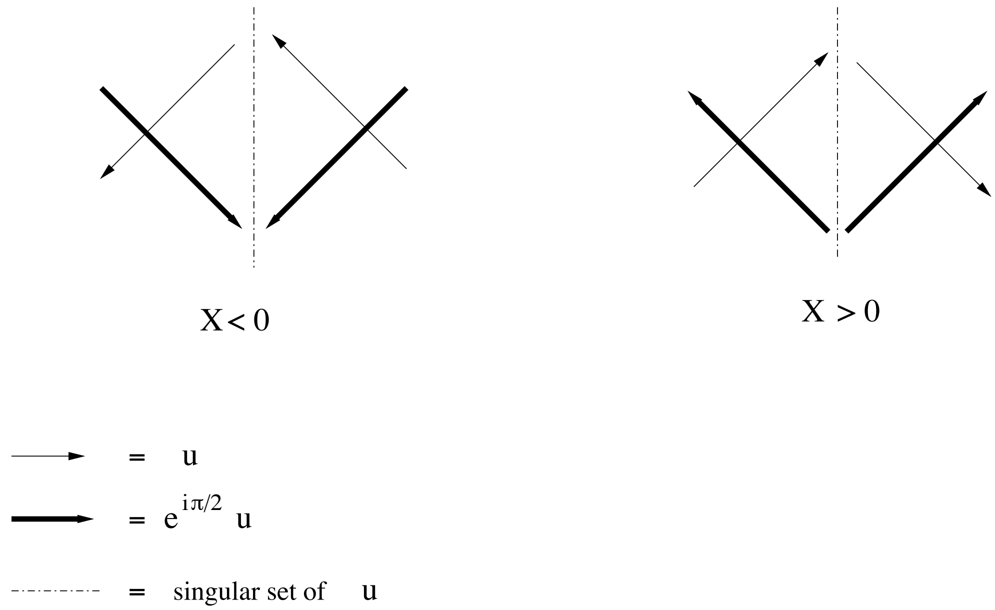

Figure 1: X is positive when iu is pointing **inwards** the singular set of u and X is negative when iu is pointing **outwards** of the singular set of u

If $|\varphi^+ - \varphi^-| < 2\pi$ in $\Omega' \subset \Omega$, we define the angle $X \in (-\pi, \pi)$ in the following way (see figure 1):

$$\begin{cases} |X| = \dfrac{|\varphi^+ - \varphi^-|}{2} \\ sgn(X) = -sgn(u^+ \cdot n^\perp). \end{cases}$$

Then we have

$$\mu_{u,\varphi}\lfloor_{\Omega'} = 2(\sin X - X\cos X)\mathcal{H}^1\lfloor_{S\cap\Omega'}.$$

The density of $\mu_{u,\varphi}$ has the sign of X.
2) If (u, φ) is a bounded family in BV and $u_\varepsilon = e^{i\varphi_\varepsilon}$ is such that $E_\varepsilon(u_\varepsilon) \leq C$ then, up to extraction, there exists $(u, \varphi) \in \mathcal{C}$ such that u and φ are in BV, for which $\varphi_\varepsilon \rightharpoonup \varphi$ in BV, $u_\varepsilon \rightharpoonup u$ in BV, and

$$\liminf_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon) \geq 2\|\mu_{u,\varphi}\| \geq 2|\partial\Omega|.$$

Remark I.1 : Observe that the sign of X and then the sign of the measure div $(\varphi u + u^\perp)$ is independent of the liftings chosen φ^+ and φ^- but only depends on u^+ and u^-, as long as we ensure $|\varphi^+ - \varphi^-| \leq 2\pi$.

Thus we have a more quantitative description of $\mu_{u,\varphi}$ in terms of the jumps of the phase of u. Observe that when X varies in $[0, \pi]$, $\sin X - X\cos X$ increases from 0 to π, and observe also that for $X \in [-\pi, \pi]$, $\sin X - X\cos X$ has the sign of X. In addition, when X is small

$$\sin X - X\cos X \sim \frac{X^3}{6}.$$

This is reminiscent of the case of (I.10) in [ADM] where the energy cost of the jumps in $\nabla\psi$ is given by (I.11). The two problems should thus have the same qualitative properties for small jumps (see also the discussion about lower semi-continuity in [ADM]). On the other hand, there is a major difference since we exhibit a cost of the jumps of φ and not only of u. If φ jumps by 2π for example, u remains continuous, but there is a cost in $\|\mu_{u,\varphi}\|$. This can be very well illustrated in the case of $\Omega = B_1(0)$. The limiting field $u_\star = \frac{1}{2\pi}\frac{\partial}{\partial\theta}$ only has a singularity at 0, but its phase must have a line singularity joining 0 to ∂B_1, and this costs $2\pi = |\partial B_1|$ in $\|\mu_{u_\star,\varphi_\star}\|$, whereas in that case, for F_ε of [ADM] or [DKMO1], the limiting energy is 0.

For general domains Ω, we prove that for the field $u_\star = \nabla^\perp \mathrm{dist}(.,\partial\Omega)$, $\int_S 2(\sin X - X\cos X) = |\partial\Omega|$. Thus, from this physical energy, we are led to a limiting functional which contains some of the geometric features of the domain, and we are led to unexpected results on the function $\mathrm{dist}(.,\partial\Omega)$. See Section V and Remark V.1.

The sign of X should play a role in the uniqueness result we expect for the minimizer for $\|\mu_{u,\varphi}\|$ in $\mathcal{C}$ (up to a reverse of sign). Following formally the kind of arguments of the proof of Lemma V.1, we can see that, for a minimizer, $\mu_{u,\varphi}$ has to be either positive or negative. Then, X is either positive everywhere, or negative everywhere. Changing u to $-u$ if necessary, we can assume X to be positive, and u to be $\nabla^\perp g$ for some $g \in C_0^{0,1}(\Omega)$. The uniqueness of the minimizer can be sketched the following way : starting from any point in the domain away from the singular set, we may follow the ray given by $u^\perp$. Because of the positivity of X, we see that it is not possible to cross the singular set (see figure 1). Then, the ray reaches the boundary without crossing the singular set. Thus, $g = dist(x,\partial\Omega)$, and $u = u_\star$.

It is tempting to imagine that there should exist a similar statement as the one of Theorem 5 for general $(u,\varphi) \in \mathcal{C}$ (or at least the ones that are limits of bounded sequences $(u_\varepsilon,\varphi_\varepsilon)$ for E_ε) without the assumption that $\varphi \in BV(\Omega)$. The main difficulty would be to prove that the measure div $(\varphi u + u^\perp)$ is supported on some set that has a nice structure, such as coutable rectifiability (coutable union of smooth curves), in order to be able to define the notions of $+$ and $-$ side, unit normal...etc. The exemple of [ADM] seems to indicate that it is too optimistic to expect the jump set to be rectifiable (finite mass). This comes from the fact that we only control X^3 (see discussion above) and not X for X small.

An adapted object to work with, that gives also another geometric interpretation of the measure div $(\varphi u + u^\perp)$ for solutions of the eikonal equation, is the following current : assume $(u,\varphi) \in \mathcal{C}$ is the L^q limit of a bounded sequence $(u_\varepsilon,\varphi_\varepsilon)$ for E_ε, we introduce the current $T_{u_\varepsilon,\varphi_\varepsilon}$ in $\Omega \times \mathbb{R}^3$ equal to

$$T_{u_\varepsilon,\varphi_\varepsilon} = \left\{ \begin{array}{c} (x, u_\varepsilon(x), tH_{u_\varepsilon}\cdot e) \in \Omega\times\mathbb{R}^3 \quad \text{where} \quad (x,t)\in\Omega\times[0,1] \\ H_{u_\varepsilon} \text{ is given by } (I.4) \text{ and } e \text{ is the unit oriented by } \nabla u_\varepsilon \end{array} \right\}$$

This current has a uniformly bounded mass as $\varepsilon \to 0$. In the case where φ_ε is uniformly

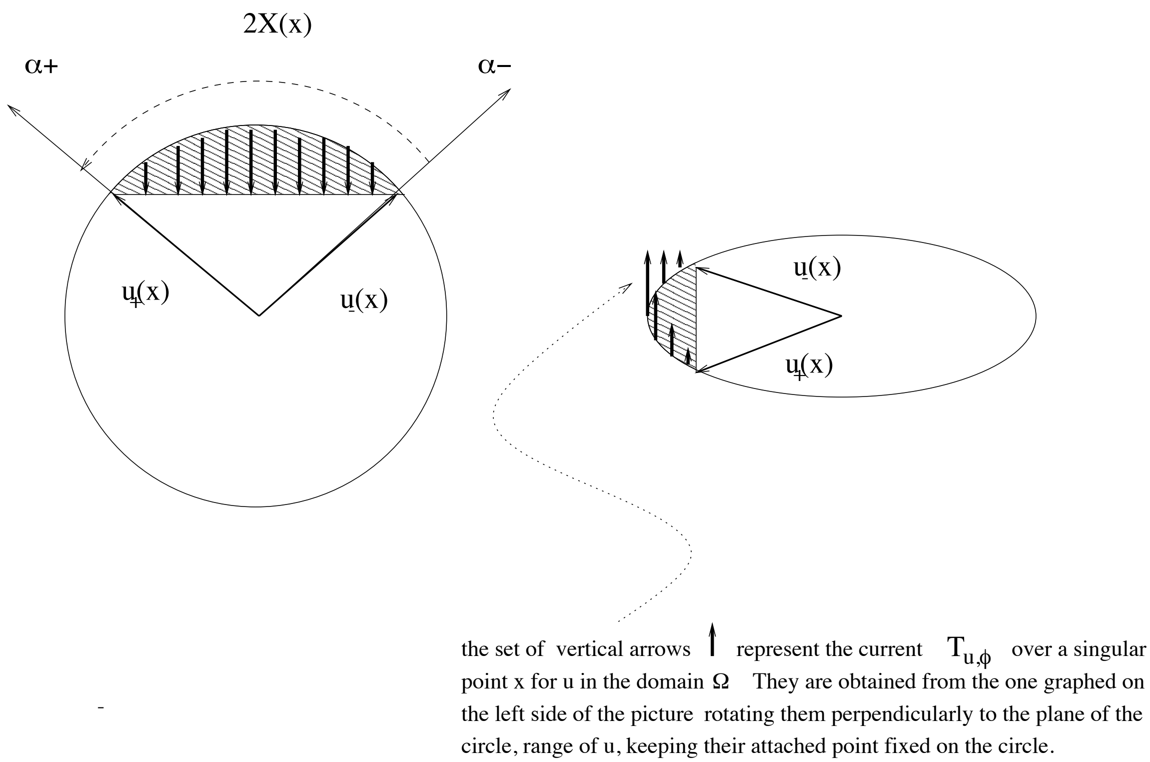

Figure 2: The current $T_{u,\varphi}$: a geometric interpretation of div $(\varphi u + u^\perp)$.

bounded in BV it converges to the following rectifiable current

$$T_{u,\varphi} = \left\{ \begin{array}{c} (x, e^{i\theta}, t\,\mathrm{sgn}(u^+ \cdot n^+) h(\theta, x)) \in \Omega \times \mathbb{R}^3 \quad \text{where} \quad (x,t) \in S \times [0,1] \quad \varphi_- \leq \theta \leq \varphi_+ \\ h(\theta, x) \text{ is the distance between } e^{i\theta} \text{ and the line } u^+, u^- \end{array} \right\}$$

(see figure 2). It is not difficult to see that

$$\text{(I.18)} \qquad M(T_{u,\varphi}) = \int_\Omega |\mathrm{div}\ (\varphi u + u^\perp)| = \|\mu_{u,\varphi}\|$$

In that case where φ is BV, we clearly have $M(\partial T_{u,\varphi}) < +\infty$. Again because of the example of [ADM] we do not expect this control of the mass of the boundary of the limit of $T_{u_\varepsilon,\varphi_\varepsilon}$ to hold in general, nevertheless we still expect the limit of $T_{u_\varepsilon,\varphi_\varepsilon}$, which has a finite mass, to be rectifiable, which should imply the countable rectifiability of the jump set of u on Ω.

We now sketch the outline of the paper. Section II contains some preliminary results, and the study of the one-dimensional problem : we expect variations of u_ε, minimizing E_ε, to be concentrated near S (jump set of $u_\star$) at a scale ε in the direction perpendicular to S. Therefore, we are led, after blow-up at the scale ε, to the one-dimensional ODE corresponding to solutions of (I.3) that depend only on one direction, which we prove to be solutions of

$$\text{(I.19)} \qquad \begin{cases} \varphi' = \sin\varphi - \sin\alpha_+ \\ \varphi(+\infty) = \alpha_+ \\ \varphi(-\infty) = \alpha_- \\ \sin\alpha_+ = \sin\alpha_-, \ 0 \le \alpha_+ - \alpha_- \le 2\pi. \end{cases}$$

This equation is the *profile* equation for the model. It is interesting to notice that (I.19) has no solution if the jump $\alpha_+ - \alpha_- > 2\pi$. This is probably related to our assumption $u \in \Lambda_{\mathcal{U}}$ for compactness.

In Section III, we prove the upper bound for $\min E_\varepsilon$, by constructing a test-configuration: it is the function $u_\star$ wherever it is smooth, and we paste the *profiles* (I.19) at a scale ε along its jump set. Then, we have to bound $\int_\Omega \frac{|H|^2}{\varepsilon}$ from above. This is done through a "projection lemma" (Lemma II.2), which permits to bound the *non-local* term $\int_\Omega |H|^2$ by *local* integrals. The assumption $\partial\Omega$ analytic by parts is needed to use a result of Choi, Choi and Moon [CCM], which asserts that in this case, the singular set of $u_\star$, called the "medial axis" of Ω, is also a finite union of analytic curves, while if $\partial\Omega$ is only assumed to be C^∞, the medial axis can be very pathological (see [CCM]). Yet, we believe that all the quantitative results ($\min E_\varepsilon \sim 2|\partial\Omega|$) would still be true without this assumption on $\partial\Omega$.

In Section IV, we prove the compactness result. Its proof is inspired from that of [ADM]. We use a convexity-type relation, the div-curl lemma, and Young measures, to prove that the limiting u must satisfy $|u| = 1$ a.e., and deduce strong convergence in $\cap_{q<\infty} L^p$, and the same for its lifting φ. The result of [JP1, JP2], which appeared after our work was completed, could allow the improvement of the compactness result for u, but apparently not for φ. Section V is devoted to Theorems 3 and 4 using the method that was already sketched, and to Theorem 5.

Notations : C is always a positive constant, independent of ε. $\mathcal{M}$ is the space of Radon measures on Ω. $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$, $u^\perp = iu = (-u_2, u_1)$. $B_r(x)$ denotes the ball of radius r centered at x.

