Gamma-convergence of gradient flows on Hilbert and metric spaces and applications

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Abstract

We are concerned with Γ-convergence of gradient flows, which is a notion meant to ensure that if a family of energy functionals depending of a parameter Γ-converges, then the solutions to the associated gradient flows converge as well. In this paper we present both a review of the abstract "theory" and of the applications it has had, and a generalization of the scheme to metric spaces which has not appeared elsewhere. We also mention open problems and perspectives.

Γ-convergence was introduced by De Giorgi in the 70's. It provides a convenient notion of convergence of a family of energy functionals E_ε to a limiting functional F, which ensures in particular that minimizers of E_ε converge to minimizers of F. In [DeG1], De Giorgi raised the question of knowing whether there was any general relation between solutions of the gradient flows of E_ε and solutions to the gradient flow of F when E_ε Γ-converges to F. Such a result is not true in general: it is easy to construct finite-dimensional examples where it fails (take for example a smooth function and perturb it by adding a sequence of functions which is small in L^∞ norm but such that the sum has many local minima). The question itself is a natural one, and of importance for a variety of (potential) applications: dynamics of singularities in materials, homogenization of evolution equations, (numerical) approximation of solutions to gradient flows, and in general asymptotic limits of PDEs.

In 2004, motivated by the convergence of the Ginzburg-Landau heat flow, we introduced with Etienne Sandier in [SS1] a notion which we called "Γ-convergence of gradient flows", which provided an abstract framework giving additional conditions on E_ε and F that ensure convergence of the gradient flows. Previously, convergence was proved on a case by case basis, usually via PDE methods. The goal in [SS1] was to provide an energy-based method, taking advantage of the Γ-convergence structure. For the sake of simplicity, the method in [SS1] was presented in the situation of Hilbert spaces for the original flows, and finite-dimensional spaces for the limiting flow. It was explained in [SS1] that with applications in mind, the appropriate situation was probably that of a (formal) Hilbert manifold structure, and that the appropriate rigorous framework was that of metric spaces using De Giorgi's "minimizing movements" i.e. gradient flows which are defined only on metric spaces, as presented in [AGS]. It was noted that the scheme could be carried out in that framework.

While the abstract result is easy to state and prove, the difficulty is displaced into proving that the hypotheses of the abstract result are satisfied for each specific problem. This has been achieved in some examples, it also raises some interesting analysis questions and encounters some difficulties, as we shall show below. Our goal here is to present an overview of these, as well as present the extension of the scheme to the more (and probably most) general setting of gradient flows on metric spaces.

The paper is organized as follows: Section 1 presents the abstract scheme following [SS1]. Section 2 presents and proves the adaptation of the abstract scheme to metric spaces, and briefly reviews other related results in the literature. Section 3 looks into applications of the scheme to a few famous evolution PDEs depending on a parameter: the heat-flow for Ginzburg-Landau vortices (following [SS1, Ku]), the Cahn-Hilliard equation (following [Le1]), the Allen-Cahn equation (drawing on [MR1]) and a prospective attempt of application to Ginzburg-Landau with large number of vortices.

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1 The abstract scheme in Hilbert spaces

In [SS1] we assumed that the we were taking the gradient flow for the C^1 functionals E_ε with respect to some Hilbert space structure X_ε, i.e. we consider solutions to

$$\partial_t u = -\nabla_{X_\varepsilon} E_\varepsilon(u) \in X_\varepsilon \tag{1.1}$$

where $\nabla_{X_\varepsilon} E_\varepsilon$ is defined via the relation $dE_\varepsilon(u) \cdot \phi = \langle \nabla_{X_\varepsilon} E_\varepsilon(u), \phi \rangle_{X_\varepsilon}$, dE_ε denoting the differential of the C^1 function E_ε. The functionals E_ε are assumed to Γ-converge to a functional F. It was noted in [SS1] that since E_ε and F need not in practice be defined on the same space, one should consider a general sense of convergence $u_\varepsilon \overset{S}{\rightharpoonup} u$, to be specified in each case, relative to which the Γ-convergence of E_ε to F holds. Following [JSt] one may model this convergence by assuming there is a continuous "projection" map π_ε from X_ε to Y such that $u_\varepsilon \overset{S}{\rightharpoonup} u$ is defined by $\pi_\varepsilon(u_\varepsilon) \to u$ in Y. In addition, to deal with Γ-limsup constructions, one may also add the existence of a "lifting" map P_ε from Y to X_ε with $\pi_\varepsilon \circ P_\varepsilon = Id$.

In [SS1], we assumed for simplicity in the rigorous results that the limiting functional F was defined over a finite-dimensional vector space Y equipped with some Hilbert scalar product, thus its gradient flow with respect to Y is given by

$$\partial_t u = -\nabla_Y F(u) \tag{1.2}$$

with the analogous notation.

We also introduced the "energy-excess" along a family of curves $u_\varepsilon(t)$ with $u_\varepsilon(t) \overset{S}{\rightharpoonup} u(t)$ by setting $D_\varepsilon(t) = E_\varepsilon(u_\varepsilon(t)) - F(u(t))$ and $D(t) = \limsup_{\varepsilon\to 0} D_\varepsilon(t)$. Similarly if $u_\varepsilon \overset{S}{\rightharpoonup} u$ then D denotes $\limsup_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) - F(u)$.

Let us now state the result in [SS1] in a slightly simplified form:

Theorem 1 (Γ-convergence of gradient flows in the Hilbert space setting - [SS1])**.** *Assume E_ε and F are as above and satisfy a Γ-liminf relation: if $u_\varepsilon \overset{S}{\rightharpoonup} u$ as $\varepsilon \to 0$ then*

$$\liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) \geq F(u).$$

Assume that the following two additional conditions hold:

1. *(Lower bound on the velocities.) If $u_\varepsilon(t) \overset{S}{\rightharpoonup} u(t)$ for all $t \in [0,T)$ then there exists $f \in L^1(0,T)$ such that for every $s \in [0,T)$*

$$\liminf_{\varepsilon\to 0} \int_0^s \|\partial_t u_\varepsilon(t)\|_{X_\varepsilon}^2 \, dt \geq \int_0^s \left(\|\partial_t u(t)\|_Y^2 - f(t) D(t) \right) dt. \tag{1.3}$$

2. *(Lower bound for the slopes) If $u_\varepsilon \overset{S}{\rightharpoonup} u$ then*

$$\liminf_{\varepsilon\to 0} \|\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon)\|_{X_\varepsilon}^2 \geq \|\nabla_Y F(u)\|_Y^2 - CD \tag{1.4}$$

   where C is a universal constant.

Let then $u_\varepsilon(t)$ be a family of solutions to (1.1) on $[0,T)$ with $u_\varepsilon(t) \overset{S}{\rightharpoonup} u(t)$ for all $t \in [0,T)$, such that

$$\forall t \in [0,T) \qquad E_\varepsilon(u_\varepsilon(0)) - E_\varepsilon(u_\varepsilon(t)) = \int_0^t \|\partial_t u_\varepsilon(s)\|_{X_\varepsilon}^2 \, ds.$$

Assume also that it is "well-prepared", i.e. $D(0) = 0$ or

$$\lim_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon(0)) = F(u(0)), \tag{1.5}$$

then u is in $H^1((0,T),Y)$ (in particular continuous in time) and is a solution to (1.2) on $[0,T)$. Moreover $D(t) = 0$ for all t (that is the solutions "remain well-prepared") and as $\varepsilon \to 0$,

$$\|\partial_t u_\varepsilon\|_{X_\varepsilon} \to \|\partial_t u\|_Y \quad \text{in } L^2(0,T)$$
$$\|\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon)\|_{X_\varepsilon} \to \|\nabla_Y F\|_Y \quad \text{in } L^2(0,T).$$

The proof of the theorem is very simple and relies on the Cauchy-Schwarz inequality combined with the two extra lower bounds (1.3)–(1.4). For simplicity, and in many examples one can take $f \equiv 0$ and $C = 0$ above. If these are not zero then they are handled via Gronwall's lemma, and proved to be zero in the end.

Proof of the theorem. Let us now present the proof in the simpler case where $f \equiv 0$ and $C \equiv 0$. Let $u_\varepsilon(t)$ be the solution to (1.1) as above, by assumption and by (1.1),

$$E_\varepsilon(u_\varepsilon(0)) - E_\varepsilon(u_\varepsilon(t)) = \int_0^t \|\partial_t u_\varepsilon(s)\|_{X_\varepsilon}^2 \, ds = \frac{1}{2}\int_0^t \|\partial_t u_\varepsilon(s)\|_{X_\varepsilon}^2 + \|\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon(s))\|_{X_\varepsilon}^2 \, ds.$$

In view of the relations (1.3) and (1.4) it follows (with Fatou's lemma) that

$$\liminf_{\varepsilon\to 0} \left(E_\varepsilon(u_\varepsilon(0)) - E_\varepsilon(u_\varepsilon(t))\right) \geq \frac{1}{2}\int_0^t \|\partial_t u(s)\|_Y^2 + \|\nabla_Y F(u(s))\|_Y^2.$$

By the Cauchy-Schwarz inequality (for real numbers) we have

$$\frac{1}{2}\int_0^t \|\partial_t u(s)\|_Y^2 + \|\nabla_Y F(u(s))\|_Y^2 \geq -\int_0^t \langle \partial_t u, \nabla_Y F(u)\rangle_Y \, ds = F(u(0)) - F(u(t)). \tag{1.6}$$

But since $u_\varepsilon(t)$ is a well-prepared solution, we have $\lim_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon(0)) = F(u(0))$. Combining the above relations, we deduce

$$\liminf_{\varepsilon\to 0} \left(-E_\varepsilon(u_\varepsilon(t))\right) \geq -F(u(t))$$

i.e.

$$\limsup_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon(t)) \leq F(u(t)).$$

But by Γ-convergence of E_ε to F the converse holds i.e.

$$\liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon(t)) \geq F(u(t)).$$

It follows that we must have

$$\lim_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon(t)) = F(u(t)) \tag{1.7}$$

and all the inequalities above must be equalities. In particular, we have equality in (1.6) and thus $\partial_t u = -\nabla_Y F(u)$ for a.e. $t \in (0,T)$. Also there must be equality in (1.3) and (1.4) for $u_\varepsilon(t)$ for almost every time, hence the last two assertions. Well-preparedness persists for all time from (1.7). The fact that $u(t) \in H^1((0,T),Y)$ is a direct consequence of (1.3). It implies in particular $u \in C^{0,\alpha}((0,T),Y)$. If f and C are present, the same can be done and one concludes using Gronwall's lemma on $D(t)$. □

Several remarks can be made at this point.

1. Although we say only a Γ-liminf relation between E_ε and F is required, in reality assuming that there is well-prepared initial data implicitly requires the Γ-limsup to hold as well.

2. Taking the spaces (and their norm) X_ε to possibly depend on ε allows for more flexibility, in particular it allows to incorporate time rescalings of the equation (1.1) into the norm (see [SS1]).