II A few preliminary results

II.1 Preliminary results on demagnetizing fields

In this section, we consider a general vector field V on $\mathbb{R}^2$, bounded with compact support. To V is associated the induced demagnetizing field H_V defined by

$$\text{(II.1)} \qquad H_V = -\nabla\Delta^{-1}\text{div } V,$$

where the operator Δ^{-1} is defined as the convolution with the kernel $2\pi\log|x-y|$. Clearly, H_V satisfies

$$\text{(II.2)} \qquad \begin{cases} \text{div } (H_V + V) = 0 \text{ in } \mathbb{R}^2 \\ \text{curl } H_V = 0 \text{ in } \mathbb{R}^2 \end{cases}$$

Lemma II.1 *1) $\forall p < \infty$, there exists C_p depending only on the support of V such that $\|H_V\|_{L^p(\mathbb{R}^2)} \leq C_p \|V\|_{L^\infty}$.*
2) For any $x \in \mathbb{R}^2$ such that $dist(x, Supp\, V) \geq 1$, $|H_V(x)| \leq \frac{C}{|x|^2}\|V\|_{L^\infty}$, and $|\Delta^{-1}\text{div } V(x)| \leq \frac{C}{|x|}\|V\|_{L^\infty}$, where the constants C depend only on the support of V.
3)

$$\text{(II.3)} \qquad \int_{\mathbb{R}^2} |H_V|^2 = \int_{\mathbb{R}^2} (\text{div } V)(\Delta^{-1}\text{div } V) = -\int_{\mathbb{R}^2} V.H_V.$$

Proof :
1) $\forall p < \infty$, $\|V\|_{L^p}$ is bounded by a constant that depends only on the support of V. Thus, standard results on singular integrals give that div $V = -$div H_V is bounded in $W^{-1,p}$ and H_V in L^p.
2) We observe, using integration by parts, that for any x such that $\text{dist}(x, suppV) \geq 1$,

$$\begin{aligned} \left|\Delta^{-1}\text{div } V(x)\right| &= \left|\int_{\mathbb{R}^2} 2\pi\log|x-y|\text{div } V(y)dy\right| \\ &= \left|-2\pi\int_{\mathbb{R}^2} V(y).\nabla_y \log|x-y|dy\right| \\ &\leq 2\pi\int_{\mathbb{R}^2} \frac{|V(y)|}{|x-y|}dy \\ &\leq \frac{C}{|x|}\|V\|_{L^\infty}|suppV|, \end{aligned} \qquad \text{(II.4)}$$

where $|suppV|$ is the volume of $suppV$. Similarly,

$$\text{(II.5)} \qquad \nabla\Delta^{-1}\text{div } V(x) = \int_{\mathbb{R}^2} L(x,y)\text{div } V(y)dy,$$

where

$$L(x,y) = 2\pi\left(\frac{x_1 - y_1}{|x-y|^2}, \frac{x_2 - y_2}{|x-y|^2}\right),$$

and

$$\begin{aligned} \left|\nabla\Delta^{-1}\text{div } V(x)\right| &= \left|\int_{\mathbb{R}^2} L(x,y)\text{div } V(y)dy\right| \\ &= \left|-\int_{\mathbb{R}^2} V(y)\cdot\nabla_y L(x,y)dy\right| \\ &\leq \frac{C}{|x|^2}\|V\|_{L^\infty}|suppV|. \end{aligned} \qquad \text{(II.6)}$$

As a consequence, H_V is in $L^2(\mathbb{R}^2)$ and assertions 1) and 2) are true. Then, using the definition of H_V and several integrations by parts,

$$\begin{aligned}
\int_{\mathbb{R}^2} |H_V|^2 &= \int_{\mathbb{R}^2} H_V \cdot \nabla \Delta^{-1} \text{div } V \\
&= \int_{\mathbb{R}^2} -(\text{div } H_V) \Delta^{-1} \text{div } V \\
&= \int_{\mathbb{R}^2} (\text{div } V) \Delta^{-1} \text{div } V \\
&= -\int_{\mathbb{R}^2} V \cdot \nabla \Delta^{-1} \text{div } V \\
&= -\int_{\mathbb{R}^2} V \cdot H_V.
\end{aligned}$$

Hence, assertion 3) is proved. □

Remark II.1 : Applying this lemma, we deduce that for any sequence of unit vector-fields u_ε, $\|H_{u_\varepsilon}\|_{L^p}$ is uniformly bounded, $\forall p < \infty$.

Next we state what we call the "projection lemma". This lemma allows to estimate integrals of demagnetizing fields of the type $\int_{\mathbb{R}^2} |H_V|^2$ knowing V, without having to compute H_V, and comparing it only with **local** integrals.

Lemma II.2 *For any V bounded with compact support, we have*

$$\int_{\mathbb{R}^2} |H_V|^2 = \min_{g \in H^1(\mathbb{R}^2, \mathbb{R})} \int_{\mathbb{R}^2} |V - \nabla^\perp g|^2. \tag{II.7}$$

This lemma is in fact the Hodge-projection theorem in L^2. In other words, the result amounts to the fact that H_V is the L^2-orthogonal projection of V on the subspace of curl-free vector fields.

Of course, these lemmas apply to the test-functions for the problem of minimizing E_ε. From them, we can deduce a restatement of the original problem. Defining, over $H^1(\mathbb{R}^2, \mathbb{R})$ the functional

$$G_\varepsilon(g) = \min_{v \in H^1(\Omega, S^1)} \int_\Omega \varepsilon |\nabla v|^2 + \int_{\mathbb{R}^2} \frac{1}{\varepsilon} |\tilde{v} - \nabla^\perp g|^2, \tag{II.8}$$

using comparison arguments, one can check that

$$\min_{H^1(\Omega, S^1)} E_\varepsilon = \min_{H^1(\mathbb{R}^2, \mathbb{R})} G_\varepsilon.$$

II.2 Study of the one dimensional problem

II.2.1 Heuristic derivation of the problem

Consider some family u_ε of uniformly bounded energy E_ε. Its limiting field is a unitary divergence-free vector field u, and it should have line singularities. Accross such a line, since u is divergence free, $u \cdot n$ remains continuous. Therefore, the argument of u in the orthonormal frame relative to the line singularity can only jump from some α_+ to some α_-, with $\sin\alpha_+ = \sin\alpha_-$. Returning to the original field u_ε, we rescale it around x_0, a point of singularity of u, and set $\hat{u_\varepsilon}(x) = u_\varepsilon(x_0 + \varepsilon x)$. We expect $\hat{u_\varepsilon}$ to behave like a function u which depends only on one variable x_2, which minimizes

$$\int_{\mathbb{R}} |\nabla u|^2 + |H|^2, \tag{II.9}$$

and satisfies

$$\lim_{x_2 \to +\infty} \text{Arg } u = \alpha_+, \qquad \lim_{x_2 \to -\infty} \text{Arg } u = \alpha_-.$$

The rescaled demagnetizing field H at the limit also depends only on one variable x_2, and is solution of

$$\begin{cases} \text{div } (H + u) = 0 \\ \text{curl } H = 0. \end{cases} \tag{II.10}$$

Writing $u = e^{i\varphi} = (\cos\varphi, \sin\varphi)$, since u depends only on x_2, div $u = (\cos\varphi)\frac{d\varphi}{dx_2}$, hence u must solve

$$\begin{cases} \dfrac{dH_2}{dx_2} = -(\cos\varphi)\dfrac{d\varphi}{dx_2} \\ \dfrac{dH_1}{dx_2} = 0. \end{cases} \tag{II.11}$$

Thus, we see that

$$\begin{cases} H_1 = 0 \\ H_2 = \sin\alpha_+ - \sin\varphi \end{cases} \tag{II.12}$$

is the only solution if we require that H tends to 0 as $|x_2| \to \infty$. We are thus led to the following one-dimensional problem : minimize

$$\int_{\mathbb{R}} |\varphi'|^2 + |\sin\varphi - \sin\alpha_+|^2. \tag{II.13}$$

We will now consider this problem (II.13) independently.

II.2.2 Study of the ordinary differential equation

In the rest of this section, we consider angles α_+ and α_- such that

$$\left\{\begin{array}{l}\alpha_+ = -\pi - \alpha_- \\ \alpha_- \in [-\frac{3\pi}{2}, -\frac{\pi}{2}), \quad \alpha_+ \in (-\frac{\pi}{2}, \frac{\pi}{2}].\end{array}\right. \tag{II.14}$$

We shall write $\sin\alpha_- = \sin\alpha_+ = \sin\alpha$ for simplicity. We consider the energy-functional

$$F(\varphi) = \int_{\mathbb{R}} |\varphi'|^2 + |\sin\varphi - \sin\alpha|^2. \tag{II.15}$$

To this functional is naturally associated the ordinary differential equation :

$$\varphi'' = (\sin\varphi - \sin\alpha)\cos\varphi, \tag{II.16}$$

and if we add to it the limit conditions we are interested in, we get the problem :

$$\left\{\begin{array}{l}\varphi'' = (\sin\varphi - \sin\alpha)\cos\varphi \\ \varphi(-\infty) = \alpha_-, \quad \varphi(+\infty) = \alpha_+.\end{array}\right. \tag{II.17}$$

We prove the following

Proposition II.1 *Let α_+ and α_- satisfy $\alpha_- < \alpha_+$ and $\sin\alpha_+ = \sin\alpha_-$.*
1) If $\alpha_+ - \alpha_- > 2\pi$, then (II.17) has no solution of finite energy F.
2) If α_+ and α_- satisfy (II.14), then (II.17) admits one solution φ_α such that $F(\varphi_\alpha) < \infty$ and $\varphi_\alpha(0) = -\frac{\pi}{2}$. φ_α has the following properties :

$$\varphi'_\alpha = \sin\alpha - \sin\varphi_\alpha. \tag{II.18}$$

There exists a constant C independent from α_+ and α_- such that

$$\left\{\begin{array}{ll}|\varphi_\alpha - \alpha_+| \leq \dfrac{C}{|x|} & \text{for } x \geq 0 \\ |\varphi_\alpha - \alpha_-| \leq \dfrac{C}{|x|} & \text{for } x \leq 0.\end{array}\right. \tag{II.19}$$

If $\alpha_+ - \alpha_- < 2\pi$, then there exists a constant C_α depending on α_+ and α_-, such that

$$\left\{\begin{array}{ll}|\varphi_\alpha - \alpha_+| \leq C_\alpha e^{-x} & \text{for } x \geq 0 \\ |\varphi_\alpha - \alpha_-| \leq C_\alpha e^{x} & \text{for } x \leq 0.\end{array}\right. \tag{II.20}$$

Denoting by $X = \frac{\alpha_+ - \alpha_-}{2} = -\frac{\pi}{2} - \alpha_-$, we have $0 < X \leq \pi$ and

$$F(\varphi_\alpha) = \int_{\mathbb{R}} (\varphi'_\alpha)^2 + (\sin\varphi_\alpha - \sin\alpha)^2 = 4(\sin X - X\cos X). \tag{II.21}$$

Proof :
- *Step 1 :* Equation (II.16) has trivial constant solutions $\varphi \equiv \alpha_+ + 2k\pi$, $\varphi \equiv \alpha_- + 2k\pi$, $k \in \mathbb{Z}$. Consider any nonconstant finite-energy solution φ of (II.16). Necessarily,

$$|\varphi''| = |\sin\varphi - \sin\alpha||\cos\varphi| \leq |\sin\varphi - \sin\alpha|,$$

hence $\varphi'' \in L^2(\mathbb{R})$. This implies that $\varphi' \in C^{0,\frac{1}{2}}(\mathbb{R})$ and φ' is uniformly continuous on $\mathbb{R}$. It also implies that $\varphi'\varphi'' \in L^1(\mathbb{R})$ and thus $(\varphi')^2(x) = \int^x 2\varphi'\varphi''$ remains bounded. We can thus assert that $(\varphi')^2$ is uniformly continuous on $\mathbb{R}$, because φ' is and it is bounded. Consequently, since $\int_{\mathbb{R}} |\varphi'|^2 < \infty$, we have $\varphi' \to 0$ at ∞.
Similarly, $(\sin\varphi - \sin\alpha)^2$ is uniformly continuous and integrable, hence must tend to 0 at ∞.
Multiplying (II.16) on both sides by φ' and integrating between x and ∞, we are led to

$$(\varphi')^2 = (\sin\varphi - \sin\alpha)^2. \tag{II.22}$$

- *Step 2 :* We claim that φ' does not vanish. Should there exist x_0 such that $\varphi'(x_0) = 0$, then from (II.22), we would have $\sin\varphi(x_0) = \sin\alpha$, and thus $\varphi''(x_0) = 0$ from (II.16). Then, $\psi \equiv \varphi(x_0)$ is a constant solution of (II.16), which satisfies

$$\begin{cases} \psi'' = (\sin\psi - \sin\alpha)\cos\psi \\ \psi(x_0) = \varphi(x_0) \\ \psi'(x_0) = 0. \end{cases} \tag{II.23}$$

But φ is also a solution of (II.23), and using the Cauchy-Lipschitz theorem on ordinary differential equations, problem (II.23) has a unique solution, thus φ is the constant solution, and we are led to a contradiction, since it was assumed to be nonconstant. The claim is proved.

- *Step 3 :* We deduce assertion 1) of the proposition. If $\alpha_+ - \alpha_- > 2\pi$, and φ solves (II.17), then $\varphi(+\infty) - \varphi(-\infty) > 2\pi$; therefore, by continuity of φ, there exists a $x_0 \in \mathbb{R}$ such that $\sin\varphi(x_0) = \sin\alpha$. But φ satisfies (II.22), hence $\varphi'(x_0) = 0$, and this is impossible from Step 2.

- *Step 4 :* We prove the existence result. First, we recall that α_+ and α_- are assumed to satisfy (II.14). We set

$$\Phi(t) = \int_{-\frac{\pi}{2}}^{t} \frac{du}{\sin\alpha - \sin u}. \tag{II.24}$$

Φ is defined on (α_-, α_+). When $u \to \alpha_+$,

$$\sin u - \sin\alpha_+ \sim (u - \alpha_+)\cos\alpha_+ - \frac{1}{2}(u - \alpha_+)^2 \sin\alpha_+, \tag{II.25}$$

hence $\Phi \to +\infty$ as $t \to \alpha_+$. Similarly, $\Phi \to -\infty$ as $t \to \alpha_-$. Furthermore,

$$\Phi'(t) = \frac{1}{\sin\alpha - \sin t} > 0 \text{ on } (\alpha_-, \alpha_+)$$

thus Φ is increasing, and defines a diffeomorphism between (α_-, α_+) and $\mathbb{R}$. We can thus consider its inverse function $\varphi(x) = \Phi^{-1}(x)$. φ is increasing from $\mathbb{R}$ to (α_-, α_+) and satisfies

$$\lim_{x\to+\infty} \varphi = \alpha_+, \ \lim_{x\to-\infty} \varphi = \alpha_-.$$

In addition,

$$\varphi'(x) = \frac{1}{\Phi'(\Phi^{-1}(x))} = \sin\alpha - \sin\varphi(x). \tag{II.26}$$

Consequently, φ solves (II.17). In addition, from (II.14), φ takes the value $-\frac{\pi}{2}$. But, for any $c \in \mathbb{R}$, $\varphi(.+c)$ is also a solution of (II.17), therefore we can find φ_α solution of (II.17) such that $\varphi_\alpha(0) = -\frac{\pi}{2}$.

- *Step 5 :* We prove the stated decay of φ_α. If $\cos\alpha_+ \neq 0$, then from (II.25), $\sin u - \sin\alpha_+ \sim_{\alpha_+} (u - \alpha_+)\cos\alpha_+$ hence

$$\Phi \sim_{\alpha_+} \frac{-1}{\cos\alpha_+}\log\,(\alpha_+ - x),$$

and therefore, $\varphi_\alpha - \alpha_+$ decays exponentially as $x \to +\infty$. The argument is similar near $-\infty$. If $\cos\alpha_+ = 0$, then from (II.25), $\sin u - \sin\alpha_+ \sim_{\alpha_+} -\frac{1}{2}(u-\alpha_+)^2\sin\alpha_+$ with $\sin\alpha_+ = 1$, hence

$$\Phi \sim_{\alpha_+} \frac{2}{\alpha_+ - x}$$

and $\varphi_\alpha - \alpha_+$ decays like $1/x$ as $x \to +\infty$, and similarly as $x \to -\infty$. It thus has a decay rate in $1/|x|$ independently from α_+ and α_-. We deduce that $(\varphi_\alpha')^2$ and $(\sin\varphi_\alpha - \sin\alpha)^2$ are in $L^1(\mathbb{R})$ and $F(\varphi_\alpha) < \infty$.

- *Step 6 :* There remains to calculate $F(\varphi_\alpha)$. From (II.26),

$$F(\varphi_\alpha) = \int_{\mathbb{R}} (\varphi_\alpha')^2 + (\sin\varphi_\alpha - \sin\alpha)^2 = 2\int_{\mathbb{R}} (\sin\alpha - \sin\varphi_\alpha)\varphi_\alpha'.$$

Performing the change of variables $\varphi_\alpha(x) = t$, we get

$$\begin{aligned} F(\varphi_\alpha) &= 2\int_{\alpha_-}^{\alpha_+} (\sin\alpha - \sin t)\,dt \\ &= 2[\cos\alpha_+ - \cos\alpha_- + (\alpha_+ - \alpha_-)\sin\alpha] \\ &= 4\cos\alpha_+ + 2(\alpha_+ - \alpha_-)\sin\alpha. \end{aligned}$$

Using the notation $X = \frac{\alpha_+ - \alpha_-}{2} = \alpha_+ + \frac{\pi}{2}$, we have $0 < X \leq \pi$, $\cos\alpha_+ = \sin X$, $\sin\alpha = -\cos X$, and the previous equality becomes

$$F(\varphi_\alpha) = 4(\sin X - X\cos X).$$

$\square$

III Upper bound for the micromagnetism energy

In this section, we prove Theorem 1, by constructing a suitable test-configuration. This test-function is obtained by deformation of the field $u_\star = \nabla^\perp \mathrm{dist}(.,\partial\Omega)$, using the one-dimensional profiles described in Section II.2.

III.1 Studying $u_\star$

Here, we mention some important results on $u_\star$ relying on the paper [CCM] by H.I. Choi, S.W. Choi and H.P. Moon. $u_\star$ is obviously divergence-free, tangent to $\partial\Omega$ (hence div $\tilde{u_\star} = 0$), and of unit norm wherever $\mathrm{dist}(.,\partial\Omega)$ is differentiable. If p is a point in Ω that admits a unique nearest-point projection on $\partial\Omega$, then $\mathrm{dist}(.,\partial\Omega)$ is differentiable near p and $u_\star$ is regular near p. Thus, $u_\star$ is singular only at points which do not admit a unique nearest-point projection on $\partial\Omega$. In [CCM], they study this set, called the medial axis of Ω. They define it as

$$\Sigma = \left\{ \begin{array}{l} p \in \Omega / \exists r \geq 0, \overline{B_r(p)} \subset \overline{\Omega}, \text{and} \\ \text{if } \overline{B_s(q)} \subset \overline{\Omega} \text{ and } \overline{B_r(p)} \subset \overline{B_s(q)}, \text{ then } \overline{B_r(p)} = \overline{B_s(q)} \end{array} \right\}. \tag{III.1}$$

One can check that Σ coincides with the set of points which do not admit a nearest-point projection on $\partial\Omega$, and is also the set of the centers of maximal disks inscribed in Ω.
If Ω is a ball, then Σ is reduced to the center. If not, and if $\partial\Omega$ is a finite union of analytic curves (which we shall sum up by "analytic by parts"), then they prove that Σ is path connected and is a finite union of real analytic curves of finite length, plus a finite number of end points or branch points called vertices. We will call $\mathcal{V}$ the set of vertices.

Lemma III.1 *If $\partial\Omega$ is analytic by parts, there exists $\varphi_\star \in BV(\Omega)$ smooth except on a finite union of analytic curves S such that the jump of $\varphi_\star$ along S is in $[0, 2\pi]$, and $u_\star = e^{i\varphi_\star}$.*

Proof : If $\partial\Omega$ has "corners", then Σ is adherent to $\partial\Omega$. If $\partial\Omega$ is analytic, consider a point p on $\partial\Omega$ such that the geodesic curvature of $\partial\Omega$ at p is positive (such a point always exists), and the normal to $\partial\Omega$ at p. Following this normal direction inside Ω, then one crosses Σ. Let us denote by T this segment joining p to Σ. $S = T \cup \Sigma$ is still a finite union of analytic curves and a finite number of vertices, and is path-connected. Since the topological degree of $u_\star$ from $\partial\Omega$ to S^1 is 1, it is possible to define a real-valued $\varphi_\star$ on $\partial\Omega\backslash\{p\}$ such that $u_\star = e^{i\varphi_\star}$ on $\partial\Omega$, and that $\varphi_\star$ is continuous except for a jump of 2π at p. It is then possible to extend $\varphi_\star$ to Ω into a BV function which is regular on $\Omega\backslash(T\cup\Sigma)$. We do it the following way : if $x \in \Omega\backslash(T\cup\Sigma)$, then x has a nearest-point projection y on $\partial\Omega$, we set $\varphi_\star(x) = \varphi_\star(y)$. Then we have

$$e^{i\varphi_\star} = u_\star = \nabla^\perp(\mathrm{dist}(.,\partial\Omega)) \text{ in } \Omega\backslash S,$$

and $\varphi_\star$ is C^∞ in $\Omega\backslash S$. For example, in the case of the ball, just consider $\varphi_\star(r,\theta) = \theta + \frac{\pi}{2}$ in polar coordinates, where the argument θ lies in $[0, 2\pi)$. Then, $\varphi_\star$ is regular everywhere except on the segment $\theta = 0$ and for $r = 0$. We easily check that $u_\star \in \mathcal{C}$.

In the general case, the jump of $\varphi_\star$ on T is exactly 2π by construction. We then consider $x \in \Sigma$ and wish to evaluate the discontinuity of $\varphi_\star$ at x across Σ. By definition of Σ, there exist at least two "rays" coming from $\partial\Omega$ crossing at x. By "ray", we mean a segment with one end point on $\partial\Omega$ and the other on Σ, normal to $\partial\Omega$, and included in Ω. By construction, $\varphi_\star$ is constant on all "rays". Let then R_1 and R_2 be two such rays crossing at x, and let x_1 and x_2 be their end points on $\partial\Omega$. We write $\beta = (\overrightarrow{xx_1}, \overrightarrow{xx_2}) \in (0, 2\pi)$. Let then l be the connected component of $\partial\Omega\backslash\{x_1, x_2\}$ that does not contain p. We thus have a closed curve, union of l, $[x, x_1]$ and $[x, x_2]$. We assume for example, that going from x_1 to x_2 on l and then from x_2 to x_1 via the segments, we follow the trigonometric orientation (see figure 3). We can write that the integral of the geodesic curvature K_g along this curve is equal to 2π:

$$\int_l K_g + \frac{\pi}{2} + (\pi - \beta) + \frac{\pi}{2} = 2\pi.$$

Here, we have used the fact that $[x, x_1]$ and $[x, x_2]$ are orthogonal to $\partial\Omega$. Then,

$$\int_l K_g = \pi - (\pi - \beta) = \beta \in (0, 2\pi).$$

But this integral is equal to $-\int_l \frac{d\tau}{ds} \cdot n$ where τ is the unit tangent on l, n the outer unit normal to l and s the parametrization with respect to the arc-length on $\partial\Omega$. But, $u_\star = -\tau$ on $\partial\Omega$, hence the previous integral is $\int_l \frac{\partial u_\star}{\partial s} \cdot n = \int_l \frac{\partial u_\star}{\partial s} \cdot u_\star^\perp = \int_l \frac{\partial \varphi_\star}{\partial s} = \varphi_\star(x_2) - \varphi_\star(x_1)$. We thus have obtained that

$$0 < \varphi_\star(x_2) - \varphi_\star(x_1) < 2\pi. \tag{III.2}$$

In addition the jump of $\varphi_\star$ at x is exactly equal to $|\varphi_\star(x_2) - \varphi_\star(x_1)|$ by construction, hence we have the desired result. $\square$

Definitions of X, α_+, α_- for $u_\star$.
If $x \in \Sigma\backslash\mathcal{V}$ then x has exactly two nearest-point projections on $\partial\Omega$, x_1 and x_2 (see[CCM]) and the direction tangent to Σ separates x_1 and x_2. We can thus choose n to be the unit normal vector to Σ at x pointing towards x_2 i.e. $n \cdot \overrightarrow{xx_2} \geq 0$. If τ is the unit tangent to Σ such that (τ, n) is a direct orthonormal frame, then this means that

$$u_\star.\tau \geq 0 \quad \text{on } (x, x_2]. \tag{III.3}$$

If $x \in T$ we orient $-\tau$ pointing towards $\partial\Omega$. By construction of T, $u_\star$ is normal to T and $u_\star = n = \tau^\perp$.

Then, at each point $x_0 \in S\backslash\mathcal{V}$, we have defined a unit normal vector n, and we denote by $\varphi_\star^+$ and $\varphi_\star^-$ the values of $\varphi_\star$ for $x \to x_0 (\overrightarrow{x_0x} \cdot n > 0)$ and $x \to x_0 (\overrightarrow{x_0x} \cdot n < 0)$ respectively. In view of the previous proof, on Σ, we have $\varphi_\star^+ = \varphi_\star(x_2)$, $\varphi_\star^- = \varphi_\star(x_1)$. Hence,

$$0 < \varphi_\star^+ - \varphi_\star^- \leq 2\pi. \tag{III.4}$$

On $T, \varphi_\star^+ - \varphi_\star^- = 2\pi$, hence (III.4) is verified $\mathcal{H}^1$-a.e. on S. If s still denotes the parameter with respect to the arc-length on S, we can then choose $\theta(s)$ such that $e^{-i\theta}\tau = (1,0)$ and

$$\theta - \frac{1}{2}(\varphi_\star^+ + \varphi_\star^- - \pi) \in [-\pi, \pi). \tag{III.5}$$

We distinguish again two cases. If $x \in T$, then there exists $k \in \mathbb{Z}$ such that $\varphi_\star^+ - \theta = \frac{\pi}{2} + 2k\pi$ and $\varphi_\star^- = \varphi_\star^+ - 2\pi = \theta + \frac{\pi}{2} + 2(k-1)\pi$. In view of (III.5), the only possibility is $k = 0$. We set

$$\alpha_+(s) = \varphi_\star^+ - \theta = \frac{\pi}{2} \qquad \alpha_-(s) = \varphi_\star^- - \theta = -\frac{3\pi}{2}.$$

If $x \in \Sigma \backslash \mathcal{V}$, then, since $u_\star$ is divergence-free, its normal component is preserved accross Σ, hence

$$\sin(\varphi_\star^+ - \theta) = \sin(\varphi_\star^- - \theta),$$

therefore

$$\exists k \in \mathbb{Z}, \quad \varphi_\star^+ - \theta = \pi - (\varphi_\star^- - \theta) + 2k\pi.$$

But, from (III.3), we also have

$$\cos(\varphi_\star^+ - \theta) \geq 0,$$

and combining this with (III.4) and (III.5), the only possibility is $k = -1$ and

$$\begin{aligned} \varphi_\star^+ - \theta &= -\pi - (\varphi_\star^- - \theta) \\ \varphi_\star^+ - \theta &\in \Big(-\frac{\pi}{2}, \frac{\pi}{2} \Big] \\ \varphi_\star^- - \theta &\in \Big(\frac{3\pi}{2}, -\frac{\pi}{2} \Big]. \end{aligned}$$

We set in all cases $\alpha_+(s) = \varphi_\star^+ - \theta$, $\alpha_-(s) = \varphi_\star^- - \theta$. They satisfy (II.14). In addition, $\alpha_-(s)$, $\alpha_+(s)$ and $\theta(s)$ are C^1 on S except on a finite number of points. If some of these excpetional points are not in $\mathcal{V}$, we add them to $\mathcal{V}$. Denoting $X = \frac{\varphi_\star^+ - \varphi_\star^-}{2}$, we have $X \in (0, \pi]$ and $X = \frac{\alpha_+ - \alpha_-}{2}$, $\cos X = -\sin\alpha$, $\sin X = -\cos\alpha_-$.

Remark III.1 : We recall that if $\partial\Omega$ is not assumed to be a finite union of analytic curves, but only C^∞ for example, then it seems more difficult to construct such a lifting of $u_\star$ because there exist such Ω's for which Σ is not of finite length or has an infinite number of branch points (see [CCM] for counter-examples).

Remark III.2 : If Ω is convex, then $\varphi_\star$ takes its values in $[l, l + 2\pi]$ for some $l \in [0, 2\pi]$.