3. When looking at the convergence of a family of gradient flows, it is not clear which is the structure with respect to which the limiting gradient flow should be taken, in other words what is the structure Y. In practice Y can be guessed by guessing for which structure the inequalities (1.3) and (1.4) can hold. In addition, even if X_ε are Hilbert spaces, the limiting gradient flow will not always be a flow with respect to a Hilbert space or a flat space. In general it can be a flow with respect to a curved space (i.e. a formal infinite dimensional manifold, where the scalar product Y_u depends on the point u). We will see examples of this phenomenon in Section 3 below (for Cahn-Hilliard and Allen-Cahn). The limiting space and energies need not even be smooth, this is the reason to go to the more general setting of metric spaces and curves of maximal slope, as in Section 2.

4. The conditions (1.3) and (1.4) are general conditions. However to prove the result of the theorem, one does not really need to prove them for all u_ε but only for families of solutions to the gradient flow, on which one may have more information.

5. The conditions (1.3) and (1.4) provide sufficient extra conditions for Γ-convergence of gradient flows. They correspond to a kind of C^1 notion of Γ-convergence in the sense that they allow to compare the C^1 structures of the energy landscapes of E_ε and F, where the gradient flows live. Of course the spaces where these flows live being different, they cannot be compared, however the sizes of the slopes or derivatives can be compared, and this suffices.

   If in (1.4) we have $C = 0$, this condition immediately implies that critical points of E_ε converge to critical points of F. (Note that Γ-convergence only ensured the convergence of global minimizers).

   A similar C^2 notion of Γ-convergence was introduced in [Se1]: it provides a sufficient condition (based on the C^2 structure of the energy landscape) to ensure that stable critical points of E_ε converge to stable critical points of F (in [Se1] this is applied to Ginzburg-Landau vortices).

6. In [SS1] we propose a constructive way of proving relation (1.4): it suffices, for any continuous $v(t)$ defined in a neighborhood of 0 and such that $v(0) = u$, to find a deformation $v_\varepsilon(t)$ of u_ε, such that

$$\limsup_{\varepsilon\to 0} \|\partial_t v_\varepsilon(0)\|_{X_\varepsilon} \le \|\partial_t v(0)\|_Y$$

   and

$$\liminf_{\varepsilon\to 0} -\frac{d}{dt}_{|t=0} E_\varepsilon(v_\varepsilon) \ge -\frac{d}{dt}_{|t=0} F(v).$$

   Indeed, if this is true, then one can take a deformation $v(t)$ with derivative $-\nabla F(u)$ at 0, and the two relations above immediately imply (1.4). In this point of view the question reduces to "lifting" a curve $v(t)$ in the space Y to a curve $v_\varepsilon(t)$ in such a way that the velocity of the lifted curve is smaller than (and in fact equal to with (1.3)) that of the original one, and that the energy decreases by more than the limiting energy, i.e. more than expected.

7. An interesting related question is to study the Γ-convergence of "action" functionals along time-dependent curves:

$$A_\varepsilon(u_\varepsilon) = \int_0^T \|\partial_t u_\varepsilon + \nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon(t))\|_{X_\varepsilon}^2 \, dt \tag{1.8}$$

with the constraint $u_\varepsilon(0) = u_\varepsilon^0$ and $u_\varepsilon(T) = u_\varepsilon^T$. Indeed, according to the theory of large deviations, in the equation $\partial_t u_\varepsilon = -\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon) +$ *noise* and when the noise is white in space and time, the switching between the states u_ε^0 and u_ε^T is most likely to happen along a curve $u_\varepsilon(t)$ which minimizes A_ε. Then, when $\varepsilon \to 0$, one just needs to identify the Γ-limit of A_ε since then minimizers of A_ε will converge to the minimizers of the Γ-limit. However, a rigorous meaning can only be given to this (and to the noisy equation) in 1 space dimension. This program, in the context of sharp-interface problems, was first presented and illustrated with numerics in E-Ren-Vanden-Eijden [ERV]. It was carried out for the action functional (formally) associated to the Allen-Cahn equation with noise in a formal way in [KORV], and at a rigorous level in [KRT, WT] in one-space dimension and [MR1] in higher dimensions (2,3).

One can check very easily that our framework is very well adapted to this question. Indeed, by expanding the scalar product in (1.8), one is led to

$$\begin{aligned} A_\varepsilon(u_\varepsilon) &= \int_0^T \|\partial_t u_\varepsilon\|_{X_\varepsilon}^2 + \|\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon)\|_{X_\varepsilon}^2 + 2\int_0^T \partial_t(E_\varepsilon(u_\varepsilon(t))\, dt \\ &= \int_0^T \|\partial_t u_\varepsilon\|_{X_\varepsilon}^2 + \|\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon)\|_{X_\varepsilon}^2 \, dt + 2(E_\varepsilon(u_\varepsilon^T) - E_\varepsilon(u_\varepsilon^0)). \end{aligned}$$

Thus it suffices to understand the Γ-convergence of $\int_0^T \|\partial_t u_\varepsilon\|_{X_\varepsilon}^2 + \|\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon)\|_{X_\varepsilon}^2$ and conditions (1.3)–(1.4) provide precisely a Γ-liminf relation on this functional.

2 Γ-convergence of gradient flows on metric spaces

It was noticed by De Giorgi that in order to define gradient flows one does not really need to have a Hilbert or Banach structure, i.e. a differentiable structure, but that one can define a weak notion using only a metric structure. The notion replacing gradient flows is then that of "curves of maximal slope". This notion was introduced in [DeGMT], then further developed in [DMT, MST, Am]. We follow here the self-contained presentation in [AGS]. The notion of gradient flows in metric spaces turns out to be useful in application, in particular for defining gradient flows over the space of probability measures equipped with the (metric) Wasserstein distance. In the past years, many interesting PDE's (such as porous media, Fokker-Planck, the Chapman-Rubinstein-Schatzman model of superconductivity ...) have been shown to fall into that class, see [Vi] chapter 23-24 and references therein. As we explained above (item 3 in the remarks above) it is a natural setting for limits of gradient flows (even of gradient flows on Hilbert spaces).

In order to generalize gradient flows to metric spaces, the starting point is to observe that if u is a solution of the gradient flow

$$\partial_t u = -\nabla\phi(u)$$

say on a Hilbert space, then u is characterized by the relation

$$\partial_t(\phi(u)) \leq -\frac{1}{2}\left(|\partial_t u|^2 + |\nabla\phi|^2\right) \tag{2.1}$$

Indeed the relation $\frac{1}{2}\left(|\partial_t u|^2 + |\nabla\phi|^2\right) \geq -\langle\partial_t u, \nabla\phi\rangle$ holds in all cases, and there is equality if and only if $\partial_t u = -\nabla\phi(u)$. (Note that this is precisely what we have used for the proof of Theorem 1 above.) Now (2.1) can naturally be extended to a metric setting provided one gives a definition to the norm of the derivative $|\partial_t u|$ (this will be called the metric derivative) and to the norm of the gradient $|\nabla\phi(u)|$ (this will be called the slope or an upper gradient).

One may also generalize this to (p, q) gradient flows where p and q are conjugate exponents in $(1, +\infty)$: solutions to $\partial_t u|\partial_t u|^{p-2} = -\nabla\phi(u)$ are characterized (via Young's inequality) by

$$\partial_t(\phi(u)) \leq -\frac{1}{p}|\partial_t u|^p - \frac{1}{q}|\nabla\phi|^q.$$

Usual gradient flows simply correspond to $p = q = 2$.

2.1 Definitions

Let us present the details and recall the main definitions from [AGS], Chapter 1. $(\mathcal{S}, d)$ is a complete metric space equipped with the distance d.

Definition 1 (Absolutely continuous curves). *$v : (a, b) \to \mathcal{S}$ is a p-absolutely continuous curve or belongs to $AC^p(a, b, \mathcal{S})$ ($p \geq 1$) if there exists an $L^p(a, b)$ function m such that*

$$d(v(s), v(t)) \leq \int_s^t m(r)\, dr \qquad \forall a < s \leq t < b.$$

In particular absolutely continuous curves are uniformly continuous functions of t.

Definition 2 (Metric derivative). *Let v be an absolutely continuous curve on (a, b). Then the limit*

$$|v'|(t) := \lim_{s \to t} \frac{d(v(s), v(t))}{|s - t|}$$

exists for a.e. $t \in (a, b)$ and is called the metric derivative of v. Moreover it is the smallest admissible function m in the definition above.

Note that $v \in AC^p$ is equivalent to $|v'| \in L^p$.

In what follows ϕ is a real-valued function on $\mathcal{S}$ with domain $D(\phi)$.

Definition 3 (Strong upper gradient). *A function $g : \mathcal{S} \to [0, +\infty]$ is a strong upper gradient for ϕ if for every absolutely continous curve $v \in AC^1(a, b, \mathcal{S})$ the function $g \circ v$ is Borel and*

$$|\phi(v(t)) - \phi(v(s))| \leq \int_s^t g(v(r))|v'|(r)\, dr \quad \forall a < s \leq t < b. \tag{2.2}$$

In particular if $g \circ v|v'| \in L^1(a, b)$ then $\phi \circ v$ is absolutely continuous and

$$|(\phi \circ v)'|(t) \leq g(v(t))|v'|(t) \quad \text{for a.e. } t \in (a, b). \tag{2.3}$$

A candidate to be an upper gradient of ϕ is its slope:

Definition 4 (Slope). *The local slope of ϕ at $v \in D(\phi)$ is defined by*

$$|\partial\phi|(v) := \limsup_{w \to v} \frac{(\phi(v) - \phi(w))_+}{d(v, w)}.$$

A theorem is that the slope $|\partial\phi|$ is always a weak upper gradient for ϕ (see [AGS] for the definition).

Definition 5 (Curve of maximal slope). *We say that a locally absolutely continuous map $u : (a, b) \to \mathcal{S}$ is a p-curve of maximal slope for the functional ϕ with respect to its weak upper gradient g if $\phi \circ u$ is a.e. equal to a nonincreasing map φ and*

$$\varphi'(t) \leq -\frac{1}{p}|u'|(t)^p - \frac{1}{q}g^q(u(t)) \quad \text{for a.e. } t \in (a, b). \tag{2.4}$$

Since (2.3) always holds, we must have

$$\varphi'(t) \geq -g(u(t))|u'|(t) \geq -\frac{1}{p}|u'|^p - \frac{1}{q}g^q(u)$$

by Young's inequality, so there must be equality in (2.4) a.e. in t. If g is a strong upper gradient then $\varphi(t)$ is absolutely continuous hence continuous and in particular the energy identity

$$\phi(u(s)) - \phi(u(t)) = \int_s^t \frac{|u'|^p}{p} + \frac{g^q(u)}{q} \tag{2.5}$$

holds.

For simplicity one may think of the most standard case $p = q = 2$. Then 2-curves of maximal slope are simply called curves of maximal slope.