III.2 Constructing the test-functions

We use the notation of the previous subsection, and the orthonormal frame (τ, n) along the line of singularities S of $u_\star$, wherever possible. We choose a γ such that $\frac{1}{2} < \gamma < 1$. We recall that S is a C^1 curve except on a finite set of points $\mathcal{V}$ and denote by S_ε the subset of S composed of all the points which are at a distance $\geq \varepsilon^\gamma$ from $\mathcal{V}$. Thus S is locally C^1 near any point of S_ε. We then define the set B_ε to be

$$B_\varepsilon = \{x = x_0 + \sigma n, |\sigma| \leq \varepsilon^\gamma, \ x_0 \in S_\varepsilon, \ n \text{ is the normal to } S_\varepsilon \text{ at } x_0\}.$$

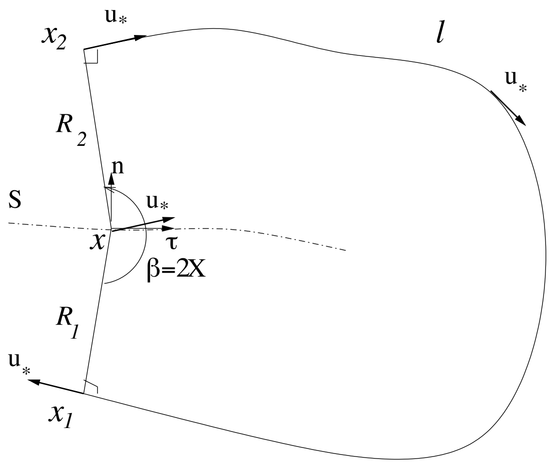

Figure 3:

For ε small enough, B_ε is thus an ε^γ-tubular neighborhood of S_ε, and each point of B_ε can be determined uniquely by coordinates (s,σ) where s corresponds to the parameter with respect to the arc-length on S and σ to the distance to S_ε. We also denote $\Omega_\varepsilon = \Omega\backslash\{x/\mathrm{dist}(x,S)\leq\varepsilon^\gamma\}$ and $V_\varepsilon = \Omega\backslash(\Omega_\varepsilon\cup B_\varepsilon)$.
We recall that the argument of $\varphi_\star$ in the local frame is $\alpha_+(s)$ as $\sigma\to 0^+$ and $\alpha_-(s)$ as $\sigma\to 0^-$, with α_+ and α_- satisfying (II.14). Let us also denote $\beta_+(s)$ the angle of $\varphi_\star$ in the orthonormal frame (τ,n) at the point (s,ε^γ), and $\beta_-(s)$ the one at $(s,-\varepsilon^\gamma)$, $\beta_+(s):=\varphi_\star(s,\varepsilon^\gamma)-\theta(s),\beta_-(s):=\varphi_\star(s,-\varepsilon^\gamma)-\theta(s)$. Of course, these β, contrarily to the α's, depend on ε. Along Σ, since $\alpha_+-\alpha_->0$, for ε small enough we have $\beta_+>-\pi/2$ and $\beta_-<-\pi/2$. Since T is a ray starting at a point of $\partial\Omega$ where the curvature is positive it is not difficult to see that for ε small enough, we also have $\beta_+<\pi/2$ and $\beta_->-3\pi/2$, although $\alpha_+=\pi/2$ and $\alpha_-=-3\pi/2$. Then, from the study of the equation (II.16) we know the existence of $\tilde\alpha^+(s)\in[-\pi/2,\pi/2)$ and $\tilde\alpha^-(s)\in[-\pi/2,\pi/2)$ such that $\varphi_{\tilde\alpha^+(s)}(\varepsilon^\gamma/\varepsilon)=\beta_+(s)$ and $\varphi_{\tilde\alpha^-(s)}(\varepsilon^\gamma/\varepsilon)=\beta_-(s)$ (a priori there is no reason why $\tilde\alpha^+(s)$ and $\tilde\alpha^-(s)$ should coincide). Using the C^1 behavior of $u_\star$ in $\Omega\backslash S$, it is clear that the following bound holds

$$|\alpha_\pm-\beta_\pm|\leq C\varepsilon^\gamma, \tag{III.6}$$

and using (II.19),

$$\left|\tilde\alpha^\pm(s)-\alpha_+\right|\leq C\varepsilon^{1-\gamma} \tag{III.7}$$

We then define the test-function u_ε in the local coordinates (s,σ), as

(III.8)
$$\begin{cases} u_\varepsilon(\sigma,\sigma)=\left(\cos(\varphi_{\tilde\alpha^\pm(s)}(\sigma/\varepsilon))\right)\tau+\left(\sin(\varphi_{\tilde\alpha^\pm(s)}(\sigma/\varepsilon))\right)n=e^{i(\psi+\theta)} & \text{for } 0\leq\pm\sigma\leq\varepsilon^\gamma \text{ in } B_\varepsilon \\ u_\varepsilon=e^{i\varphi_\star} \text{ in } \Omega_\varepsilon \end{cases}$$

In V_ε, we extend the phase of u_ε in such a way that it remains continuous, and that $|\nabla u_\varepsilon| \le \frac{C}{\varepsilon}$.
We also define in B_ε

(III.9)
$$\begin{cases} K_\varepsilon(s,\sigma) = \left(\cos(\varphi_{\tilde{\alpha}^{\pm}(s)}(\sigma/\varepsilon))\right)\tau + (\sin(\beta_\pm(\sigma/\varepsilon^\gamma) + \alpha_\pm(1-\sigma/\varepsilon^\gamma)))\, n \quad \text{for } 0 \le \pm\sigma \le \varepsilon^\gamma \text{ in } B_\varepsilon \\ K_\varepsilon = u_\star \quad \text{in } \Omega_\varepsilon \end{cases}$$

Notice that K_ε is not unit-valued. We extend K_ε on V_ε in the same way as u_ε, with $|K_\varepsilon| \le 1$. We thus end up with two vector-fields u_ε and K_ε which are continuous on Ω and have the following properties :

Lemma III.2

$$\begin{aligned} &\|\nabla u_\varepsilon\|_{L^\infty(\Omega)} \le \frac{C}{\varepsilon} \\ &|\mathrm{div}\ K_\varepsilon| \le O(1) \quad \textit{in } \Omega\backslash V_\varepsilon \\ &|\mathrm{div}\ K_\varepsilon| \le \frac{1}{\sup(\varepsilon, dist(x,S))} \quad \textit{for } dist(x,S) \le \varepsilon^\gamma. \end{aligned}$$

Proof : The upper-bound for ∇u is straightforward from the construction. In B_ε, we have, for $0 \le \pm\sigma \le \varepsilon^\gamma$,

$$\begin{aligned} \mathrm{div}\ K_\varepsilon &= \nabla\left(\cos\varphi_{\tilde{\alpha}^{\pm}(s)}\left(\sigma/\varepsilon\right)\right)\cdot\tau + \left(\cos\varphi_{\tilde{\alpha}^{\pm}(s)}\left(\sigma/\varepsilon\right)\right)\mathrm{div}\ \tau \\ &+ \nabla\left(\sin(\beta_\pm\sigma/\varepsilon^\gamma + \alpha_\pm(1-\sigma/\varepsilon^\gamma))\right)\cdot n + \sin(\beta_\pm\sigma/\varepsilon^\gamma + \alpha_\pm(1-\sigma/\varepsilon^\gamma))\mathrm{div}\ n, \end{aligned}$$

and from this expression, it is clear using (III.6) that, as $\varepsilon \to 0$, $|\mathrm{div}\ K_\varepsilon| = O(1)$ in $\Omega\backslash V_\varepsilon$.
$\square$

III.3 Evaluation of their energy - Proof of Theorem 1

We prove Theorem 1. The proof is divided into several lemmas.

Lemma III.3

$$\frac{1}{\varepsilon}\int_\Omega |u_\varepsilon - K_\varepsilon|^2 + \varepsilon\int_{B_\varepsilon}|\nabla u_\varepsilon|^2 \le \int_S 4(\sin X(s) - X(s)\cos X(s))ds + o(1).$$

Proof : First, observe that in Ω_ε, $u_\varepsilon = K_\varepsilon = u_\star$, hence

$$\frac{1}{\varepsilon}\int_\Omega |u_\varepsilon - K_\varepsilon|^2 = \frac{1}{\varepsilon}\int_{\Omega\backslash\Omega_\varepsilon}|u_\varepsilon - K_\varepsilon|^2,$$

and, since $|u_\varepsilon - K_\varepsilon| \le 2$ on V_ε, and $\gamma > \frac{1}{2}$,

$$\begin{aligned} \frac{1}{\varepsilon}\int_\Omega |u_\varepsilon - K_\varepsilon|^2 &\le \frac{1}{\varepsilon}\int_{B_\varepsilon}|u_\varepsilon - K_\varepsilon|^2 + C\frac{\varepsilon^{2\gamma}}{\varepsilon} \\ &\le \frac{1}{\varepsilon}\int_{B_\varepsilon}|u_\varepsilon - K_\varepsilon|^2 + o(1). \end{aligned} \tag{III.10}$$

From the definitions (III.8) and (III.9) of u_ε and K_ε,

$$\begin{aligned}\frac{1}{\varepsilon}\int_{B_\varepsilon}|u_\varepsilon - K_\varepsilon|^2 \quad &\leq \quad \sum_{\pm}\frac{1}{\varepsilon}\int_{0\leq\pm\sigma\leq\varepsilon^\gamma}\left|\sin\varphi_{\tilde\alpha^\pm(s)}\left(\sigma/\varepsilon\right) - \sin\tilde\alpha^\pm(s)\right|^2 \\ &\quad + \sum_{\pm}\frac{1}{\varepsilon}\int_{0\leq\pm\sigma\leq\varepsilon^\gamma}|\sin\tilde\alpha^\pm(s) - \sin(\beta_\pm\sigma/\varepsilon^\gamma + \alpha_\pm(1-\sigma/\varepsilon^\gamma))|^2. \end{aligned} \tag{III.11}$$

Using (III.7) we have

$$\frac{1}{\varepsilon}\int_{0\leq\pm\sigma\leq\varepsilon^\gamma}|\sin\tilde\alpha^\pm(s) - \sin(\beta_\pm\sigma/\varepsilon^\gamma + \alpha_\pm(1-\sigma/\varepsilon^\gamma))|^2 \leq \frac{C}{\varepsilon}\,|B_\varepsilon|\;|\varepsilon^{1-\gamma}|^2 = O(\varepsilon^{1-\gamma})$$

and (III.11) becomes

$$\frac{1}{\varepsilon}\int_{B_\varepsilon}|u_\varepsilon - K_\varepsilon|^2 \leq \sum_{\pm}\frac{1}{\varepsilon}\int_{0\leq\pm\sigma\leq\varepsilon^\gamma}\left|\sin\varphi_{\tilde\alpha^\pm(s)}\left(\sigma/\varepsilon\right) - \sin\tilde\alpha^\pm(s)\right|^2 + o(1). \tag{III.12}$$

On the other hand,

$$\begin{aligned}&\int_{0\leq\pm\sigma\leq\varepsilon^\gamma}\varepsilon|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}\left|\sin\varphi_{\tilde\alpha^\pm(s)}\left(\sigma/\varepsilon\right) - \sin\tilde\alpha^\pm(s)\right|^2 \\ &\qquad = \int_{0\leq\pm\sigma\leq\varepsilon^\gamma}\frac{1}{\varepsilon}\left|\nabla\varphi_{\tilde\alpha^\pm(s)}\left(\sigma/\varepsilon\right)\right|^2 + \frac{1}{\varepsilon}\left|\sin\varphi_{\tilde\alpha^\pm(s)}\left(\sigma/\varepsilon\right) - \sin\tilde\alpha^\pm(s)\right|^2\,ds\,d\sigma \\ &\qquad + \int_{0\leq\pm\sigma\leq\varepsilon^\gamma}\varepsilon\left|\frac{\partial}{\partial s}\left(\varphi_{\tilde\alpha^\pm(s)}(\sigma/\varepsilon)\right)\right|^2\,ds\,d\sigma. \end{aligned} \tag{III.13}$$

$\frac{\partial}{\partial s}\left(\varphi_{\tilde\alpha^\pm(s)}(\sigma/\varepsilon)\right) = \frac{d\tilde\alpha^\pm(s)}{ds}\frac{\partial\varphi_{\alpha(s)}}{\partial\alpha}\left(\sigma/\varepsilon\right)$ remains bounded, since α is C^1 on $S\backslash\mathcal{V}$. Furthermore, by definition of φ_α,

$$\nabla\varphi_{\alpha(s)}(x) = \sin\alpha(s) - \sin\varphi_{\alpha(s)}(x). \tag{III.14}$$

Inserting this into (III.13), we obtain

$$\begin{aligned}&\int_{0\leq\pm\sigma\leq\varepsilon^\gamma}\varepsilon|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}\left|\sin\varphi_{\tilde\alpha^\pm(s)}\left(\sigma/\varepsilon\right) - \sin\tilde\alpha^\pm(s)\right|^2 \\ &= \frac{2}{\varepsilon}\int_{0\leq\pm\sigma\leq\varepsilon^\gamma}\left(\sin\tilde\alpha^\pm(s) - \sin\varphi_{\tilde\alpha^\pm(s)}\left(\sigma/\varepsilon\right)\right)\nabla\varphi_{\tilde\alpha^\pm(s)}\left(\sigma/\varepsilon\right)\,ds\,d\sigma + o(1),\end{aligned}$$

where we have used a coarea formula relative to π_S, the orthogonal projection from B_ε into S, and the fact that $||\pi_S^* ds| - 1| \leq C_S\varepsilon^\gamma$ and that $\gamma > 1/2$ (where C_S involves informations on the curvature of S). Performing the change of variable $y = \sigma/\varepsilon$,

$$\int_{|\sigma|\leq\varepsilon^\gamma} \varepsilon|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}\left|\sin\varphi_{\tilde\alpha^{sgn(\sigma)}(s)}\left(\frac{\sigma}{\varepsilon}\right) - \sin\tilde\alpha^{sign(\sigma)}(s)\right|^2$$

$$= 2\int_s\int_{|y|\leq\varepsilon^{\gamma-1}} \left(\sin\tilde\alpha^{sgn(y)}(s) - \sin\varphi_{\tilde\alpha^{sgn(y)}(s)}(y)\right)\nabla\varphi_{\tilde\alpha^{sign(\sigma)}(y)}(y)\,dy\,ds + o(1)$$

$$\leq 2\int_s\left(\int_{-\infty}^{+\infty} \left(\sin\tilde\alpha^{sgn(y)}(s) - \sin\varphi_{\tilde\alpha^{sgn(y)}(s)}(y)\right)\nabla\varphi_{\tilde\alpha^{sgn(y)}(s)}(y)\,dy\right)ds + o(1)$$

$$\leq \int_s 4(\sin X(s) - X(s)\cos X(s))\,ds + o(1), \tag{III.15}$$

where we have used (II.21) and the fact that $\tilde\alpha^{\pm} + \frac{\pi}{2}$ tend to $X = \alpha_+ + \frac{\pi}{2}$ as ε tends to 0. Combining (III.10), (III.12), and (III.15), we get the result. □

Lemma III.4 *Denoting by H_{u_ε} the magnetic field induced by $\tilde u_\varepsilon$, we have*

$$\frac{1}{\varepsilon}\int_{\mathbb{R}^2}|H_{u_\varepsilon}|^2 \leq \frac{1}{\varepsilon}\int_\Omega |u_\varepsilon - K_\varepsilon|^2 + o(1).$$

We postpone the proof of this lemma until later, in order to state the last lemma

Lemma III.5

$$\varepsilon\int_{\Omega\setminus B_\varepsilon}|\nabla u_\varepsilon|^2 = o(1) \quad \textit{as } \varepsilon \to 0.$$

Proof : First, by definition, $u_\varepsilon = u_\star$ in Ω_ε, hence $\varepsilon\int_{\Omega_\varepsilon}|\nabla u_\varepsilon|^2 \to 0$.
Secondly, the contribution coming from V_ε is negligible, because there $|\nabla u_\varepsilon| \leq C\varepsilon^{-1}$, and hence

$$\varepsilon\int_{V_\varepsilon}|\nabla u_\varepsilon|^2 \leq C\varepsilon\frac{\varepsilon^{2\gamma}}{\varepsilon^2} = \varepsilon^{2\gamma-1} = o(1).$$

This completes the proof of the lemma. □

If we assume the result of Lemma III.4 to be true, then in view of Lemmas III.3 and III.5, we have constructed some $u_\varepsilon \in H^1(\Omega, S^1)$ for which

(III.16)
$$\varepsilon\int_\Omega|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}\int_{\mathbb{R}^2}|H_{u_\varepsilon}|^2 \leq \int_S 4\left(\sin X(s) - X(s)\cos X(s)\right)\,ds + o(1) \qquad \text{as } \varepsilon \to 0.$$

If we assume that $\int_S 4(\sin X - X\cos X) = 2|\partial\Omega|$, which shall be proved in Lemma V.2, then the first part of Theorem 1 is established.

Proof of Lemma III.4 : The idea relies on the projection lemma, Lemma II.2.

For simplicity, we drop the subscripts ε. We denote by H_K the magnetic field induced by the configuration $\tilde{K}_\varepsilon$ (which is K_ε extended by 0 outside Ω. Let g achieving

$$\min_{g\in H^1(\mathbb{R}^2,\mathbb{R})}\int_{\mathbb{R}^2}|\tilde{K}-\nabla^\perp g|^2,$$

then, as seen in Section II, $H_K=\tilde{K}-\nabla^\perp g$. Furthermore, the projection lemma ensures that

$$\begin{aligned}\left(\int_{\mathbb{R}^2}|H_u|^2\right)^{\frac{1}{2}} &\le \left(\int_{\mathbb{R}^2}|\tilde{u}-\nabla^\perp g|^2\right)^{\frac{1}{2}}\\ &\le \left(\int_{\mathbb{R}^2}|\tilde{u}-\tilde{K}|^2\right)^{\frac{1}{2}}+\left(\int_{\mathbb{R}^2}|\tilde{K}-\nabla^\perp g|^2\right)^{\frac{1}{2}}\\ &\le \left(\int_{\Omega}|u-K|^2\right)^{\frac{1}{2}}+\left(\int_{\mathbb{R}^2}|H_K|^2\right)^{\frac{1}{2}}.\end{aligned}\tag{III.17}$$

Consequently, the lemma shall be proved if we show that

$$\frac{1}{\varepsilon}\int_{\mathbb{R}^2}|H_K|^2\to 0.\tag{III.18}$$

We now prove (III.18). From Lemma III.2, $\|\mathrm{div}\ K\|_{L^\infty}\le C\varepsilon^{-1}$, on the other hand div $K=0$ in Ω_ε, hence K satisfies the hypotheses of Lemma II.1. We decompose $\Omega\backslash\Omega_\varepsilon$ as $B_\varepsilon\cup V_\varepsilon$. As seen in Lemma II.1,

$$\begin{aligned}|(\Delta^{-1}\mathrm{div}\ K)(x)| &= \left|\int_{\Omega\backslash\Omega_\varepsilon}2\pi\log|x-y|\mathrm{div}\ K(y)dy\right|\\ &\le \int_{B_\varepsilon}|2\pi\log|x-y|\mathrm{div}\ K(y)dy|+\int_{V_\varepsilon}|2\pi\log|x-y|\mathrm{div}\ K(y)dy|.\end{aligned}\tag{III.19}$$

If we assume for simplicity that B_ε is a rectangle in cartesian coordinates (x_1,x_2) of the form $\{(x_1,x_2)/|x_1|\le L,|x_2|\le\varepsilon^\gamma\}$, then we deduce, using the L^∞ bounds on div K of Lemma III.2,

$$\begin{aligned}\int_{B_\varepsilon}|2\pi\log|x-y|\mathrm{div}\ K(y)dy| &\le C\left|\int_{y_2=-L}^{L}\int_{y_1=0}^{\varepsilon^\gamma}\log|x-y|dy\right|\\ &\le C\varepsilon^\gamma|\log\varepsilon|.\end{aligned}\tag{III.20}$$

The contribution of V_ε can be bounded as follows :

$$\begin{aligned}\int_{V_\varepsilon}|2\pi\log|x-y|\mathrm{div}\ K(y)dy| &\le C\int_{y_2=-\varepsilon^\gamma}^{\varepsilon^\gamma}\int_{y_1=0}^{\varepsilon^\gamma}\frac{|\log|x-y||}{\sup(\varepsilon,y_1)}dy_1\,dy_2\\ &\le \varepsilon^\gamma|\log\varepsilon|^2.\end{aligned}\tag{III.21}$$

Combining (III.19)—(III.21), we get the following L^∞ bound :

$$|(\Delta^{-1}\text{div } K)(x)| \leq C\varepsilon^\gamma |\log \varepsilon|^2 \text{ in } \Omega\backslash\Omega_\varepsilon, \tag{III.22}$$

the general case follows easily from a change of coordinates. On the other hand, as seen in Lemma II.1,

$$\begin{aligned}\int_{\mathbb{R}^2} |H_K|^2 &= \int_{\mathbb{R}^2} (\text{div } K)(\Delta^{-1}\text{div } K)\\ &= \int_{\Omega\backslash\Omega_\varepsilon} (\text{div } K)(\Delta^{-1}\text{div } K),\end{aligned}$$

using the fact that div $K \equiv 0$ in Ω_ε. But, thanks to (III.22),

$$\begin{aligned}\int_{V_\varepsilon} (\text{div } K)(\Delta^{-1}\text{div } K) &\leq C\varepsilon^{2\gamma}\|\text{div } K\|_{L^\infty}\|\Delta^{-1}\text{div } K\|_{L^\infty(\Omega\backslash\Omega_\varepsilon)}\\ &\leq C\varepsilon^{2\gamma}\varepsilon^{-1}\varepsilon^\gamma|\log \varepsilon|^2\\ &\leq C\varepsilon^{3\gamma-1}|\log \varepsilon|^2 = o(\varepsilon),\end{aligned}$$

since $\gamma > 1/2$. Similarly, from (III.22) again,

$$\int_{B_\varepsilon} (\text{div } K)(\Delta^{-1}\text{div } K) \leq C\varepsilon^\gamma\varepsilon^\gamma|\log \varepsilon|^2.$$

Since $\gamma > 1/2$, we conclude that

$$\int_{\Omega\backslash\Omega_\varepsilon} (\text{div } K)(\Delta^{-1}\text{div } K) = o(\varepsilon)$$

hence

$$\frac{1}{\varepsilon}\int_{\mathbb{R}^2} |H_K|^2 = o(1).$$

□

In order to complete the proof of Theorem 1, there remains to prove the following

Lemma III.6 *There exists an admissible covering $\mathcal{U}$ such that the u_ε constructed in (III.8) belong to $\Lambda_{\mathcal{U}}$ for ε small enough.*

Proof : Let us return to the $\varphi_\star$ defined in Lemma III.1, so that $u_\star = e^{i\varphi_\star}$. We recall that, on Σ, $\varphi_\star^+ - \varphi_\star^- \in (0, 2\pi)$, hence if $x_0 \in \Sigma$, it has an open neighborhood U_{x_0} such that

$$\sup_{U_{x_0}} \varphi_\star - \inf_{U_{x_0}} \varphi_\star \leq 2\pi.$$

We also chose T to be such that the geodesic curvature of $\partial\Omega$ at $T \cap \partial\Omega$ is positive. Thus, we can make sure that there exists an open neighbourhood U_T of T in which $\sup_{U_T} \varphi_\star -$

$\inf_{U_T} \varphi_\star \leq 2\pi$. It is also obvious that, since $\varphi_\star$ is continuous in $\Omega \backslash S$, $\Omega \backslash S$ can be covered by open sets U on which $\sup_U \varphi_\star - \inf_U \varphi_\star \leq 2\pi$. We can then extract a finite covering $\cup_{i\in I} U_i$ of Ω such that $\forall i \in I$, $\sup_{U_i} \varphi_\star - \inf_{U_i} \varphi_\star \leq 2\pi$, and such that $\forall i,j \in I, U_i \cap U_j$ is diffeomorphic to $\mathbb{R}^2$. This condition guarantees that $u_\star$ has local liftings $\psi_i \in L^\infty(U_i)$ satisfying

$$u_\star = e^{i\psi_i} \text{ in } U_i, \quad \psi_i \in [l_i, l_i + 2\pi] \text{ with } l_i \in [0, 2\pi].$$

Of course $\psi_i - \psi_j$ is constant on $U_i \cap U_j$ connected, and equal to $2\pi k_{ij}$ for some $k_{ij} \in \mathbb{Z}$. The k_{ij} satisfy the cocycle relations of Definition I.1.

On the other hand u_ε is defined by (III.8) hence it has a lifting

$$\text{(III.23)} \qquad \begin{cases} \varphi_\varepsilon = \psi + \theta & \text{in } B_\varepsilon \\ \varphi_\varepsilon = \varphi_\star & \text{in } \Omega_\varepsilon. \end{cases}$$

By monotonicity of φ_α (see (II.21)), we have $\beta_- \leq \varphi_{\tilde{\alpha}^{sgn(\sigma)}}(\sigma/\varepsilon) \leq \beta_+$. By construction of ψ (see (III.8)), we thus have

$$\sup_{B_\varepsilon} \psi - \inf_{B_\varepsilon} \psi \leq \sup_s (\beta_+(s) - \beta_-(s)).$$

Therefore, on each U_i such that $U_i \cap S \neq \varnothing$, for ε small enough,

$$\text{(III.24)} \qquad \sup_{U_i} \varphi_\varepsilon - \inf_{U_i} \varphi_\varepsilon \leq \sup_{U_i} \varphi_\star - \inf_{U_i} \varphi_\star \leq 2\pi.$$

(The phase of u_ε can be extended to V_ε in such a way that this property also holds.) If U_i does not intersect S, then $u_\varepsilon = u_\star$ on U_i (for ε small enough), hence (III.24) also holds. This property suffices to ensure that $u_\varepsilon \in \Lambda_{\mathcal{U}}$ for ε small enough. □

Remark III.3 : If Ω is convex, then the test-function u_ε also satisfies the same property as $\varphi_\star$: $\varphi_\star \in [l, l+2\pi]$ and Ω is a "suitable covering". In the general case, the suitable covering depends only on $u_\star$, hence on Ω.

IV The compactness result - Proof of Theorem 2

This section is devoted to the proof of Theorem 2. The proof of compactness relies on a similar method as that developed in [ADM]. We identify a jacobian structure in order to use the compensated-compactness lemma and combine it with the use of Young measures. Let us consider a sequence of functions $u_n \in \Lambda_{\mathcal{U}}$ for some $\mathcal{U}$, such that $E_{\varepsilon_n}(u_n)$ remains bounded, where ε_n is a sequence converging to 0. Let φ_n be the $H^1(\Omega) \cap L^\infty$ lifting of u_n given by Lemma I.1. As seen in Section I (I.12), (I.16), we have

$$\text{(IV.1)} \qquad \|H_n\|_{L^2(\mathbb{R}^2)} \to 0$$

$$\text{(IV.2)} \qquad C \geq E_{\varepsilon_n}(u_n) \geq 2\int_\Omega |\nabla \varphi_n \cdot H_n| \geq 2\int_\Omega |\text{div } (\varphi_n u_n + u_n^\perp + \varphi_n H_n)|.$$

The assumption of boundedness on $E_{\varepsilon_n}(u_n)$ can be replaced as well by the two conditions $H_n \to 0$ in some L^p and $\int_\Omega |\nabla\varphi_n \cdot H_n|$ bounded, without any change in the proof, and with the same conclusions.

The quantity to study is

$$v_n = (H_n + u_n)\varphi_n + u_n^\perp, \tag{IV.3}$$

which is equal to the sum of a local term $u_n\varphi_n + u_n^\perp$ and a non-local term $H_n\varphi_n$ which tends to 0 in L^q, $q < \infty$. The hypothesis on v_n is

$$\int_\Omega |\text{div } v_n| \leq C. \tag{IV.4}$$

Since $|u_n| = 1$, u_n is uniformly bounded in $L^\infty(\Omega)$, hence, extracting a subsequence if necessary, we can assume that it converges weakly in $L^\infty(\Omega)$ to some u. We are going to prove that $|u| = 1$ a.e. in Ω. This constitutes the compactness result :
Claim : If $|u| = 1$ a.e. in Ω, then $u_n \to u$ strongly in $L^q(\Omega)$ $(\forall 1 \leq q < \infty)$.
Proof of the claim : If $|u| = 1$ a.e, then

$$\int_\Omega |u_n| = |\Omega| = \int_\Omega |u|.$$

Thus $\|u_n\|_{L^2(\Omega)} \to \|u\|_{L^2(\Omega)}$, while $u_n \rightharpoonup u$ in $L^2(\Omega)$. It is standard that this implies strong convergence of u_n to u in L^2, and in L^q $(\forall q < \infty)$, from the bound $|u_n| \leq 1$.

In view of this claim, it suffices to prove $|u| = 1$ a.e. in Ω.

Lemma IV.1 *For almost every $x_0 \in \Omega$, there exist sequences u'_k, φ'_k, H'_k in $B_1(0) = B_1$ such that*

$$|u'_k| = 1 \text{ a.e. in } B_1 \tag{IV.5}$$

$$u'_k \rightharpoonup u(x_0) \text{ in } L^\infty(B_1) \tag{IV.6}$$

$$u'_k = e^{i\varphi'_k} \text{ with } \varphi'_k \in [l_k, l_k + 2\pi] \text{ and } \lim_{k\to\infty} l_k = l \tag{IV.7}$$

$$\text{div } (H'_k + u'_k) = 0 \text{ and curl } H'_k = 0 \text{ in } B_1 \tag{IV.8}$$

$$\|H'_k\|_{L^p(B_1)} \to 0, \quad \forall 1 \leq p < \infty \tag{IV.9}$$

$$\int_{B_1} |\text{div } (u'_k\varphi'_k + (u'_k)^\perp + H'_k\varphi'_k)| \to 0 \tag{IV.10}$$

Proof : This lemma relies on a blow-up argument.
Let v_n be as in (IV.3). From Lemma II.1, (H_n) is uniformly bounded in $\cap_p L^p$. Since $u_n \in \Lambda_\mathcal{U}$, φ_n is uniformly bounded in $L^\infty(\Omega)$, and since x_0 is in some $U_j, j \in I$, we can assume that φ_n takes its values in $[l_n, l_n + 2\pi]$ in a neighborhood of x_0. Therefore, v_n is bounded in L^2 for example and we can assume that $v_n \rightharpoonup v$ in $L^2(\Omega)$. From (IV.4), div v_n is

bounded in $\mathcal{M}$ (the space of Radon measures), hence, modulo a subsequence, it converges weakly to some measure in $\mathcal{M}$ and this measure is necessarily div v by uniqueness of the distribution limit. Thus div $v \in \mathcal{M}$ and

$$\int_\Omega |\text{div } v| \le \liminf_{n\to\infty} \int_\Omega |\text{div } v_n| \le C. \tag{IV.11}$$

For almost every $x_0 \in \Omega$,

$$\limsup_{r\to 0} \frac{1}{r^2} \int_{B_r(x_0)} |\text{div } v| < \infty. \tag{IV.12}$$

If this was false, we could contradict (IV.11) by a covering argument. From now on, we consider that x_0 satisfies this condition and also x_0 is chosen to be a Lebesgue point for the limit u.
We define the dilated functions, for any $r > 0$,

$$\begin{cases} u_n^r(x) = u_n(x_0 + rx) \\ v_n^r(x) = v_n(x_0 + rx) \\ \varphi_n^r(x) = \varphi_n(x_0 + rx) \\ H_n^r(x) = H_n(x_0 + rx) \end{cases} \tag{IV.13}$$