The notion of curve of maximal slope is thus a natural analogue of the notion of gradient flow or steepest descent curve. The existence of curves of maximal slope is obtained via the notion of "generalized minimizing movements" of De Giorgi i.e. taking the limit as $\tau \to 0$ of the semi-discrete implicit Euler scheme for steepest descent which is given by

$$U_\tau^k = \operatorname{argmin}\left(\phi(v) + \frac{1}{p\tau^{p-1}} d^p(v, U_\tau^{k-1})\right). \tag{2.6}$$

Let $(\mathcal{S}_\varepsilon, d_\varepsilon)$ be a family of metric spaces equipped with distances d_ε, and $(\mathcal{S}, d)$ be another metric space. d naturally induces a topology on $\mathcal{S}$, however we also consider a possibly weaker topology σ on $\mathcal{S}$. We are given functionals Φ_ε on $\mathcal{S}_\varepsilon$ and Φ on $\mathcal{S}$. We assume there is a sense of convergence S of $u_\varepsilon \in \mathcal{S}_\varepsilon$ to $u \in \mathcal{S}$ which can be general and with respect to which we know the Γ-liminf convergence of Φ_ε to Φ:

$$u_\varepsilon \xrightarrow{S} u \Longrightarrow \qquad \liminf_{\varepsilon \to 0} \Phi_\varepsilon(u_\varepsilon) \geq \Phi(u). \tag{2.7}$$

To be specific, as above we may assume that there exists a map π_ε from $\mathcal{S}_\varepsilon$ to $\mathcal{S}$ such that $u_\varepsilon \xrightarrow{S} u$ means $\pi_\varepsilon(u_\varepsilon) \xrightarrow{\sigma} u$. The need to use here a possibly weaker topology than that induced by d on $\mathcal{S}$ is well explained in [AGS], as well as in the examples we shall see below.

2.2 Abstract result and proof

Slopes and metric derivatives will be computed in $\mathcal{S}_\varepsilon$ and $\mathcal{S}$ with respect to the metrics d_ε and d. To emphasize this dependence, we will place a d_ε or d subscript next to them.

Theorem 2 (Γ-convergence of gradient flows in the metric setting)**.** *Let Φ_ε and Φ be functionals defined on metric spaces $(\mathcal{S}_\varepsilon, d_\varepsilon)$ and $(\mathcal{S}, d)$ respectively, and such that (2.7) holds. Let g_ε and g be strong upper gradients of Φ_ε and Φ respectively. Assume in addition the relations*

1. *(Lower bound on the metric derivatives) If $u_\varepsilon(t) \xrightarrow{S} u(t)$ for $s \in (0,T)$ then*

$$\forall s \in [0,T) \quad \liminf_{\varepsilon \to 0} \int_0^s |u'_\varepsilon|^p_{d_\varepsilon}(t)\,dt \geq \int_0^s |u'|^p_d(t)\,dt. \tag{2.8}$$

2. *(Lower bound on the slopes) If $u_\varepsilon \xrightarrow{S} u$ then*

$$\liminf_{\varepsilon \to 0} g_\varepsilon(u_\varepsilon) \geq g(u). \tag{2.9}$$

Let then $u_\varepsilon(t)$ be a p-curve of maximal slope on $(0,T)$ for Φ_ε with respect to g_ε, such that $u_\varepsilon(t) \xrightarrow{S} u(t)$, which is well-prepared in the sense that

$$\lim_{\varepsilon \to 0} \Phi_\varepsilon(u_\varepsilon(0)) = \Phi(u(0)).$$

Then u is a p-curve of maximal slope for Φ with respect to g and

$$\lim_{\varepsilon \to 0} \Phi_\varepsilon(u_\varepsilon(t)) = \Phi(u(t)) \quad \forall t \in [0,T)$$

$$g_\varepsilon(u_\varepsilon) \to g(u) \quad \text{in } L^p_{loc}(0,T)$$
$$|u'_\varepsilon|_{d_\varepsilon} \to |u'|_d \quad \text{in } L^p_{loc}(0,T).$$

Proof. The proof follows the same steps as in Theorem 1. First note that since u_ε are p-curves of maximal slopes they are in $AC^p(0,T,\mathcal{S}_\varepsilon)$ (see Remark 1.3.3 in [AGS]). From (2.8) it follows that $|u'|$ is in $L^p_{loc}[0,T)$ and thus u is in $AC^p(0,T,\mathcal{S})$. From (2.5) since g_ε is a strong upper gradient

$$\Phi_\varepsilon(u_\varepsilon(0)) - \Phi_\varepsilon(u_\varepsilon(t)) = \int_0^t \frac{1}{p}|u'_\varepsilon|^p_{d_\varepsilon}(s) + \frac{1}{q}g^q_\varepsilon(u_\varepsilon(s))\,ds.$$

Using the inequalities (2.8) and (2.9) and Fatou's lemma, it follows that for all $t \in [0,T)$

$$\liminf_{\varepsilon\to 0} \Phi_\varepsilon(u_\varepsilon(0)) - \Phi_\varepsilon(u_\varepsilon(t)) \geq \int_0^t \frac{1}{p}|u'|^p_d(s) + \frac{1}{q}g^q(u(s))\,ds. \tag{2.10}$$

From Young's inequality we deduce

$$\int_0^t \frac{1}{p}|u'|^p_d(s) + \frac{1}{q}g^q(u(s))\,ds \geq \int_0^t |u'|_d(s)g(u(s))\,ds. \tag{2.11}$$

In addition, by the definition of a strong upper gradient we have (2.2), so

$$\int_0^t |u'|_d(s)g(u(s))\,ds \geq \Phi(u(0)) - \Phi(u(t)). \tag{2.12}$$

On the other hand from the well-preparedness assumption (2.10) and (2.11) become

$$\liminf_{\varepsilon\to 0} -\Phi_\varepsilon(u_\varepsilon(t)) \geq -\Phi(u(t))$$

But the converse inequality holds by (2.7) so we have

$$\lim_{\varepsilon\to 0} \Phi_\varepsilon(u_\varepsilon(t)) = \Phi(u(t)) \tag{2.13}$$

and there must be equality in all the inequalities above. In particular there is equality in (2.11) and (2.12), that is

$$\frac{1}{p}|u'|^p_d(s) + \frac{1}{q}g^q(u(s)) = |u'|_d(s)g(u(s)) \quad \text{a.e. in } s \in (0,T)$$

and $|u'|_d(s)g(u(s)) = -(\Phi \circ u)'$. The fact that $\Phi \circ u$ is a.e. nonincreasing immediately follows from (2.13) and the fact that $\Phi_\varepsilon \circ u_\varepsilon$ is a.e. nonincreasing (since u_ε is a p-curve of maximal slope). This proves that u is a p-curve of maximal slope for Φ with respect to g. The other relations easily follow from that chain of equalities as well.

□

Several remarks can be made at this point

1. There is obviously a question of compactness of sequences of solutions which we have left aside here as well as in the previous theorem.

2. This can easily be particularized to λ-convex functions following [Or1].

3. One can particularize this scheme to Banach spaces by using Section 1.4 in [AGS]. For example, if ϕ is a (non C^1) convex functional on a Banach space X and $\partial\phi$ is the Fréchet subdifferential of ϕ, we have that

$$|\partial\phi|(u) := \min\{\|\xi\|_{X^*}, \xi \in \partial\phi(v)\}$$

i.e. the element of $\partial\phi$ with smallest norm, is a strong upper gradient (see [AGS] Section 1.4). This way we can treat the passage to the limit in differential inclusions for nonsmooth functionals, which correspond to p-curves of maximal slope according to Proposition 1.4.1 in [AGS].

4. The metric space setting should be a good generalization of the (formal) Hilbert manifold setting, by taking the geodesic distance on the Hilbert manifold. An interesting example where this correspondance is known is the case of the space of probability measures equipped with the 2-Wasserstein distance (see [AGS, Vi]). Otto observed that this space, initially just a metric space, is a formal Hilbert manifold with Hilbert metric at μ being given by L^2_μ. This was made rigorous in [AGS]. This space appears as a natural limiting space for gradient-flows of Ginzburg-Landau with large number of vortices, as we shall see in Section 3.4.

2.3 Related results in the literature

The convergence of curves of maximal slope under Γ-convergence actually holds with no extra assumptions when the functionals are ϕ-convex: this was proved in [DMT].

A later paper that addressed the same type of questions is [Ji], but there it was restricted to a certain class of parabolic flows.

In [Or1], Ortner also looked into something very close: he proposed to approximate curves of maximal slope via the minimizing movement scheme, by Γ-convergence approximation i.e. taking a sequence of functionals ϕ_n which approximate ϕ by Γ-converging to it. He proved that the discrete solutions to the associated Euler scheme (2.6) associated to ϕ_n converge to a curve of maximal slope for ϕ, under two additional conditions:

$$\liminf_{n\to\infty} d_n(u_n, v_n) \geq d(u, v)$$

if $u_n \rightharpoonup u$ and $v_n \rightharpoonup v$ which is the counterpart of (1.3), and the lower semi-continuity of the slopes

$$\liminf_{n\to\infty} |\partial\phi_n(u_n)| \geq |\partial\phi(u)| \tag{2.14}$$

which is the analogue to (1.4). Then he particularized it to λ-convex functionals, noticing that the condition of λ-convexity together with the Γ-convergence of the ϕ_n to ϕ ensure that (2.14) holds. In [Or2] the presentation was a bit different: he stated an abstract convergence theory for convergence of the gradient flows of λ-convex functionals on Hilbert spaces (sufficient to the purpose of that paper), analogous to the one we present here. The motivation was the convergence of local minima during numerical approximation: gradient flows are then seen as a selection criteria. It was then applied to an atomistic energy approximation.

In the paper [MRS], a program similar to ours is carried out for "rate-independent processes". These are evolutions which arise in the modelling of fracture and plasticity and which can essentially be viewed as the limit case $p = 1$ of the $p-$curves of maximal slope described above.

Another natural and important question is whether similar results hold for Hamiltonian flows, i.e. under which conditions solutions to Hamiltonian flows of a family of energies E_ε (Γ-converging to F) converge to solutions of the Hamiltonian flow of F. An abstract convergence result, using convergence conditions on the underlying symplectic structures, is proposed in [Mi]. It is applied to some atomistic models and homogenization of wave equations.

3 A review of some applications

3.1 Ginzburg-Landau vortices

The first motivation for the method introduced in [SS1] was the derivation of the dynamical law of vortices for solutions to the parabolic Ginzburg-Landau equation, with or without applied magnetic field, this application was detailed in [SS1]. This recovered by a different method, a result that was known [Li, JSo] in the case without magnetic field; and proved a new result in the case with magnetic field. For simplicity we present here the application in the case without magnetic field. The Ginzburg-Landau energy functional in that case is

$$F_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{(1-|u|^2)^2}{\varepsilon^2}, \tag{3.1}$$

and its gradient flow under consideration is the parabolic PDE

$$\frac{\partial_t u}{|\log \varepsilon|} = \Delta u + \frac{u}{\varepsilon^2}(1-|u|^2). \tag{3.2}$$

Here Ω is a two-dimensional smooth bounded domain (simply connected), ε is a (small) material constant, and $u : \Omega \to \mathbb{C}$ is the "order-parameter" in physics. This model is a simplified version of the Ginzburg-Landau model of superconductivity (it also serves in the modelling of superfluidity and Bose-Einstein condensates). A key feature is the existence of vortices, i.e. isolated zeros of u with nonzero winding number $d \in \mathbb{Z}$ of $u/|u|$ around. In the limit $\varepsilon \to 0$ the vortices become point singularities of $u/|u|$. Each vortex has a divergent energetic cost of at least $\pi|d|\log\frac{1}{\varepsilon}$.