Observe that

$$\begin{cases} \text{div } v_n^r = r\text{div } v_n(x_0 + rx) \\ \text{div } u_n^r = r\text{div } u_n(x_0 + rx) = -r\text{div } H_n(x_0 + rx) = -\text{div } H_n^r, \end{cases} \tag{IV.14}$$

and

$$\int_{B_1} |\text{div } v_n^r| = \frac{1}{r} \int_{B_r(x_0)} |\text{div } v_n|. \tag{IV.15}$$

Since div $v_n \to$ div v in $\mathcal{M}$, we have for a fixed r,

$$\lim_{n\to\infty} \int_{B_1} |\text{div } v_n^r| = \frac{1}{r} \lim_{n\to\infty} \int_{B_r(x_0)} |\text{div } v_n| \le \frac{1}{r} \int_{B_{2r}(x_0)} |\text{div } v|. \tag{IV.16}$$

Hence, in view of (IV.12), there exists $C > 0$ such that

$$\forall r > 0, \quad \lim_{n\to\infty} \int_{B_1} |\text{div } v_n^r| \le Cr. \tag{IV.17}$$

By assumption, $H_n \to 0$ in $L^2(\Omega)$ and it is bounded in all L^p, hence converges to 0 in $\cap_{p<\infty} L^p$. Let $p > 2$, for fixed $r < r_0$ and for r_0 small enough

$$\int_{B_1} |H_n(x_0 + rx)|^p = \frac{1}{r^2} \int_{B_r(x_0)} |H_n|^p \le \frac{1}{r^2} \int_{B_{r_0}(x_0)} |H_n|^p \to 0 \ \text{ as } n \to \infty. \tag{IV.18}$$

Let us now take any sequence $r_k \to 0$. In view of (IV.17) and (IV.18), we may choose a sequence $n_k \to \infty$ such that

$$\lim_{k\to\infty} \int_{B_1} |\text{div } v_{n_k}^{r_k}| = 0. \tag{IV.19}$$

and such that also

$$\lim_{k\to\infty} \int_{B_1} |H_{n_k}^{r_k}|^p = \lim_{k\to\infty} \int_{B_1} |H_{n_k}(x_0 + r_k x)|^p = 0 \tag{IV.20}$$

We could also have chosen n_k such that $u_{n_k}^{r_k} \rightharpoonup u(x_0)$. Indeed, let e_j be a Hilbert basis for the L^2 scalar product on B_1 such that $e_j \in L^\infty$. We have

$$\begin{aligned} \int_{B_1} e_j(x)[u_n^{r_k}(x) - u(x_0)] &= \frac{1}{{r_k}^2} \int_{B_{r_k}(x_0)} e_j(x/r_k)[u_n(x) - u(x_0)] \\ &= \frac{1}{r_k^2} \int_{B_{r_k}(x_0)} e_j(x/r_k)[u_n(x) - u(x)] + \frac{1}{r_k^2} \int_{B_{r_k}(x_0)} e_j(x/r_k)[u(x) - u(x_0)] \end{aligned} \tag{IV.21}$$

Since x_0 is a Lebesgue point of u,

$$\frac{1}{r_k^2} \int_{B_{r_k}(x_0)} e_j(x/r_k)[u(x) - u(x_0)] \to 0, \quad \text{as } k \to \infty.$$

Then, using the weak convergence of u_n to u, we deduce that for a fixed $r > 0$,

$$\lim_{n\to+\infty} \int_{B_1} e_j(x)[u_n^r(x) - u(x_0)] = \frac{1}{r^2} \int_{B_r(x_0)} e_j(x/r)[u(x) - u(x_0)]$$

so that we may find a subsequence n_k^1 such that $\int_{B_1} e_1(x)[u_{n_k^1}^{r_k}(x) - u(x_0)] \to 0$. From n_k^1, we extract a subsequence n_k^2 such that the previous affirmation holds for e_2 and e_1 and so on. Then, following the standard diagonal argument, for $n_k = n_k^k$ we have

$$\forall j < +\infty \qquad \int_{B_1} e_j(x)[u_{n_k}^{r_k}(x) - u(x_0)] \longrightarrow 0$$

Hence, $u_{n_k}^{r_k}$ converges weakly in $L^2(B_1)$ to $u(x_0)$, as $k \to \infty$.

Then we define

$$\left\{ \begin{array}{l} u'_k = u_{n_k}^{r_k} \\ \varphi'_k = \varphi_{n_k}^{r_k} \\ H'_k = H_{n_k}^{r_k}, \end{array} \right. \tag{IV.22}$$

and we have $u'_k = e^{i\varphi'_k}$. In addition, $\varphi'_k(x) \in [l_{n_k}, l_{n_k} + 2\pi]$ for k large and for $x \in B_1$. Extracting again if necessary, we can assume that l_{n_k} converges to some $l \in \mathbb{R}$ as $k \to \infty$.

Assertions (IV.5), (IV.6), (IV.7), (IV.8) and (IV.9) of the lemma are verified. Assertion (IV.10) follows from the definition of H'_k and from (IV.19). Indeed,

$$\begin{aligned}(\text{div } v_{n_k}^{r_k})(x) &= r_k(\text{div } v_{n_k})(x_0 + r_k x) \\ &= r_k \text{div } (\varphi_{n_k} u_{n_k} + u_{n_k}^{\perp} + \varphi_{n_k} H_{n_k})(x_0 + r_k x) \\ &= \text{div } (\varphi'_k u'_k + (u'_k)^{\perp} + \varphi'_k H'_k)(x),\end{aligned}$$

and in view of (IV.19),

$$\int_{B_1} |\text{div } (\varphi'_k u'_k + (u'_k)^{\perp} + \varphi'_k H'_k)| \to 0$$

which is (IV.10). □

Under the same hypotheses, we may now find l_0 such that

$$l_0 \in [l, l + 2\pi], \tag{IV.23}$$

where l satisfies (IV.7), and

$$e^{-il_0} u(x_0) = (\alpha, 0), \quad \alpha = |u(x_0)|. \tag{IV.24}$$

Lemma IV.2 *Let (u'_k, φ'_k, H'_k) be given by Lemma IV.1, and define*

$$\left\{ \begin{array}{l} A_k = e^{-il_0}(u'_k)^{\perp} \\ B_k = e^{-il_0}\varphi'_k u'_k + e^{-il_0}(u'_k)^{\perp} \\ C_k = e^{-il_0}\varphi'_k u'_k, \end{array} \right. \tag{IV.25}$$

then, modulo a subsequence, we have

$$\left\{ \begin{array}{l} A_k \rightharpoonup (0, \alpha) = A \quad \textit{weakly in } L^{\infty}(B_1) \\ C_k \rightharpoonup (\beta, \gamma) \quad \textit{weakly in } L^{\infty}(B_1) \\ B_k \rightharpoonup (\beta, \gamma + \alpha) = B \quad \textit{weakly in } L^{\infty}(B_1), \end{array} \right. \tag{IV.26}$$

and moreover

$$1 = A_k \cdot B_k \rightharpoonup A \cdot B = \alpha(\gamma + \alpha) \quad \textit{in } \mathcal{D}'(B_1). \tag{IV.27}$$

Proof :
- *Step 1 :* From (IV.6) and (IV.24), $A_k \rightharpoonup (0, \alpha) = A$ weakly in $L^{\infty}(B_1)$. $\varphi_n u_n$ is bounded in $L^{\infty}(\Omega)$ hence converges weakly to some function f, up to extraction. Since $C_k = e^{-il_0}\varphi'_k u'_k$, if x_0 is chosen to be a Lebesgue point of f, we can assume exactly as for u that C_k converges weakly in $L^{\infty}(B_1)$ to $e^{-il_0} f(x_0) = (\beta, \gamma)$. We immediately deduce that $B_k \rightharpoonup (\beta, \gamma + \alpha)$ weakly in $L^{\infty}(B_1)$.

- *Step 2 :* We prove (IV.27). It is a direct consequence of the compensated-compactness lemma of Murat and Tartar (see [Mu2] and [Ta]), which says that if $A_k \rightharpoonup A$ in L^2, $B_k \rightharpoonup B$ in L^2, curl A_k and div B_k are compact in H^{-1}, then $A_k \cdot B_k \rightharpoonup A \cdot B$. We just need to check that these hypotheses hold for our A_k and B_k. Their weak convergence in $L^2(B_1)$ has already been established. Furthermore,

$$\text{curl } A_k = -e^{-il_0}\text{div } u_k' = e^{-il_0}\text{div } H_k' \to 0 \quad \text{in } H^{-1}(B_1)$$

from the fact that $\|H_k'\|_{L^2(B_1)} \to 0$.

$$\begin{aligned} \text{div } B_k &= e^{-il_0}\text{div } (\varphi_k' u_k' + (u_k')^\perp) \\ &= e^{-il_0}\text{div } (\varphi_k' u_k' + (u_k')^\perp + \varphi_k' H_k') - e^{-il_0}\text{div } (\varphi_k' H_k'). \end{aligned}$$

In view of (IV.10), div $(\varphi_k' u_k' + (u_k')^\perp + \varphi_k' H_k') \to 0$ in $L^1(B_1)$. On the other hand this quantity is bounded in $W^{-1,p}$ for all $p < \infty$, hence by a result of Murat (see [Mu1]) it is compact in H^{-1}. Meanwhile div $(\varphi_k' H_k') \to 0$ in $H^{-1}(B_1)$ as previously. Hence, div $B_k \to 0$ in $H^{-1}(B_1)$, and we conclude that A_k and B_k satisfy the desired properties. We have thus established that $A_k \cdot B_k \rightharpoonup A \cdot B$ in $\mathcal{D}'(B_1)$. □

Lemma IV.3 *$\alpha = 1$ and the Young measure generated by φ_k' at 0 is supported in $\{l_0, l_0 + 2\pi\}$ or $\{l_0 - 2\pi, l_0\}$.*

Proof : We are now in a position to prove that $\gamma\alpha + \alpha^2 = 1$ implies that $\alpha = 1$. Following the approach of [ADM], since φ_k' is uniformly bounded in $L^\infty(B_1)$, we can introduce ν, the Young measure it generates at 0. Recall that from (IV.7), φ_k' takes values in $[l_k, l_k + 2\pi]$ with $l_k \to l$. Therefore, ν is supported in $[l, l+2\pi]$. Thus, in view of (IV.23), ν is supported in $[l_0 - 2\pi, l_0 + 2\pi]$. By definition of Young measures, we have the following relations (in view of (IV.25)-(IV.26)):

$$\int_{l_0-2\pi}^{l_0+2\pi} d\nu = 1 \tag{IV.28}$$

$$\alpha = \int_{l_0-2\pi}^{l_0+2\pi} \cos(y - l_0) d\nu(y) = \int_{-2\pi}^{2\pi} (\cos t) d\nu(t + l_0) \tag{IV.29}$$

$$0 = \int_{l_0-2\pi}^{l_0+2\pi} \sin(y - l_0) d\nu(y) = \int_{-2\pi}^{2\pi} (\sin t) d\nu(t + l_0) \tag{IV.30}$$

$$\gamma = \int_{l_0-2\pi}^{l_0+2\pi} y \sin(y - l_0) d\nu(y) = \int_{-2\pi}^{2\pi} (t \sin t) d\nu(t + l_0). \tag{IV.31}$$

We claim that

$$\forall t \in [-2\pi, 2\pi], \quad t\sin t + 2\cos t \le 2. \tag{IV.32}$$

This can be checked by looking at the variations of $t \mapsto t\sin t + 2\cos t$ on $[-2\pi, 2\pi]$. Inserting (IV.32) into (IV.31), we are led to

$$\begin{aligned} \gamma = \int_{-2\pi}^{2\pi} (t\sin t) d\nu(t + l_0) &\le 2\int_{l_0-2\pi}^{l_0+2\pi} d\nu - 2\int_{-2\pi}^{2\pi} (\cos t) d\nu(t + l_0) \\ &\le 2 - 2\alpha, \end{aligned} \tag{IV.33}$$

where we have used the fact that ν is a probability measure, hence positive. It then suffices to insert this relation into $\gamma\alpha + \alpha^2 = 1$ to get $1 \leq 2\alpha - 2\alpha^2 + \alpha^2$ equivalent to $(1-\alpha)^2 \leq 0$. Therefore, $\alpha = |u(x_0)| = 1$. In view of (IV.29), ν is necessarily supported in $\{l_0, l_0 - 2\pi, l_0 + 2\pi\}$. Since it is also supported in $[l, l+2\pi]$, its support is either $\{l_0, l_0+2\pi\}$ or $\{l_0, l_0 - 2\pi\}$. From Lemma IV.1, this is true for almost every $x_0 \in \Omega$, thus $|u| = 1$ a.e. and $u_n \to u$ strongly in $\cap_{1\leq q<\infty} L^q$. $\square$

Let us consider a general $u \in C^1(B_1, S^1)$ and $\varphi \in C^1(B_1, \mathbb{R})$ such that $u = e^{i\varphi}$. Then, we define for any $l \in \mathbb{R}$ and $\eta > 0$, the truncated function

$$\text{(IV.34)} \qquad \begin{cases} T_\eta\varphi = \varphi & \text{if } \varphi \in [l+\eta, l+2\pi-\eta] \\ T_\eta\varphi = l+\eta & \text{if } \varphi \leq l+\eta \\ T_\eta\varphi = l+2\pi-\eta & \text{if } \varphi \geq l+2\pi-\eta \end{cases}$$

We will write

$$\text{(IV.35)} \qquad T_\eta u = exp(iT_\eta\varphi).$$

Lemma IV.4 *Let $u_k \in H^1(B_1, S^1)$ and $\varphi_k \in H^1(B_1, \mathbb{R})$ be such that $u_k = e^{i\varphi_k}$ and that (IV.8)—(IV.10) hold. Then, for any choice of $l \in \mathbb{R}$, for any $\eta_0 > 0$, there exists a sequence $\eta_k \in [\frac{1}{2}\eta_0, \eta_0]$ such that (for a subsequence), $T_{\eta_k}\varphi_k$ converges strongly in $\cap_{1\leq q<\infty} L^q$ and*

$$T_{\eta_k} u_k \to v \ \textit{in} \ \cap_{1\leq q<\infty} L^q(B_1), \ \textit{with} \ \text{div } v = 0, |v| = 1.$$

Proof :
- *Step 1 :* Let us denote $U_k^\eta = \{x \in B_1 / l+\eta \leq \varphi_k \leq l+2\pi-\eta\}$. We claim we can choose $\eta_k \in [\frac{1}{2}\eta_0, \eta_0]$ such that

$$\text{(IV.36)} \qquad \int_{\partial U_k^{\eta_k}} |H_k \cdot n| \to 0 \text{ as } k \to \infty.$$

Indeed, using the coarea formula we have

$$\int_{\varphi_k \in [l+\frac{1}{2}\eta_0, l+\eta_0] \cup [2\pi+l+\frac{1}{2}\eta_0, 2\pi+l+\eta_0]} |\nabla\varphi_k \cdot H_k| = \int_{\frac{1}{2}\eta_0}^{\eta_0} d\eta \int_{\partial U_k^\eta} |H_k \cdot n|.$$

But the left-hand side tends to zero by (IV.10). Indeed, $\nabla\varphi \cdot H = \text{div } (u\varphi + u^\perp + H\varphi)$ for H^1 functions, because C^∞ functions are dense in $H^1(\Omega, S^1)$. Hence, using the mean-value theorem on the right-hand side, we get (IV.36).

- *Step 2 :* We prove that $\|\text{div } T_{\eta_k} u_k\|_{W^{-1,q}(B_1)} \to 0$.
Indeed, let $\xi \in C_0^\infty(B_1)$. Since $T_{\eta_k}\varphi_k$ is constant in $B_1 \backslash U_k^{\eta_k}$,

$$\int_{B_1} \text{div } (T_{\eta_k} u_k)\xi = \int_{U_k^{\eta_k}} \text{div } (u_k)\xi.$$

From (IV.8),

$$\text{(IV.37)} \qquad \begin{aligned} \int_{B_1} \text{div } (T_{\eta_k} u_k)\xi &= -\int_{U_k^{\eta_k}} \text{div } (H_k)\xi \\ &= \int_{U_k^{\eta_k}} H_k \cdot \nabla\xi - \int_{\partial U_k^{\eta_k}} \xi(H_k \cdot n). \end{aligned}$$

Let μ_k denote the measure defined by $\mu_k(\xi) = \int_{\partial U_k^{\eta_k}} \xi(H_k \cdot n)$. By (IV.36), $\mu_k \to 0$ in the sense of measures, hence it converges to 0 in $W^{-1,q}(B_1)$ for any $q < 2$. Also, from condition (IV.9),

$$\left| \int_{U_k^{\eta_k}} H_k \cdot \nabla\xi \right| \leq o(1) \|\nabla\xi\|_{L^q(B_1)}, \;\; \forall q < \infty.$$

Hence, we deduce from (IV.37) that div $(T_{\eta_k} u_k)$ converges to 0 in $W^{-1,q}(B_1)$ for any $q < 2$. Since $|T_{\eta_k} u_k| = 1$, it is also bounded in $W^{-1,q}(B_1)$ for any $q < \infty$, hence it converges to 0 in $W^{-1,q}(B_1)$ for any $q < \infty$.

- *Step 3 :* Let χ_k be the characteristic function of U_k and

$$D_k = T_{\eta_k}\varphi_k(T_{\eta_k} u_k + H_k) + (T_{\eta_k} u_k)^{\perp} + T_{\eta_k}\varphi_k \;\, H_k \;(1 - \chi_k).$$

We prove that div $D_k \to 0$ in the sense of measures.
Let $\xi \in C_0^\infty(B_1)$. We have to consider

$$\text{(IV.38)} \qquad \begin{aligned} \int_{B_1} \xi \text{div } D_k &= \int_{U_k} \xi \left(\nabla\varphi_k \cdot u_k + \text{div } u_k^{\perp}\right) \\ &+ \int_{B_1} \xi T_{\eta_k}\varphi_k \text{div } (T_{\eta_k} u_k + H_k) \\ &+ \int_{U_k} \xi \nabla\varphi_k \cdot H_k + \int_{B_1 \backslash U_k} (\nabla\xi \cdot H_k) T_{\eta_k}\varphi_k \end{aligned}$$

The first integral of the three identically vanishes since $T_{\eta_k} u_k = exp(iT_{\eta_k}\varphi_k)$ and all the functions involved are regular enough. On the other hand, div $(T_{\eta_k} u_k) = -$div H_k in $U_k^{\eta_k}$ and 0 in $B_1 \backslash U_k^{\eta_k}$. Hence, we can rewrite (IV.38) as

$$\text{(IV.39)} \qquad \begin{aligned} \int_{B_1} \xi \text{div } D_k &= -\int_{B_1 \backslash U_k^{\eta_k}} \xi T_{\eta_k}\varphi_k \text{div } H_k + \int_{U_k} \xi \nabla\varphi_k \cdot H_k \\ &\quad -\int_{B_1 \backslash U_k^{\eta_k}} T_{\eta_k}\varphi_k (\nabla\xi \cdot H_k) \\ &= -\int_{\partial U_k^{\eta_k}} \xi (T_{\eta_k}\varphi_k) H_k \cdot n + \int_{U_k^{\eta_k}} \xi \nabla\varphi_k \cdot H_k, \end{aligned}$$

where we have used the fact that $T_{\eta_k}\varphi_k$ is constant in each connected component of $B_1\backslash U_k^{\eta_k}$, and integration by parts. We observe that first

$$\left|\int_{B_1\backslash U_k^{\eta_k}} T_{\eta_k}(\varphi_k \nabla\xi\cdot H_k)\right| \leq C\|\nabla\xi\|_\infty \int_{B_1}|H_k| \longrightarrow 0$$

moreover

$$\int_{\partial U_k^{\eta_k}} |(T_{\eta_k}\varphi_k)H_k\cdot n| = C\int_{\partial U_k^{\eta_k}} |H_k\cdot n| \to 0$$

by (IV.36). On the other hand,

$$\int_{U_k^{\eta_k}} |\nabla\varphi_k\cdot H_k| \leq \int_{B_1}|\nabla\varphi_k\cdot H_k| \to 0$$

by (IV.10). Hence, we deduce from (IV.39) that div D_k tends to 0 in the sense of measures.

- *Step 4 :* Let $E_k = T_{\eta_k}\varphi_k\ H_k\ (1-\chi_k)$. We have div $E_k \to 0$ in $H^{-1}(B_1)$. Indeed

$$\left|\int_{B_1}\xi \text{ div } E_k\right| = \left|\int_{B_1\backslash U_k^{\eta_k}} T_{\eta_k}\varphi_k\nabla\xi\cdot H_k\right| \leq C\|\xi\|_{H^1}\ \|H_k\|_{L^2}\to 0.$$

- *Step 5 :* Since D_k in bounded in all L^q, div D_k is also bounded in $W^{-1,q}(B_1)$ for any $q<\infty$, and applying again the result of [Mu1], we deduce with the previous step that it converges to 0 strongly in $H^{-1}(B_1)$. Thus div $(D_k - E_k)$ converges to 0 strongly in $H^{-1}(B_1)$ Denoting $G_k = (T_{\eta_k}u_k)^\perp$, curl G_k converges to 0 strongly in $H^{-1}(B_1)$ from Step 2. Arguing as in Step 2 of the proof of Lemma IV.2, using the div-curl lemma, we obtain that $G_k\cdot(D_k-E_k)\rightharpoonup G\cdot(D-E)$, where G, D and E are the weak limits of G_k, D_k and E_k. Since $G_k\cdot(D_k-E_k)\to 1$, we deduce $G\cdot(D-E)=1$.

- *Step 6 :* By construction, $T_{\eta_k}\varphi_k$ takes its values in $[l+\frac{1}{2}\eta_0, l+2\pi-\frac{1}{2}\eta_0]$. Following the arguments used above, we can deduce that the Young measure it generates at 0 is a Dirac measure, hence $T_{\eta_k}\varphi_k$ converges strongly in $L^q(\Omega)$ for all $q<\infty$. We deduce that $T_{\eta_k}u_k$ also converges strongly to some v in $\cap_{1\leq q<\infty}L^q(B_1)$. Necessarily $|v|=1$ a.e., and div $v=0$ from Step 1. $\square$.

We now complete the proof of Theorem 2. Applying Lemma IV.4 to the sequences u_k', φ_k', H_k' of Lemma IV.1 and to the l given in (IV.7), we obtain strong convergence of $T_{\eta_k}\varphi_k'$ and $T_{\eta_k}u_k'$, for any choice of $\eta_0>0$. We may consider that η_k converges to some η. Denoting again by ν the Young measure φ_k' generates at 0, the Young measure generated by $T_{\eta_k}(\varphi_k')$ at 0 is $T_\eta^*\nu$. On the other hand, from Lemma IV.3, ν is supported, say, in $\{l_0, l_0+2\pi\}$, hence it is $t\delta_{l_0}+(1-t)\delta_{l_0+2\pi}$ for some $t\in[0,1]$. But by strong convergence of $T_{\eta_k}\varphi_k'$, $T_\eta^*\nu$ is a Dirac mass, hence $T_\eta^*(t\delta_{l_0}+(1-t)\delta_{l_0+2\pi})$ is a Dirac mass, and we conclude that t must be 0 or 1, and ν is a Dirac mass too. We deduce that φ'_k converges a.e. to its weak limit φ, hence by Lebesgue's theorem, it converges in L^q for all $q < \infty$. Assuming Proposition V.1, we deduce Theorem 2.

V Study of the Γ-limit

V.1 The limiting problem and the energy lower bound

We recall that $\mathcal{C}$ was defined in Definition I.2, it is the natural set of admissible limiting configurations as explained in the introduction. In particular, it contains $u_\star$. Let $u : \Omega \mapsto S^1$ satisfying div $\tilde{u} = 0$. Then, there exists a Lipschitz-continuous function $g \in C^{0,1}(\Omega, \mathbb{R})$ such that

$$\text{(V.1)} \qquad \begin{cases} u = \nabla^\perp g & \text{in } \Omega \\ g = 0 & \text{on } \partial\Omega. \end{cases}$$

Indeed, div $\tilde{u} = 0$ means that div $u = 0$ in Ω and $u \cdot n = 0$ on $\partial\Omega$, where n is the outer unit normal to $\partial\Omega$ The existence of g satisfying $u = \nabla^\perp g$ in Ω is quite standard. g is then constant on $\partial\Omega$, and can be chosen to be 0 on $\partial\Omega$.

Lemma V.1 *Assume* $\varphi \in L^1(\Omega, \mathbb{R})$ *and* $u : \Omega \to S^1$ *are such that* div $\tilde{u} = 0$ *and* div $(\varphi u + u^\perp) \in \mathcal{M}$*, then*

$$\|\text{div } (\varphi u + u^\perp)\| \geq |\partial\Omega|.$$

Notice that for this lemma, u and φ are not necessarily related.
Proof : Let us consider

$$\text{(V.2)} \qquad \begin{cases} f_\varepsilon(s) = -1 & \text{if } s \leq -\varepsilon \\ f_\varepsilon(s) = 1 & \text{if } s \geq \varepsilon \\ f_\varepsilon(s) = \frac{s}{\varepsilon} & \text{if } -\varepsilon \leq s \leq \varepsilon. \end{cases}$$

Since div $(\varphi u + u^\perp) \in \mathcal{M}$ and $f_\varepsilon \in L^\infty$, we can consider

$$\text{(V.3)} \qquad \left| \int_\Omega f_\varepsilon \text{div } (\varphi u + u^\perp) \right| \leq \|\text{div } (\varphi u + u^\perp)\|.$$

Since $g = 0$ on $\partial\Omega$ and $f_\varepsilon(0) = 0$, we may write, after an integration by parts,

$$\text{(V.4)} \qquad \int_\Omega f_\varepsilon(g) \text{div } (\varphi u + u^\perp) = -\int_\Omega (f_\varepsilon)'(g) \nabla g . (\varphi u + u^\perp).$$

The right-hand side is well-defined, indeed, since $\varphi \in L^1$, $\varphi u + u^\perp \in L^1$, while $f'_\varepsilon(g) \nabla g \in L^\infty$. Now, $f'_\varepsilon(s) = \frac{1}{\varepsilon} \mathbf{1}_{[-\varepsilon, \varepsilon]}$ and $u = \nabla^\perp g$, hence (V.4) becomes

$$\text{(V.5)} \qquad \int_\Omega f_\varepsilon(g) \text{div } (\varphi u + u^\perp) = \frac{1}{\varepsilon} \int_{-\varepsilon \leq g \leq \varepsilon} |\nabla g|^2.$$

We may use the co-area formula, and since $|\nabla g| = 1$, this can be rewritten as

$$\text{(V.6)} \qquad \int_\Omega f_\varepsilon(g)\text{div}\,(\varphi u + u^\perp) = \frac{1}{\varepsilon}\int_{-\varepsilon}^{\varepsilon} \mathcal{H}^1(\{g = s\} \cap \Omega)ds.$$

Then, we claim that

$$\text{(V.7)} \qquad \liminf_{\varepsilon \to 0} \frac{1}{\varepsilon}\int_0^\varepsilon \mathcal{H}^1(\{|g| = s\} \cap \Omega)\, ds \geq |\partial\Omega|.$$

Indeed, let V_ε be $\{x \in \Omega / \text{dist}(x, \partial\Omega) \leq \varepsilon\}$. Since $|\nabla g| = 1$ a.e, by the mean value theorem $|g| \leq \varepsilon$ in V_ε. Hence

$$\int_{V_\varepsilon} |\nabla g| \leq \int_{|g| \leq \varepsilon} |\nabla g|$$

and applying the co-area formula again

$$\frac{1}{\varepsilon}|V_\varepsilon| \leq \frac{1}{\varepsilon}\int_{|g|\leq\varepsilon} |\nabla g| = \frac{1}{\varepsilon}\int_{-\varepsilon}^{\varepsilon} \mathcal{H}^1(\{g = s\} \cap \Omega)\, ds.$$

Therefore

$$\liminf_{\varepsilon \to 0} \frac{1}{\varepsilon}\int_{-\varepsilon}^{\varepsilon} \mathcal{H}^1(\{g = s\} \cap \Omega)\, ds \geq \liminf_{\varepsilon \to 0}\left(\frac{1}{\varepsilon}|V_\varepsilon|\right) \geq |\partial\Omega|,$$

thus we have (V.7). On the other hand,

$$\text{(V.8)} \qquad \forall \varepsilon > 0 \quad \left|\int_\Omega f_\varepsilon(g)\text{div}\,(\varphi u + u^\perp)\right| \leq \|\text{div}\,(\varphi u + u^\perp)\|.$$

Thus, combining (V.3), (V.6), (V.7), we have

$$\text{(V.9)} \qquad \|\text{div}\,(\varphi u + u^\perp)\| \geq |\partial\Omega|.$$

□

In particular, $\inf_{(u,\varphi)\in\mathcal{C}} \|\mu_{u,\varphi}\| \geq |\partial\Omega|$. The fact that $u_\star$ and $-u_\star$ achieve this minimum shall be proved in the next subsection, Lemma V.2.