This functional was studied in detail by Bethuel-Brezis-Hélein [BBH] and some subsequent works, in the situation where the total number of vortices remains bounded independently of ε (this is ensured by a bound $C|\log \varepsilon|$ on $F_\varepsilon(u_\varepsilon)$). Then one may extract limiting vortices $a_1, \cdots, a_n$ with corresponding nonzero integer degrees $d_1, \cdots, d_n$. The functionals F_ε are defined on $H^1(\Omega, \mathbb{C})$ or $H^1_g(\Omega, \mathbb{C})$ if a boundary condition $u = g$ (with $|g| = 1$) on $\partial\Omega$ is imposed (then H^1_g corresponds to H^1 functions with trace g on the boundary). The limiting space is that of configurations of points + degrees $(\mathbf{a}, \mathbf{d}) \in \Omega^n \times \mathbb{Z}^n_*$ where the total number of points is fixed to n. An appropriate sense of convergence to consider here is $u_\varepsilon \overset{S}{\rightharpoonup} u := (\mathbf{a}, \mathbf{d})$ if

$$\operatorname{curl} \langle iu_\varepsilon, \nabla u_\varepsilon \rangle \rightharpoonup 2\pi \sum_{i=1}^n d_i \delta_{a_i} \quad \text{in } \mathcal{D}'(\Omega)$$

where δ_x is the Dirac mass at x, and $\langle , \rangle$ denotes the scalar product in $\mathbb{C}$ as identified with $\mathbb{R}^2$. Equivalently one could take the definition $u_\varepsilon \overset{S}{\rightharpoonup} (\mathbf{a}, \mathbf{d})$ if $u_\varepsilon / |u_\varepsilon|$ converges to u_*, an S^1-valued map with singularities a_i of degrees d_i.

The results proved in [BBH] can be phrased in Γ-convergence terms as follows. There exists a function $W_g(\mathbf{a}, \mathbf{d})$ called "renormalized energy", defined over $\Omega^n \times \mathbb{Z}^n_*$ such that if $u_\varepsilon \in H^1_g(\Omega)$, modulo a subsequence $u_\varepsilon \overset{S}{\rightharpoonup} (\mathbf{a}, \mathbf{d})$ and

$$\liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) - \pi \sum_{i=1}^n |d_i| |\log \varepsilon| \geq W_g(\mathbf{a}, \mathbf{d}).$$

A matching upper bound also holds. This renormalized energy corresponds to the interaction energy between the vortices left when one substracts the divergent core energy $\pi |d_i| |\log \varepsilon|$ of each vortex.

The same can be done when the Dirichlet boundary condition is replaced by a Neumann boundary condition, leading to a modified renormalized energy. We will all call them W for simplicity.

Restricting to the situation where the limiting degrees d_i are all ± 1 (this class is preserved under the gradient-flow, at least for short times), we may say that $E_\varepsilon(u) := F_\varepsilon(u) - \pi n |\log \varepsilon|$ Γ-converges to $F = W$, which is defined on a *finite-dimensional* space. We may thus look into applying Theorem 1 to the family E_ε. (3.2) is the gradient flow of F_ε or E_ε for the $L^2(\Omega)$ structure rescaled as follows: take $\| \cdot \|^2_{X_\varepsilon} = \frac{1}{|\log \varepsilon|} \| \cdot \|^2_{L^2(\Omega)}$. Indeed one may check that

$$\nabla_{X_\varepsilon} E_\varepsilon(u) = -|\log \varepsilon| \left(\Delta u + \frac{u}{\varepsilon^2}(1 - |u|^2)^2 \right). \tag{3.3}$$

The limiting space of configurations $(\mathbf{a}, \mathbf{d})$ with d_i fixed to ± 1, can be identified to Ω^n, and we equip it with the rescaled Euclidean structure on $(\mathbb{R}^2)^n$ given by $\| \cdot \|^2_Y = \frac{1}{\pi} | \cdot |^2_{\mathbb{R}^{2n}}$. In the framework of Theorem 1, the two extra conditions are here specifically (dropping the $D(t)$):
if $u_\varepsilon(t) \overset{S}{\rightharpoonup} (\mathbf{a}, \mathbf{d})$ with $d_i = \pm 1$,

$$\liminf_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} \int_0^s \| \partial_t u_\varepsilon \|^2_{L^2(\Omega)}(t)\, dt \geq \frac{1}{\pi} \int_0^s |\partial_t a_i|^2\, dt \tag{3.4}$$

and (in view of (3.3))

$$\liminf_{\varepsilon\to 0} \frac{1}{|\log \varepsilon|} \int_\Omega |\log \varepsilon|^2 \left|\Delta u + \frac{u}{\varepsilon^2}(1-|u|^2)\right|^2 \ge \|\nabla_Y W(\mathbf{a},\mathbf{d})\|_Y^2 = \pi |\nabla W(\mathbf{a},\mathbf{d})|^2. \tag{3.5}$$

The relation (3.4), which relates the velocity of the curve $u(t)$ to the velocity of the underlying vortices, is not true in general, however it is true when one restricts to u_ε which has n limiting vortices of degrees ± 1 and such that

$$F_\varepsilon(u) \le \pi n |\log \varepsilon| + C. \tag{3.6}$$

This is sufficient since we only focus on well-prepared configurations, i.e. such that $E_\varepsilon(u) \le C$, which implies (by decrease of the energy) that $F_\varepsilon(u) \le \pi n|\log \varepsilon| + C$ for all subsequent times, as long as the number of vortices remains equal to n. This analysis thus breaks down when some vortices collide or exit from the domain. The inequality (3.4) under the assumption (3.6) was first proved in [Je], and is also a corollary of a more general lower bound proved in [SS3] which has the form of a "product estimate", and which we will see again in other examples below.

On the other hand the relation (3.5) which relates the slope of E_ε to the gradient of the renormalized energy, was proved in [Li]. We gave a proof in [SS1] using the approach outlined after Theorem 1, item 6, i.e. via the construction of a deformation. With these two conditions proved, one recovers the dynamical law of vortices obtained in [JSo, Li] by PDE methods.

Theorem 3. *Let u_ε be a family of solutions to (3.2) with either Dirichlet or Neumann boundary condition, such that* $\mathrm{curl}\,\langle iu_\varepsilon, \nabla u_\varepsilon\rangle(0) \rightharpoonup 2\pi \sum_{i=1}^n d_i \delta_{a_i^0}$ *as $\varepsilon \to 0$ where a_i^0 are distinct points in Ω and $d_i = \pm 1$. Assume also $u_\varepsilon(0)$ is well-prepared in the sense*

$$F_\varepsilon(u_\varepsilon(0)) = \pi n |\log \varepsilon| + W(\mathbf{a}^0, \mathbf{d}) + o(1) \quad \text{as } \varepsilon \to 0.$$

Then there exists a time $T_* > 0$ such that $\mathrm{curl}\,\langle iu_\varepsilon, \nabla u_\varepsilon\rangle(t) \rightharpoonup 2\pi \sum_{i=1}^n d_i \delta_{a_i(t)}$ *for all $t \in [0, T_*)$ and*

$$\frac{da_i}{dt} = -\frac{1}{\pi}\partial_i W(\mathbf{a}(t), \mathbf{d}), \quad a_i(0) = a_i^0$$

with the d_i's remaining constant. T_* is the minimum of the collision time and the exit time (in the Neumann case) under this law. Moreover, the solution "remains well-prepared" in time.

We make several remarks.

1. If (3.2) is scaled differently in time, then it is shown that the limiting vortices either do not move at all, or converge instantaneously to a critical configuration of W (i.e. such that $\nabla W = 0$). This is naturally obtained by including the time rescaling in the metric X_ε.

2. In [SS1] the method above was also applied to the case of the parabolic Ginzburg-Landau equation but with a gauge field and an applied magnetic field. That result was new, and in addition did not follow from the existing methods for the case without magnetic field.

3. As mentioned above, this method is naturally limited to the interval of time before collisions or exits. This is because at those instances, the solutions cease to be "well-prepared" and instantaneous energy excess appears, causing in particular (3.4) to break down. We will see this is a common feature in the applications. On the other hand, the limiting behaviour of vortices has been understood passed collision times, via an energy-based method (such as here) in [Se2], and via PDE methods [BOS2, BOS3, BOS4]. This required a better understanding of the possible vortex structure for configurations for which $\limsup_{\varepsilon\to 0}\|\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon)\|_{X_\varepsilon} < \infty$, and obtaining a stronger lower bound than (3.5) when energy-excess appears due to collisions.

This approach of Γ-convergence of gradient flows was also followed by M. Kurzke [Ku] for deriving the dynamical law of boundary vortices in a thin-film model for micromagnetics. The functional there is

$$\frac{1}{2}\int_\Omega |\nabla m|^2 + \frac{1}{2\varepsilon}\int_{\partial\Omega} (m\cdot\nu)^2$$

where $m \in H^1(\Omega, S^1)$ and ν is the normal to the boundary. Using a lifting $m = e^{iu}$, the functional gets transformed into

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2\varepsilon}\sin^2(u-g)$$

where $\nu = ie^{ig}$, and this time $u \in H^1(\Omega, \mathbb{R})$. The function g, phase of ν, must jump by -2π, but the paper [Ku] considers the more general case where it jumps by $-2\pi D$ for some integer D. The functions u_ε with energy bounded by $C|\log\varepsilon|$ converge to harmonic functions on Ω with singularities on the boundary, more precisely to functions u_* satisfying

$$\begin{cases} \Delta u_* = 0 & \text{in } \Omega\\ \sin(u_* - g) = 0 & \text{on } \partial\Omega\\ u_* \text{ jumps by } -\pi d_i \text{ at } a_i \in \partial\Omega \end{cases}$$

The sense of convergence is then defined as follows: $u_\varepsilon \stackrel{S}{\rightharpoonup} (\mathbf{a}, \mathbf{d})$ if u converges to the u_* above. A renormalized energy $W(\mathbf{a}, \mathbf{d})$ exists just as in the above case of Ginzburg-Landau, with the main difference that it is defined for configurations of points + degrees on the boundary only. The result in [Ku] is then the analogue as the one for Ginzburg-Landau vortices, more precisely: if $u_\varepsilon(t)$ are solutions of the L^2 gradient flow for E_ε,

$$\begin{cases} \lambda_\varepsilon \partial_t u_\varepsilon = \Delta u_\varepsilon & \text{in } \Omega\times(0,T)\\ \frac{\partial u_\varepsilon}{\partial\nu} = -\frac{1}{2\varepsilon}\sin 2(u_\varepsilon - g) & \text{on } \partial\Omega. \end{cases} \tag{3.7}$$

which are initially well-prepared and have initial vortices with degrees $d_i = \pm 1$, then the limiting evolution law of $(\mathbf{a}(t), \mathbf{d})$ is

$$\frac{da_i}{dt} = -\frac{2}{\pi}\partial_{a_i} W(\mathbf{a}(t), \mathbf{d})$$

if $\lambda_\varepsilon = \frac{1}{|\log\ \varepsilon|}$. If $\lambda_\varepsilon \ll \frac{1}{|\log\ \varepsilon|}$ the vortices converge to their initial locations, i.e. there is no motion; and if $\lambda_\varepsilon \gg \frac{1}{|\log\ \varepsilon|}$ they converge instantaneously to a collection of boundary points $\mathbf{b}$ with $\nabla W(\mathbf{b}, \mathbf{d}) = 0$, i.e. to a critical point of W. The two conditions for Γ-convergence of gradient flows are proved through arguments analogous to the above arguments from [SS1]: proof of a "product estimate" for (1.3), and proof of (1.4) via construction of a deformation.