We have the following lower semi-continuity result :

Proposition V.1 *Let $u_\varepsilon \in \Lambda_\mathcal{U}$ be such that $E_\varepsilon(u_\varepsilon) \leq C$, $u_\varepsilon \to u$ in L^q ($\forall q < \infty$), and $\varphi_\varepsilon \to \varphi$ in L^q ($\forall q < \infty$) where φ_ε is a lifting of u_ε given by Lemma I.1, then $(u, \varphi) \in \mathcal{C}$ and*

$$\liminf_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon) \geq 2\|\mu_{u,\varphi}\| \geq 2|\partial\Omega|.$$

Again here, the hypotheses could be replaced by $\|H_\varepsilon\|_{L^2} \to 0$ and $\int_\Omega |\nabla u_\varepsilon||H_\varepsilon| \le C$.
Proof : Let φ_ε be the $H^1(\Omega,\mathbb{R})$-lifting of $u_\varepsilon \in \Lambda_{\mathcal{U}}$. The hypotheses of the proposition imply that $H_\varepsilon \to 0$ in $L^2(\Omega)$. Clearly this implies that div $(\tilde{u_\varepsilon}) = -$div $H_\varepsilon \to 0$ in H^{-1}, hence div $\tilde{u} = 0$. Since $u_\varepsilon \in \Lambda_{\mathcal{U}}$, φ_ε is uniformly bounded in L^∞, hence, up to extraction, it converges weakly to some φ. Again as in (I.11), the boundedness of $E_\varepsilon(u_\varepsilon)$ implies $\int_\Omega |\mu_\varepsilon| \le C$, where

$$\mu_\varepsilon = \text{div } (\varphi_\varepsilon u_\varepsilon + u_\varepsilon^\perp + \varphi_\varepsilon H_\varepsilon)$$

Since u_ε and φ_ε converge strongly in L^q, and H_ε strongly in L^2, we have

$$\varphi_\varepsilon u_\varepsilon + u_\varepsilon^\perp + \varphi_\varepsilon H_\varepsilon \rightharpoonup \varphi u + u^\perp \quad \text{in } \mathcal{D}'(\Omega),$$

hence $\mu_\varepsilon \rightharpoonup$ div $(\varphi u + u^\perp)$ in $\mathcal{D}'(\Omega)$. On the other hand, μ_ε is bounded in $\mathcal{M}$ by assumption, hence it converges weakly in $\mathcal{M}$ to div $(\varphi u + u^\perp)$. Thus div $(\varphi u + u^\perp) \in \mathcal{M}$, and $(u,\varphi) \in \mathcal{C}$. With Lemma V.1,

$$|\partial\Omega| \le \int_\Omega |\text{div } (\varphi u + u^\perp)| \le \liminf_{\varepsilon\to 0} \int_\Omega |\mu_\varepsilon| \le \frac{1}{2} \liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon).$$

□

We deduce Theorem 4 from Proposition V.1 and Theorem 2.

Remark V.1 : Using similar integration by parts as in (V.3) with the measure $\mu_{u_\star,\varphi_\star}$, we obtain the following isoperimetric-type inequality :

$$|\Omega| = \int_\Omega dist(x,\partial\Omega)\text{div } (\varphi_\star u_\star + u_\star^\perp) \le |\partial\Omega| \max_\Omega \text{dist}(.,\partial\Omega).$$

V.2 The limiting problem revisited : case of BV functions

In the case of BV functions, the limiting functional $u \mapsto \|\mu_{u,\varphi}\|$ can be expressed much more simply. Suppose $\varphi \in BV$. We use the notations of [ADM], Section 2.

The approximate limit of φ at x, denoted by $\tilde{\varphi}(x)$, is, when it exists, the unique $z \in \mathbb{R}$ satisfying

$$\lim_{r\to 0} \frac{1}{r^2} \int_{B_r(x)} |\varphi(y) - z| dy = 0.$$

We denote by S' the set of approximate discontinuity points, i.e. the set of points where the approximate limit does not exist. The one-sided approximate limits $\varphi^+(x)$ and $\varphi^-(x)$ are numbers a and b in $\mathbb{R}$ such that

$$\lim_{r\to 0} \int_{B_r^+(x)} |\varphi(y) - a| dy = 0 \text{ and } \lim_{r\to 0} \int_{B_r^-(x)} |\varphi(y) - b| dy = 0,$$

where $B_r^\pm = \{y \in B_r(x) / \pm < y - x, n > \ge 0\}$ are the two half-balls corresponding to some unit-vector n. We denote by $S \subset S'$ the set of approximate jump points i.e. points of S'

for which $\varphi^\pm$ exist for some unit vector $n(x)$. The Radon measure $D\varphi$ can be split into three mutually singular parts :

$$D\varphi = \nabla\varphi\mathcal{L}^2 + D_c\varphi + (\varphi^+ - \varphi^-)\otimes n\mathcal{H}^1\lfloor_S, \tag{V.10}$$

where $\mathcal{L}^2$ is the Lebesgue measure, and $D_c\varphi$ is the Cantor part of $D\varphi$. Also $\mathcal{H}^1(S'\backslash S) = 0$. If $D_c\varphi = 0$, then φ belongs to the subspace $SBV(\Omega)$ of $BV(\Omega)$.

Proof of Theorem 5 :
1) If $\varphi \in BV$, according to Volpert's chain rule (cf [ADM] Proposition 3.6 for instance), then $\cos\varphi$ and $\sin\varphi$ are in BV, hence $u \in BV$. It is then easy to check that if, in addition, div $u = 0$ in Ω, div $(\varphi u + u^\perp)$ is automatically a Radon measure with div $(\varphi u) = D\varphi\cdot u$. Indeed, consider $\xi \in C_0^\infty(\Omega)$,

$$\int_\Omega \xi \text{div } (\varphi u) = -\int_\Omega \varphi u\cdot\nabla\xi = -\int_\Omega \varphi\text{div } (\xi u). \tag{V.11}$$

(Since div $u = 0$, one can check that $\forall\xi \in C_0^\infty(\Omega)$, $u.\nabla\xi = $ div (ξu) in $\mathcal{D}'(\Omega)$.) since $\xi u \in L^\infty$, by definition of BV,

$$\left|\int_\Omega \varphi \text{ div } (\xi u)\right| \le \|\xi u\|_{L^\infty}\int_\Omega |D\varphi| \le \|\xi\|_{L^\infty}\int_\Omega |D\varphi|, \tag{V.12}$$

and

$$-\int_\Omega \varphi \text{ div } (\xi u) = \int_\Omega \xi(D\varphi\cdot u). \tag{V.13}$$

From (V.11) and (V.12), we deduce that div (φu) is a Radon measure on Ω and from (V.13), that, in the frame of BV functions

$$\text{div } (\varphi u) = D\varphi\cdot u. \tag{V.14}$$

Since $u \in BV$, div $(u^\perp)$ is also automatically a Radon measure with div $(u^\perp) = D_2u_1 - D_1u_2$. From Volpert's chain rule, we have

$$\begin{cases} D(\cos\varphi) = -\sin\varphi\nabla\varphi - \sin\tilde\varphi D_c\varphi + (\cos\varphi^+ - \cos\varphi^-)\otimes n\mathcal{H}^1\lfloor_S \\ \\ D(\sin\varphi) = \cos\varphi\nabla\varphi + \cos\tilde\varphi D_c\varphi + (\sin\varphi^+ - \sin\varphi^-)\otimes n\mathcal{H}^1\lfloor_S. \end{cases} \tag{V.15}$$

The condition div $u = 0$ means that $D_1(\cos\varphi) + D_2(\sin\varphi) = 0$. Since $D(\cos\varphi)$ and $D(\sin\varphi)$ are split into mutually singular measures, we can consider their jump part (which is supported on S) and deduce that

$$\left((\cos\varphi^+ - \cos\varphi^-)n_1 + (\sin\varphi^+ - \sin\varphi^-)n_2\right)\mathcal{H}^1\lfloor_S = 0$$

where n_1 and n_2 are the cartesian coordinates of n. In other words, $((u^+ - u^-)\cdot n)\,\mathcal{H}^1\lfloor_S = 0$, or

$$u^+ \cdot n = u^- \cdot n \quad \mathcal{H}^1\text{- a.e. on } S. \tag{V.16}$$

Hence, we can write $u\cdot n = u^+\cdot n = u^-\cdot n$ on S without ambiguity. This means simply that for a divergence-free vector-field, the normal component is preserved across a jump line.

Returning to (V.14), we have

$$\text{div }(\varphi u) = D\varphi\cdot u = \nabla\varphi\cdot u + D_c\varphi\cdot\tilde{u} + (\varphi^+ - \varphi^-)(n\cdot u)\mathcal{H}^1\lfloor_S.$$

On the other hand, in view of (V.15),

$$\begin{aligned}\text{div } u^\perp &= D_2u_1 - D_1u_2 = D_2\cos\varphi - D_1\sin\varphi\\ &= -u\cdot\nabla\varphi - u\cdot D_c\varphi + \big((\cos\varphi^+ - \cos\varphi^-)n_2 - (\sin\varphi^+ - \sin\varphi^-)n_1\big)\,\mathcal{H}^1\lfloor_S.\end{aligned}$$

Therefore,

$$\begin{aligned}\text{div }(\varphi u + u^\perp) &= \big((\varphi^+ - \varphi^-)(u\cdot n) + (\cos\varphi^+ - \cos\varphi^-)n_2 - (\sin\varphi^+ - \sin\varphi^-)n_1\big)\,\mathcal{H}^1\lfloor_S\\ &= \big((\varphi^+ - \varphi^-)(u\cdot n) + (u^+ - u^-)\cdot n^\perp\big)\,\mathcal{H}^1\lfloor_S. \qquad \text{(V.17)}\end{aligned}$$

Hence, $\mu_{u,\varphi}$ is only supported on the jump set S of φ.

If $|\varphi^+ - \varphi^-| < 2\pi$, then necessarily $u^+ \neq u^-$ pointwise (otherwise $\varphi^+ = \varphi^-$ and there would be no jump), and $u^+\cdot n^\perp = -u^-\cdot n^\perp$. It is left to the reader that in this case $u\cdot n = \cos X$, and $(u^+ - u^-)\cdot n^\perp = 2u^+\cdot n^\perp = 2\sin X$, where $X = -\frac{1}{2}sgn(u^+\cdot n^\perp)|\varphi^+ - \varphi^-|$.

2) If φ_ε is bounded in BV and $E_\varepsilon(u_\varepsilon) \le C$ where $u_\varepsilon = e^{i\varphi_\varepsilon}$, u_ε is also bounded in BV. Since BV embedds compactly into L^p, for all $p<2$, we can assume, up to extraction, that $u_\varepsilon \to u$ strongly in L^p $(p<2)$. Since $|u_\varepsilon| = 1$ a.e., we deduce that $u_\varepsilon \to u$ in $\cap_{1\le q<\infty} L^q$. Similarly, we can assume $\varphi_\varepsilon \rightharpoonup \varphi$ weakly in BV and $\varphi_\varepsilon \to \varphi$ strongly in L^p $(p<2)$. Up to extraction, $\varphi_\varepsilon \to \varphi$ almost everywhere, hence $e^{i\varphi_\varepsilon} \to e^{i\varphi}$ a.e., and by Lebesgue's theorem $u_\varepsilon \to e^{i\varphi}$ in L^1, hence $u = e^{i\varphi}$ ae. As in Proposition V.1, we are led to

$$\liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) \ge \liminf_{\varepsilon\to 0}\int_\Omega |\mu_\varepsilon| \ge \|\text{div }(\varphi u + u^\perp)\| = \|\mu_{u,\varphi}\|.$$

$\square$

We can then prove

Lemma V.2 *For $u_\star = e^{i\varphi_\star}$ constructed in Lemma III.1, $\varphi_\star \in BV$ and*

$$\mu_{u_\star,\varphi_\star} = 2(\sin X - X\cos X)\mathcal{H}^1\lfloor_S, \qquad \textit{where } X = \tfrac{1}{2}(\varphi_\star^+ - \varphi_\star^-) \ge 0$$

$$\mu_{u_\star,\varphi_\star} \ge 0$$

$$\|\mu_{u_\star,\varphi_\star}\| = |\partial\Omega|.$$

Proof : We return to the notations of Lemma III.1. The singular set of $\varphi_\star$ is made of two parts : the medial axis Σ and the ray T. If $x \in \Sigma$ is not a vertex, then we recall that we chose n to be the normal to Σ pointing towards x_2 ie $n.\frac{\overrightarrow{xx_2}}{|xx_2|} \geq 0$ and thus $\varphi_\star^+ > \varphi_\star^-$ in the frame of (V.10) with $X = \frac{1}{2}(\varphi_\star^+ - \varphi_\star^-) \in]0,\pi[$. On the other hand, from (III.3), $u_\star^+ \cdot n^\perp \leq 0$ and $u_\star^- \cdot n^\perp = -u_\star^+ \cdot n^\perp \geq 0$,

$$(u_\star^+ - u_\star^-) \cdot n^\perp = 2u_\star^+ \cdot n^\perp = -2\cos(\varphi_\star^+ - \theta) = -2\cos\alpha_+ = -2X\sin X,$$

while

$$(\varphi_\star^+ - \varphi_\star^-)(u_\star \cdot n) = (\alpha_+ - \alpha_-)\sin\alpha = 2X\cos X.$$

Thus, $x \mapsto \sin x - x\cos x$ being nonnegative on $[0,\pi]$,

$$\left((\varphi_\star^+ - \varphi_\star^-)(u_\star \cdot n) - (u_\star^+ - u_\star^-) \cdot n^\perp\right) \mathcal{H}^1\lfloor_\Sigma = 2(\sin X - X\cos X)\mathcal{H}^1\lfloor_\Sigma \geq 0$$

There remains to see that $\mu_{u_\star,\varphi_\star} \geq 0$ on T. On T, we chose $n = u_\star$, $\varphi_\star^+ - \varphi_\star^- = 2\pi$ and

$$\mu_{u_\star,\varphi_\star}\lfloor_T = (\varphi_\star^+ - \varphi_\star^-)(u_\star \cdot n)\mathcal{H}^1\lfloor_T = 2\pi\mathcal{H}^1\lfloor_T = 2(\sin X - X\cos X)\mathcal{H}^1\lfloor_T \geq 0$$

with $X = \frac{1}{2}(\varphi_\star^+ - \varphi_\star^-) = \pi$. We conclude that $\mu_{u_\star,\varphi_\star} \geq 0$ since $\mu_{u_\star,\varphi_\star}$ is supported on $S = T \cup \Sigma$. Consequently, arguing as previously,

$$\int_\Omega |\mu_{u_\star,\varphi_\star}| = \int_\Omega \mu_{u_\star,\varphi_\star} = \int_\Omega \operatorname{div}\,(\varphi_\star u_\star + u_\star^\perp).$$

Since $\varphi_\star \in BV$, this can be integrated as

$$\int_{\partial\Omega} (\varphi_\star u_\star + u_\star^\perp) \cdot n = \int_{\partial\Omega} -u_\star \cdot \tau = |\partial\Omega|$$

in view of the definition of $u_\star$ ($u_\star \cdot \tau = -1$ on $\partial\Omega$). We conclude that

$$||\mu_{u_\star,\varphi_\star}|| = 2\int_S |\sin X - X\cos X| d\mathcal{H}^1 = |\partial\Omega|.$$

$\square$

Theorem 3 is a consequence of Lemma V.1 and Lemma V.2.

Remark V.2 : In Lemma III.1, there was a large possibility of choice of $\varphi_\star$ since T could be any ray joining Σ to $\partial\Omega$. The results of Lemma V.1 are independent of this choice, which is interesting in itself : the result $\mu_{u_\star,\varphi_\star} \geq 0$, $\mu_{u_\star,\varphi_\star} = \int_S 2\sin X - X\cos X = |\partial\Omega|$ is true independently of the choice of T. This is made possible by the fact that, as we change T, the contribution of T to the limiting energy (which is 2π times its length) can vary, but the contribution along Σ varies also to compensate this change. Indeed, although u^+ and u^- are independent of T along Σ, as we move T, X changes between the two contact points of the initial and final T. Observe that the sign of X remains unchanged.

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