3.2 Cahn-Hilliard

This section covers the work on Nam Le [Le1], who applied the scheme in a formal manner to the convergence of the well-known Cahn-Hilliard equation (which is a model for phase-separation phenomena):

$$\tag{3.8} \begin{cases} \partial_t u_\varepsilon = -\Delta v_\varepsilon & \text{in } \Omega \\ v_\varepsilon = \varepsilon \Delta u_\varepsilon - \frac{1}{\varepsilon} f(u_\varepsilon) & \text{in } \ \Omega \\ \frac{\partial u_\varepsilon}{\partial \nu} = \frac{\partial v_\varepsilon}{\partial \nu} = 0 & \text{on } \partial\Omega \\ u_\varepsilon(x, 0) = u_\varepsilon^0(x) \end{cases}$$

Here u_ε and v_ε are real-valued functions on $\Omega \times [0, +\infty)$, Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$, $f(u) = -2u(1-u^2)$ is the derivative of the double-well potential $W(u) = \frac{1}{2}(1-u^2)^2$. As $\varepsilon \to 0$, the phases $u_\varepsilon \sim 1$ and $u_\varepsilon \sim -1$ become separated by a sharp interface $\gamma(t)$, and solutions to the Cahn-Hilliard equation converge to solutions to the Mullins-Sekerka motion (also called two-phase Hele-Shaw), in the sense that v_ε converges to v solving the following free-boundary problem

$$\tag{3.9} \begin{cases} \Delta v = 0 & \text{in } \Omega \backslash \gamma(t) \\ v = \sigma \kappa & \text{on } \ \gamma(t) \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \ \partial\Omega \\ \partial_t \gamma = \frac{1}{2}\left[\frac{\partial v}{\partial \nu}\right]_{\gamma(t)} & \text{on } \gamma(t) \\ \gamma(0) = \gamma_0. \end{cases}$$

Here κ is the mean curvature of the hypersurface $\gamma(t)$, $\sigma = \int_{-1}^{1} \sqrt{W(s)/2}\, ds = \frac{2}{3}$, $\left[\frac{\partial v}{\partial \nu}\right]_{\gamma(t)}$ denotes the jump of the normal derivative of v accross the hypersurface $\gamma(t)$, and γ_0 is the interface separating $\{u^0 = 1\}$ and $\{u^0 = -1\}$ where $u_0 \in BV(\Omega, \{\pm 1\})$ is the limit (after extraction if necessary) of the initial u_ε^0.

The convergence of solutions of Cahn-Hilliard to Mullins Sekerka motion was derived formally by Pego via matched asymptotic expansions in [Pe], and proved rigorously by Alikakos-Bates-Chen in [ABC] under the assumption of a smooth classical solution to (3.9) (see also other references in [Le1]). It is well-known on the other hand that the Cahn-Hilliard equation is a gradient-flow. More precisely it is the H^{-1} gradient-flow of the Allen-Cahn (or Modica-Mortola) energy

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega \varepsilon|\nabla u|^2 + \frac{(1-u^2)^2}{\varepsilon}. \tag{3.10}$$

It is well-known [MM] that E_ε Γ-converges the perimeter functional. On the other hand, one can observe that (3.9) is a gradient flow for the perimeter functional. So the question of convergence of (3.8) to (3.9) can be phrased as proving that solutions of the H^{-1} gradient flow of (3.10) converge to solutions of the gradient flow (for some structure to be determined) of the Γ-limit of (3.10).

Let us now describe more precisely how to fit this problem into the abstract framework, following [Le1]. Using the notation of Theorem 1, the functionals E_ε are as in (3.10), defined over $H^1(\Omega)$. The structure X_ε should be taken to be $H_n^{-1}(\Omega)$ defined as follows: letting $(H^1(\Omega))^*$ denote the dual to $H^1(\Omega)$, and $\langle,\rangle$ denote the pairing between $H^1(\Omega)$ and $(H^1(\Omega))^*$, then

$$H_n^1(\Omega) = \left\{ f \in (H^1(\Omega))^* | \exists g \in H^1(\Omega), \text{such that } \langle f, \varphi\rangle = \int \nabla g \cdot \nabla \varphi \quad \forall \varphi \in H^1(\Omega) \right\}$$

Formally, H_n^{-1} consists of all distributions f of the form $f = \Delta u$ where $u \in H^1(\Omega)$ and and $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$. The function g in the definition above can be chosen to have mean zero and is then denoted $-\Delta_n^{-1} f$. $H_n^{-1}(\Omega)$ is equipped with the Hilbert space inner product

$$\langle u, v\rangle_{H_n^{-1}(\Omega)} = \int_\Omega \nabla(\Delta_n^{-1} u)\cdot \nabla(\Delta_n^{-1} v).$$

The structure X_ε is then taken to be this Hilbert space $H_n^{-1}(\Omega)$. Simple calculations give indeed that

$$\nabla_{H_n^{-1}(\Omega)} E_\varepsilon(u) = -\Delta\left(-\varepsilon\Delta u + \frac{1}{\varepsilon} f(u)\right).$$

So (3.8) is the gradient flow of E_ε for the $H_n^{-1}(\Omega)$ structure. Moreover

$$\|\nabla_{H_n^{-1}(\Omega)} E_\varepsilon(u)\|^2_{H_n^{-1}(\Omega)} = \|\Delta v_\varepsilon\|^2_{H_n^{-1}(\Omega)} = \|\nabla v_\varepsilon\|^2_{L^2(\Omega)} \tag{3.11}$$

where v_ε is as in (3.8).

Next, the limiting space should be taken as the space of finite perimeter hypersurfaces, which are sufficiently regular (at this point things are formal). It is seen as a formal Hilbert manifold and the structure Y_γ to be taken at each γ on that space is $H_n^{-1/2}(\gamma)$ defined as follows: for every $\tilde{f} \in H^1(\Omega)$ such that

$$\begin{cases} \Delta \tilde{f} = 0 & \text{in } \Omega \backslash \gamma \\ \tilde{f} = f & \text{on } \gamma \\ \frac{\partial \tilde{f}}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

we set

$$\|f\|_{H^{1/2}(\gamma)} = \|\nabla \tilde{f}\|_{L^2(\Omega)}$$

(this definition is unambiguous by trace theory if γ is regular enough). We note that $\|f\|_{H^{1/2}(\gamma)} = 0$ iff $f = cst$ on γ, and we take the quotient under the relation $f_1 \sim f_2$ if $f_1 - f_2 = cst$ on γ. Then $H_n^{1/2}(\gamma) = H^{1/2}(\gamma)/\sim$ becomes a Hilbert space equipped with this norm ($H_n^{1/2}(\gamma)$ is thus roughly speaking the space of traces on γ of $H^1(\Omega)$). For $f \in H^{1/2}(\gamma)$ we define

$$\Delta_\gamma f = -\left[\frac{\partial \tilde{f}}{\partial \nu}\right]_\gamma$$

and we may check that

$$\langle u, v \rangle_{H_n^{1/2}(\gamma)} = -\int_\gamma v \Delta_\gamma u \, d\mathcal{H}^{N-1}. \tag{3.12}$$

Finally $H_n^{-1/2}(\gamma)$ is defined as the dual of $H_n^{1/2}(\gamma)$. We may check that $H_n^{-1/2}(\gamma)$ is a Hilbert space with the inner product

$$\langle u, v \rangle_{H_n^{-1/2}(\gamma)} = \langle \Delta_\gamma^{-1} u, \Delta_\gamma^{-1} v \rangle_{H_n^{1/2}(\gamma)}. \tag{3.13}$$

The limiting functional defined on the limiting space Y_γ is

$$F(\gamma) = 2\sigma \mathcal{H}^{N-1}(\gamma).$$

It is the Γ-limit of E_ε as $\varepsilon \to 0$, relative to the sense of convergence $u_\varepsilon \overset{S}{\rightharpoonup} \gamma$ if $u_\varepsilon \to u$ in L^1 and $\gamma = \partial\{u = 1\}$. The gradient of $F(\gamma)$ with respect to the structure $\|\cdot\|_{Y_\gamma}^2 = 4\|\cdot\|_{H_n^{-1/2}(\gamma)}^2$ can be computed and is found (if γ is a C^3 parametrized hypersurface) to be

$$\nabla_{Y_\gamma} F(\gamma) = \frac{1}{2}\Delta_\gamma(\sigma\kappa) n$$

where n is the unit normal to γ. Moreover, using (3.12)–(3.13),

$$\|\nabla_{Y_\gamma} F(\gamma)\|_{Y_\gamma}^2 = \|\Delta_\gamma(\sigma\kappa)\|_{H_n^{-1/2}(\gamma)}^2 = \sigma^2 \|\kappa\|_{H_n^{1/2}(\gamma)}^2. \tag{3.14}$$

So if γ is C^3 the Mullins-Sekerka law (3.9) coincides with the gradient flow of F for the structure Y_γ. We are thus (under regularity assumptions on γ) in the framework of Theorem 1, formally (since the limiting space is infinite dimensional).

The first extra condition (1.3) in Theorem 1 here becomes showing that if $u_\varepsilon(t) \xrightharpoonup{S} \gamma(t)$ on $[0,T)$ then for all $0 \le s < T$

$$\int_0^s \|\partial_t u_\varepsilon\|^2_{H_n^{-1}(\Omega)}(t)\,dt \ge 4\int_0^s \|\partial_t \gamma(t)\|^2_{H_n^{-1/2}}\,ds \tag{3.15}$$

and the second condition (1.4) becomes (in view of (3.11)–(3.14)) showing that if $u_\varepsilon \xrightharpoonup{S} \gamma$, then

$$\liminf_{\varepsilon\to 0} \|\nabla v_\varepsilon\|^2_{L^2(\Omega)} \ge \sigma^2 \|\kappa\|^2_{H_n^{1/2}(\gamma)}. \tag{3.16}$$

The first condition (3.15) is a direct consequence of the assumed convergence and weak lower semi-continuity. The second condition is much more interesting: it bounds from below the H^1 norm of the chemical potential v_ε by the $H_n^{1/2}$ norm of the curvature of the limiting interface. This actually led in [Le1] to the formulation of a Γ-convergence conjecture: is it true that as $\varepsilon \to 0$

$$u \mapsto \int_\Omega \left|\nabla\left(\varepsilon\Delta u - \frac{1}{\varepsilon} f(u)\right)\right|^2 \quad \Gamma-\text{converges to} \quad \gamma \mapsto \sigma^2\|\kappa\|^2_{H_n^{1/2}(\gamma)}? \tag{3.17}$$

A related conjecture was made by De Giorgi: is is true that

$$u \mapsto \int_\Omega \frac{1}{\varepsilon}\left|\varepsilon\Delta u - \frac{1}{\varepsilon} f(u)\right|^2 \quad \Gamma-\text{converges to } 2\sigma\|\kappa\|^2_{L^2(\gamma)}? \tag{3.18}$$

The latter question was answered positively (under the additional conditions that $E_\varepsilon(u_\varepsilon) \le C$) by Röger and Schätzle [RS] in dimensions 2 and 3. We will see in Section 3.3 that it comes out naturally from the question of convergence of Allen-Cahn to mean-curvature motion. Now (3.17) can be seen as a higher-order derivative analogue of (3.18), and comes out naturally (as we just saw) from the question of convergence of Cahn-Hilliard to Mullins-Sekerka.

In [Le1] (3.17) is proved in dimension $N \le 3$ (or in higher dimension provided an equipartition result holds) if γ is C^3. (There was another assumption of constant multiplicity which can in fact be removed, according to [Sc]: $\|\kappa\|_{H_n^{1/2}(\gamma)} < \infty$ implies the constant multiplicity – see [Le2] for the details). Note that a full Γ-convergence result is not really needed to use our scheme, only a Γ-liminf relation is needed. However in [Le1] Le also proved the Γ-limsup, which is a question of independent interest.

With the result (3.17) and (3.15), it is established in [Le1], in dimension $N \le 3$, that if u_ε is a sequence of solutions to (3.8) with

1. $E_\varepsilon(u_\varepsilon) \le C$
2. $u_\varepsilon(0) \xrightharpoonup{S} \gamma_0$
3. $\lim_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon(0)) = F(\gamma_0)$

then there exists $T_* > 0$ such that on $[0, T_*)$, we have $u_\varepsilon(t) \xrightarrow{S} \gamma(t)$ and provided $\gamma(x,t) \in C^3$ in space-time, then γ is a solution to (3.9) in the classical sense. Moreover, T_* can be chosen as the minimum of the self-collision time and the exit time from Ω under the evolution (3.9). We thus encounter the same limitation as in the case of collisions or exit for Ginzburg-Landau vortices.
The method of this proof was extended by N. Le in [Le2] to treat the case of Ohta-Kawasaki equation, a nonlocal variant of Cahn-Hilliard (related to the modelling of "diblock copolymers"), for which it is noteworthy that no maximum principle holds. The results obtained there are new.

Of course, the study described above leaves open many questions:

1. Is the conjecture (3.17) true in all dimensions without assumptions?

2. We have worked in a formal Hilbert manifold setting, however it would be better to work on the associated metric space where the distance is the formal geodesic distance for $H_n^{-1/2}(\gamma)$. Is it possible to find a suitable definition for this metric space that does not require regularity of γ and that bypasses the geodesic definition? Is it possible to find a suitable weak notion of solutions coinciding with curves of maximal slope on that metric space so that something like Theorem 2 could be applied, without having to assume any regularity on γ? In particular does an upper bound on the left-hand side of (3.17) allow to obtain the sufficient regularity on γ?

3.3 Allen-Cahn

It is interesting to look into the case of the Allen-Cahn equation in light of the other cases we have examined: Ginzburg-Landau and Cahn-Hilliard, and in the light of the closely related recent progress in [MR1] on the Allen-Cahn action functional. We use the notation of Section 3.2. The Allen-Cahn equation is the PDE

$$\partial_t u = \Delta u - \frac{1}{\varepsilon^2} f(u) \tag{3.19}$$

where u is real valued.

Just like Cahn-Hilliard, it is a gradient-flow of the (real-valued) Allen-Cahn energy (3.10), but for an L^2 structure (like Ginzburg-Landau). The structure X_ε to consider is easily seen to be given by $\|\cdot\|^2_{X_\varepsilon} = \varepsilon\|\cdot\|^2_{L^2(\Omega)}$ structure. Indeed,

$$\nabla_{X_\varepsilon} E_\varepsilon(u) = -\Delta u + \frac{1}{\varepsilon^2} f(u).$$

It is a result due to several authors [DMS, BK, Ch, ESS, I] that the interfaces (between the values ± 1) of solutions to the Allen-Cahn equation converge to solutions of mean curvature motion (in the sense of Brakke), formally

$$\partial_t \gamma = H \tag{3.20}$$

where H is the mean curvature vector of γ. Note a very recent paper [Sa] gives a very nice and short proof of this convergence result, building on the result of [RS]. The result of [MR1] on the Γ-convergence of the Allen-Cahn action functional (see item 7 of the remarks of Section 1) also yields as a byproduct this convergence result. [MR2] also extends the study to the convergence of the Allen-Cahn equation with forcing right-hand side.

One may check that (3.20) is the formal gradient flow of the perimeter functional $F(\gamma) = 2\sigma\mathcal{H}^{N-1}(\gamma)$ (itself Γ-limit of E_ε in the same sense as in Section 3.2), with respect to structure $\|\cdot\|_{Y_\gamma}^2 = 2\sigma\|\cdot\|_{L^2_\gamma}^2$. This is more precisely defined as follows: tangent vectors to a hypersurface γ are given by vector fields X defined on γ and one sets (if γ is regular enough)

$$\|X\|_{Y_\gamma}^2 = 2\sigma \int_\gamma |X|^2.$$

The first variation of the area of γ is equal to its mean curvature vector (oriented outwards), this is equivalent to saying that the differential of $\mathcal{H}^{N-1}(\gamma)$ along the direction X is

$$-\int_\gamma X \cdot H,$$

where H is oriented inwards. So we may indeed write (formally)

$$\nabla_{Y_\gamma} F(\gamma) = -H.$$

In the scheme of Γ-convergence of gradient flows, the two extra conditions are in this setting: if $u_\varepsilon(t) \stackrel{S}{\rightharpoonup} \gamma(t)$,

$$\liminf_{\varepsilon\to 0} \int_0^s \varepsilon\|\partial_t u_\varepsilon\|_{L^2(\Omega)}^2(t)\,dt \geq \int_0^s \|\partial_t\gamma\|_{Y_\gamma}^2(t)\,dt = 2\sigma\int_0^s \int_{\gamma(t)} |\partial_t\gamma|^2\,dt. \tag{3.21}$$

and if $u_\varepsilon \stackrel{S}{\rightharpoonup} \gamma$

$$\liminf_{\varepsilon\to 0} \int_\Omega \varepsilon \left|\Delta u_\varepsilon - \frac{1}{\varepsilon^2} f(u_\varepsilon)\right|^2 \geq 2\sigma \int_\gamma |H|^2. \tag{3.22}$$

It turns out that both these relations are known to be true from [RS, MR1], once phrased in a suitable setting. As we mentioned earlier (3.22) was obtained in [RS] as part of the proof of the De Giorgi conjecture of Γ-convergence of the left-hand side of (3.22) to the Wilmore functional. (3.21) is proved in [MR1]. The problem with the way they are phrased is that first of all the interface γ need not be smooth, so one should use the theory of varifolds and view its mean curvature H in the generalized sense; second of all γ may have foldings and "hidden boundaries" corresponding to higher multiplicity of the associated varifold. Because of that we should resort to the definition of L^2 flows introduced in [MR1], which allows to prove (3.22) even with multiplicity.

Let us now present the framework of [MR1]:

Definition 6. *Let μ^t be any family of integer rectifiable Radon measures such that $\mu := \mathcal{L}^1 \otimes \mu^t$ defines a Radon measure on $[0,T] \times \Omega$ and such that μ^t has a weak mean curvature $H(t,\cdot) \in L^2(\mu^t)$ for almost all $t \in (0,T)$. If there exists a positive constant C and a vector field $v \in L^2(\mu, \mathbb{R}^N)$ such that $v(t,x) \perp T_x\mu^t$ for μ-almost all $(t,x) \in [0,T] \times \Omega$ and*

$$\left| \int_0^T \int_\Omega (\partial_t \eta + \nabla\eta \cdot v) d\mu^t \, dt \right| \leq C \|\eta\|_{C^0}$$

for all $\eta \in C_c^1((0,T) \times \bar{\Omega})$, then μ^t is called an L^2 flow and v is called a generalized velocity vector.

They also show that any generalized velocity is (in a set of good points) uniquely determined by the evolution μ^t. With this definition the definition of mean curvature flow becomes

$$v = H$$

where v is a generalized velocity vector and H a generalized mean curvature vector, and it is similar to Brakke's formulation.

Note that in view of the proof of Theorem 1 it suffices to prove rather that along families of solutions we have

$$\liminf_{\varepsilon \to 0} \left(\int_0^s \varepsilon \|\partial_t u_\varepsilon\|^2_{L^2(\Omega)}(t) + \int_\Omega \varepsilon \left| \Delta u_\varepsilon(t) - \frac{1}{\varepsilon^2} f(u_\varepsilon(t)) \right|^2 dt \right) \geq 2\sigma \int_0^s \int_{\gamma(t)} |\partial_t \gamma|^2 + 2\sigma \int_{\gamma(t)} |H|^2.$$

So we may assume, without loss of generality, that first $E_\varepsilon(u_\varepsilon(t)) \leq C$ and second

$$\limsup_{\varepsilon \to 0} \int_0^s \left(\varepsilon \|\partial_t u_\varepsilon\|^2_{L^2(\Omega)}(t) + \int_\Omega \varepsilon \left| \Delta u_\varepsilon(t) - \frac{1}{\varepsilon^2} f(u_\varepsilon(t)) \right|^2 \right) dt < \infty. \tag{3.23}$$

Let us then set for such u_ε,

$$\mu_\varepsilon^t = \frac{1}{2\sigma} \left(\frac{1}{2} |\nabla u_\varepsilon(t)|^2 + \frac{1}{\varepsilon} W(u_\varepsilon(t)) \right)$$

which is for each $t \in (0,T)$ a Radon measure on $\bar{\Omega}$ (the energy measure). Then for a subsequence, as $\varepsilon \to 0$, $\mu_\varepsilon^t \to \mu^t$ for almost all t, where from [RS, HT], μ^t are integer rectifiable and have a (generalized) mean curvature. It is a main theorem of [MR1] that under (3.23) and $E_\varepsilon(u_\varepsilon(t)) \leq C$, μ^t admit a generalized velocity $v \in L^2(\mu, \mathbb{R}^N)$ and that

$$\liminf_{\varepsilon \to 0} \int_0^s \int_\Omega \varepsilon |\partial_t u_\varepsilon|^2 \, dt \geq 2\sigma \int_{[0,s] \times \Omega} |v|^2 \, d\mu. \tag{3.24}$$

This is the rigorous version of (3.21).

The rigorous version of (3.22) is the main theorem of [RS] which states that in dimension $N \le 3$, if $E_\varepsilon(u_\varepsilon) \le C$ and $\int_\Omega \varepsilon \left|\Delta u_\varepsilon - \frac{1}{\varepsilon^2} f(u_\varepsilon)\right|^2 \le C$ then defining μ_ε as above, μ_ε converges up to extraction to μ which is an integer multiplicity rectifiable varifold, with a generalized mean curvature vector H and

$$\liminf_{\varepsilon\to 0} \int_\Omega \varepsilon \left|\Delta u_\varepsilon - \frac{1}{\varepsilon^2} f(u_\varepsilon)\right|^2 \ge 2\sigma \int_\Omega |H|^2 \, d\mu. \tag{3.25}$$

Finally, in order to deduce in dimension ≤ 3 the convergence of Allen-Cahn to the Brakke flow (as defined in [MR1]) from the scheme, it suffices to show (using the definition of the velocity and mean curvature) that

$$\int_0^s \int_\Omega H \cdot v \, d\mu = \int_\Omega d\mu^0 - \int_\Omega d\mu^s,$$

which is true for regular flows. Note that here $\int_\Omega d\mu$ replaces the perimeter functional.

To finish this subsection, we point to an interesting negative result for a problem also related to mean curvature flow. In [NV] it is shown that the functionals defined over sets of finite perimeter:

$$F_\varepsilon(E) = Per(E) + \frac{1}{\varepsilon} \int_E U(\frac{x}{\varepsilon}) \, dx$$

where U is a given function, Γ-converge as $\varepsilon \to 0$ to an anisotropic perimeter functional (with anisotropy depending on U), while the solutions to the gradient flows of F_ε *do not* converge to the solution of limiting (anisotropic curvature) flow. This demonstrates again the need for extra conditions to ensure the convergence of gradient flows, and the possibility of failure of these conditions.

3.4 Ginzburg-Landau with large number of vortices

In this section we move to purely formal considerations and conjectures.

When the assumption $F_\varepsilon(u) \le C|\log \varepsilon|$ is dropped in the Ginzburg-Landau energy with or without magnetic field, the number of vortices can blow up as $\varepsilon \to 0$. This is for example the case for minimizers of the energy with magnetic field, with an external field of strength $|\log \varepsilon|$ (see [SS4]).

In this situation, configurations are best described through their limiting vortex density μ, limit of the measures

$$\frac{2\pi \sum_i d_i \delta_{a_i}}{\sum_i |d_i|},$$

where a_i's are the vortex locations and d_i their degrees. An equivalent (and more rigorous) way of expressing this is to consider the sense of convergence $(u_\varepsilon, A_\varepsilon) \overset{S}{\rightharpoonup} \mu$ if

$$\frac{1}{h_{\text{ex}}} \left(\operatorname{curl} \langle iu_\varepsilon, \nabla_{A_\varepsilon} u_\varepsilon\rangle + \operatorname{curl} A_\varepsilon\right) \rightharpoonup \mu$$

in the weak sense of measures, where ∇_A denotes $\nabla - iA$ (see [SS4], Chapter 6). The (full) Ginzburg-Landau energy with magnetic field is

$$G_\varepsilon(u, A) = \frac{1}{2}\int_\Omega |\nabla_A u|^2 + |\mathrm{curl}\, A - h_{\mathrm{ex}}|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2. \tag{3.26}$$

Here $A : \Omega \to \mathbb{R}^2$ is the magnetic potential and h_{ex} is the intensity of the applied magnetic field, assumed here for simplicity to be equal to $\lambda|\log \varepsilon|$, where λ is a fixed constant.

The Γ-convergence of this full Ginzburg-Landau energy as $\varepsilon \to 0$ was established in [SS5] (see also [SS4]). The result is that in this situation $G_\varepsilon/h_{\mathrm{ex}}^2$ Γ-converges (for the sense $(u_\varepsilon, A_\varepsilon) \stackrel{S}{\rightharpoonup} \mu$ above) to the functional

$$F_\lambda(\mu) = \frac{1}{2\lambda}\int_\Omega |\mu| + \frac{1}{2}\int_\Omega |\nabla h_\mu|^2 + |h_\mu - 1|^2 \tag{3.27}$$

where h_μ is deduced from μ through

$$\begin{cases} -\Delta h_\mu + h_\mu = \mu & \text{in } \Omega \\ h_\mu = 1 & \text{on } \partial\Omega. \end{cases} \tag{3.28}$$

The purely dissipative evolution of the vortices is given by solving the time-dependent parabolic Ginzburg-Landau, as proposed by Gorkov-Eliashberg

$$\begin{cases} \partial_t u + iu\Phi = \nabla_A^2 u + \frac{u}{\varepsilon^2}(1-|u|^2) & \text{in } \Omega \\ \partial_t A + \nabla\Phi = \nabla^\perp h + \langle iu, \nabla_A u\rangle & \text{in } \Omega \\ \langle iu, \nabla_A u\rangle \cdot \nu = 0 & \text{on } \partial\Omega \\ h = h_{\mathrm{ex}} & \text{on } \partial\Omega \end{cases} \tag{3.29}$$

We will not go into details of this equation, suffice it to say that this is the version of (3.2) with magnetic field and gauge-invariance, and that it is the gradient flow of (3.26) with respect to the $L^2(\Omega) \times L^2(\Omega)$ structure.

Chapman-Rubinstein-Schatzman formally established in [CRS] that in the $\varepsilon \to 0$ limit, the evolution of the limiting $\mu(t)$ should be given by the equation

$$\partial_t \mu - \mathrm{div}\ (\nabla h_\mu |\mu|) = 0. \tag{3.30}$$

With L. Ambrosio we established in [AS] that, restricted to the space of positive measures μ, (3.30) is the gradient flow of (3.27) for the 2-Wasserstein structure on positive (without loss of generality probability) measures, and we established some existence and uniqueness results using that structure, following the method of [AGS]. In [AMS] we extend this approach to the case of general signed measures (but with partial results only). For the definition of the Wasserstein metric, the reader can refer to [AGS, Vi]. Deriving rigorously (3.30) as the $\varepsilon \to 0$ limit of (3.29) is still a challenging open problem. Let us see how this question would fit in our Γ-convergence of gradient flows framework.

First of all, the Γ-convergence result we have is for $E_\varepsilon := G_\varepsilon/h_{\rm ex}^2$. We saw that (3.29) is the $L^2\times L^2$ gradient flow for G_ε, so it is the gradient flow for F_ε for the X_ε structure given by

$$\|(\cdot,\cdot)\|^2_{X_\varepsilon} = \frac{1}{h_{\rm ex}^2}\|(\cdot,\cdot)\|^2_{L^2(\Omega)\times L^2(\Omega)}.$$

The formal Hilbert manifold structure associated to the Wasserstein metric is the L^2_μ scalar product at each point μ. It is established in [AGS] that $\mu(t)$ is an absolutely continuous curve on the space of probabilities if and only if it solves a continuity equation

$$\partial_t\mu + \mathrm{div}\ (v_t\mu(t)) = 0$$

with $v_t\in L^2_{\mu(t)}$. The metric derivative (with respect to the Wasserstein distance) in the sense of Section 2 is $|\mu'|(t) = \|v_t\|_{L^2_{\mu(t)}}$. (Note the similarity with L^2 flows in Section 3.3.) On the other hand, we know from [SS3] that the $\mu(t)$ obtained as limit of $(u_\varepsilon(t), A_\varepsilon(t))$ solution to (3.29) (or even not necessarily solution) solves a continuity equation

$$\partial_t\mu + \mathrm{div}\ V = 0$$

where V is the "Jacobian velocity" associated to μ. The condition (1.3) in the scheme is thus (formally) in this setting

$$\liminf_{\varepsilon\to 0}\frac{1}{h_{\rm ex}^2}\int_0^s |\partial_t u_\varepsilon|^2 + |\partial_t A_\varepsilon|^2\, dt \ge \int_0^s |\mu'|^2(t)\, dt = \int_0^s \left\|\frac{V}{\mu}\right\|^2_{L^2_\mu} dt. \tag{3.31}$$

To prove that this holds along solutions of (3.29), we can use try to use the product-estimate established in [SS3] Theorem 3, which states that for any $X\in C^0_c([0,s]\times\Omega,\mathbb{R}^2)$, we have (without magnetic field, to simplify matters)

$$\liminf_{\varepsilon\to 0}\frac{1}{h_{\rm ex}^2|\log\,\varepsilon|^2}\left(\int_{[0,s]\times\Omega}|\partial_t u_\varepsilon|^2\int_{[0,s]\times\Omega}\int_\Omega |X\cdot\nabla u_\varepsilon|^2\right)\ge\left|\int_{[0,s]\times\Omega} V\cdot X^\perp\right|^2 \tag{3.32}$$

On the other hand we expect that

$$\frac{1}{h_{\rm ex}|\log\,\varepsilon|}\int_\Omega |X\cdot\nabla u_\varepsilon|^2 \sim \frac12\frac{1}{h_{\rm ex}|\log\,\varepsilon|}\int_\Omega |X|^2|\nabla u_\varepsilon|^2\sim\frac12\int_\Omega|X|^2\,d\mu. \tag{3.33}$$

At this point we see that the scaling in (3.31) is not right (unless $2|\log\,\varepsilon| = h_{\rm ex}$ i.e. $\lambda=2$). In order to fix this we need to rescale the original equation (3.29) in time by a factor of $\lambda/2$ before we can get convergence to the desired limiting flow (this is an illustration of how the scheme indicates which time-scaling to use). So instead of (3.29) we consider the same equation with left-hand sides $\frac{\lambda}{2}\partial_t u$ and $\frac{\lambda}{2}\partial_t A$. Then it is the gradient flow for E_ε for (the right) X_ε which is $\frac{1}{2h_{\rm ex}|\log\,\varepsilon|}\|(\cdot,\cdot)\|^2_{L^2\times L^2}$, and we can hope that (3.31) is true: it would follow from (3.32) by duality, just like in Section 3.3.

The relation (3.33) should hold provided there is no excess energy. Indeed, first of all $\int|\nabla u_\varepsilon|^2$ should not really be understood as the energy over the whole domain but only as the concentration near the vortices (the defect measure for the L^2 convergence of ∇u_ε as in [SS3]). Then the first equality in (3.33) is expected to hold if there is no excess energy, as in Corollary 4 of [SS3]. This corresponds to the fact that then the vortex core energy is isotropic (because the optimal case is that of radial vortices), then $\int |\nabla u \cdot e|^2 = \int |\nabla u \cdot e^\perp|^2 = \frac{1}{2}\int |\nabla u|^2$ along any unit direction e. This is the equivalent of the "equipartition of energy" for Allen-Cahn. The second equality in (3.33) corresponds to establishing that $\frac{1}{2h_{\mathrm{ex}}|\log\,\varepsilon|}|\nabla u_\varepsilon|^2 \rightharpoonup \frac{1}{2}\mu$ in the sense of measures. This is the same as saying that the energy measure (the limit of the defect measure of $\frac{1}{2h_{\mathrm{ex}}|\log\,\varepsilon|}|\nabla u_\varepsilon|^2$) coincides with the topological measure, which is the same as saying there is no excess energy.

The second relation (1.4) in the scheme is here (formally): if $(u_\varepsilon, A_\varepsilon) \overset{S}{\rightharpoonup} \mu$ then

$$\frac{1}{2h_{\mathrm{ex}}|\log\,\varepsilon|}\int_\Omega |\Delta u + \frac{u}{\varepsilon^2}(1-|u|^2)|^2 + |\nabla^\perp h + \langle iu, \nabla_A u\rangle|^2 \geq \int_\Omega |\nabla h_\mu|^2\,d\mu. \tag{3.34}$$

Indeed the formal gradient of (3.27) with respect to the Wasserstein metric is ∇h_μ, which should be measured in L^2_μ. But the right-hand side of (3.34) does not make sense if μ is not a regular measure. In [AS], we show how this can be replaced by a slope that allows to give a weaker meaning to $\int_\Omega |\nabla h_\mu|^2\,d\mu$. The relation (3.34) can potentially be proved following the method of [SS2]: this works formally, provided the same conditions as above hold, i.e. the energy measure and the topological measure can be identified.

This example thus has a lot of ressemblance with those examined above. In particular the issues that one faces to establish the sufficient conditions (1.3)-(1.4) are similar questions of equipartition of energy, energy-excess, and identification of the energetic and topological measures. In the case of Allen-Cahn and Cahn-Hilliard this is the question of multiplicity of interfaces, and in the case of Ginzburg-Landau this is the question of collisions or exit of vortices, which can happen instantaneously (in the case of large number of vortices) or not. One gets into a difficult loop: the Γ-convergence of gradient flows will establish that no energy-excess appears, but to prove it holds, we need to control precisely the energy-excess.

It shouldn't be a surprise to encounter these issues, which are usual in the study of these problems. In some cases one can succeed in getting around them or solving them, as done for example in [I, BOS1, MR1, WT, SS2].

Note that this scheme of Γ-convergence of gradient flows was initially introduced to understand via a general underlying principle "why" solutions to gradient flows converge to their limiting counterpart. All in all, even when it cannot be applied rigorously, this scheme provides an indication of inequalities (the conditions (1.3)–(1.4)) which should hold and thus opens the way to many potential conjectures. In all the examples we have examined (Ginzburg-Landau with finite number of vortices, Allen-Cahn, Cahn-Hilliard), it turns out that these conjectures have already been proved or seem to be true. Finally, giving rigorous applications of this scheme, in particular applying it in the metric space setting, and finding other examples of interest, also potentially leads to many open questions.

References

[ABC] N. Alikakos, P. Bates, X. Chen, Convergence of the Cahn-Hilliard equation to the Hele-Shaw model. *Arch. Rational Mech. Anal. 128* (1994), no. 2, 165-205.

[Am] L. Ambrosio, Minimizing movements, *Rend. Accad. Naz. Sci, XL Mem. Mat. Appl. (5)* (1995) 191-246.

[AGS] L. Ambrosio, N. Gigli, G. Savaré, *Gradient flows in metric spaces and in the space of probability measures.* Lectures in Mathematics ETH Zürich. Birkhäuser, 2008.

[AMS] L. Ambrosio, E. Mainini, S. Serfaty, Gradient flow of the Chapman-Rubinstein-Schatzman model for signed vortices, preprint.

[AS] L. Ambrosio, S. Serfaty, A gradient flow approach to an evolution problem arising in superconductivity. *Comm. Pure Appl. Math. 61* (2008), no. 11, 1495–1539.

[BBH] F. Bethuel, H. Brezis, F. Hélein, *Ginzburg-Landau vortices*, Birkhäuser, 1994.

[BOS1] F. Bethuel, G. Orlandi, D. Smets, Convergence of the parabolic Ginzburg-Landau equation to motion by mean curvature. *Ann. of Math. (2) 163* (2006), no. 1, 37–163.

[BOS2] F. Bethuel, G. Orlandi, D. Smets, Collisions and phase-vortex interactions in dissipative Ginzburg-Landau dynamics. *Duke Math. J. 130* (2005), no. 3, 523–614.

[BOS3] F. Bethuel, G. Orlandi, D. Smets, Quantization and motion law for Ginzburg-Landau vortices. *Arch. Ration. Mech. Anal. 183* (2007), no. 2, 315–370.

[BOS4] F. Bethuel, G. Orlandi, D. Smets, Dynamics of multiple degree Ginzburg-Landau vortices. *Comm. Math. Phys. 272* (2007), no. 1, 229–261.

[BK] L. Bronsard, R. V. Kohn, Motion by mean curvature as the singular limit of Ginzburg-Landau dynamics, *J. Differential Equations 90* (1991), 211–237.

[CRS] S. J. Chapman, J. Rubinstein, M. Schatzman, A mean-field model of superconducting vortices. *European J. Appl. Math. 7* (1996), no. 2, 97–111.

[Ch] X. Chen, Generation and propagation of interfaces for reaction-diffusion equations,*J. Differential Equations 96* (1992), 116–141.

[DeG1] E. De Giorgi, New problems in Γ-convergence and G-convergence. *Free Boundary Problems,* Proc. of seminar held in Pavia. Pivia, September-October 1979, Rome, Francesco Severi, 1980, Vol. II 183-194.

[DeG2] E. De Giorgi, New problems on minimizing movements. Boundary value problems for partial differential equations and applications, 81–98, RMA Res. Notes Appl. Math., 29, Masson, Paris, 1993.

[DeGMT] E. De Giorgi, A. Marino, M. Tosques, Problems of evolution in metric spaces and maximal decreasing curve, *Att Accad Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8), 68*, (1980), 180-187.

[DMT] M. Degiovanni, A. Marino, M. Tosques, Evolution equations with lack of convexity, *Nonlinear Anal., 9* (1985) 1401-1443.

[DMS] P. de Mottoni, M. Schatzman, Development of interfaces in R^N. *Proc. Roy. Soc. Edinburgh Sect. A 116* (1990), no. 3-4, 207–220.

[ERV] W. E, W. Ren, E. Vanden-Eijnden, Minimum action method for the study of rare events. *Comm. Pure Appl. Math. 57* (2004), no. 5, 637–656.

[ESS] L. C. Evans, H. M. Soner, P. E. Souganidis, Phase transitions and generalized motion by mean curvature. *Comm. Pure Appl. Math. 45* (1992), no. 9, 1097–1123

[HT] J. Hutchinson, Y. Tonegawa, Convergence of phase interfaces in the van der Waals-Cahn-Hilliard theory. *Calc. Var. Partial Differential Equations 10* (2000), no. 1, 49–84.

[I] T. Ilmanen, Convergence of the Allen-Cahn equation to Brakke's motion by mean curvature, *J. Differential Geom, 38* (1993), 417–461.

[Je] R. L. Jerrard, Vortex dynamics for the Ginzburg-Landau wave equation, *Calc. Var. Partial Differential Equations 9* (1999), no. 1, 1–30.

[JSo] R. L. Jerrard, H. M. Soner, Dynamics of Ginzburg-Landau vortices. *Arch. Rational Mech. Anal. 142* (1998), no. 2, 99–125.

[JSt] R. L. Jerrard, P. Sternberg, Critical points via Gamma-convergence: general theory and applications, *Jour. Eur. Math. Soc.*, **11**, (2009), no. 4, 705-753.

[Ji] H. Jian, A relation between Γ-convergence of functionals and their associated gradient flows. *Sci. China Ser. A 42* (1999), no. 2, 133–139.

[KORV] R. V. Kohn, F. Otto, M. G. Reznikoff, E. Vanden-Eijnden, Action minimization and sharp-interface limits for the stochastic Allen-Cahn equation, *Comm. Pure Appl. Math. 60* (2007), no. 3, 393–438.

[KRT] R. V. Kohn, M. G. Reznikoff, Y. Tonegawa, Sharp-interface limit of the Allen-Cahn action functional in one space dimension. *Calc. Var. Partial Differential Equations 25* (2006), no. 4, 503–534.

[Ku] M. Kurzke, The gradient flow motion of boundary vortices. *Ann. Inst. H. Poincaré Anal. Non Linéaire 24* (2007), no. 1, 91–112.

[Le1] N. Le, A gamma-convergence approach to the Cahn-Hilliard equation. *Calc. Var. Partial Differential Equations 32* (2008), no. 4, 499–522.

[Le2] N. Le, On the convergence of the Ohta-Kawasaki Equation to Motion by Nonlocal Mullins-Sekerka Law, to appear in *SIAM. J. Math. Analysis.*

[Li] F. H. Lin, Some dynamical properties of Ginzburg-Landau vortices,*Comm. Pure Appl. Math. 49,* 323–359, 1996.

[MST] A. Marino, C. Saccon, M. Tosques, Curves of maximal slope and parabolic variational inequalities on nonconvex constraints, *Ann Scuola Norm. Sup. Pisa Cl. Sci. (4)*, *16* (1989), 281-330.

[Mi] A. Mielke, Weak convergence methods for Hamiltonian multiscale problems. *DCDS(A) Discrete Contin. Dyn. Syst. Ser. A, 20*, (2008), no. 1, 53-79.

[MRS] A. Mielke, T. Roubicek, U. Stefanelli. Γ-limits and relaxations for rate-independent evolutionary problems. *Calc. Var. PDE*, *31*, (2008), 387-416.

[MM] L. Modica, S. Mortola, Il limite nella Γ-convergenza di una famiglia di funzionali ellittici. *Boll. Un. Mat. Ital. A (5)* 14 (1977), no. 3, 526–529.

[MR1] L. Mugnai, M. Röger, The Allen-Cahn action functional in higher dimensions. *Interfaces Free Bound. 10* (2008), no. 1, 45–78.

[MR2] L. Mugnai, M. Röger, Convergence of the perturbed Allen-Cahn equations to forced mean curvature flow, preprint.

[NV] M. Novaga, E. Valdinocci, Closed curves of prescribed curvature and a pinning effect, to appear in *Networks Heter. Media.*

[Or1] C. Ortner, Two Variational Techniques for the Approximation of Curves of Maximal Slope, Technical report NA05/10, Oxford University Computing Laboratory, Oxford, UK, 2005.

[Or2] C. Ortner, Gradient flows as a selection procedure for equilibria of nonconvex energies. *SIAM J. Math. Anal. 38* (2006), no. 4, 1214–1234

[Pe] R. L. Pego, Front migration in the nonlinear Cahn-Hilliard equation. Proc. Roy. Soc. London Ser. A 422 (1989), no. 1863, 261-278.

[RS] M. Röger, R. Schätzle, On a modified conjecture of De Giorgi. *Math. Z. 254* (2006), no. 4, 675–714.

[SS1] E. Sandier, S. Serfaty, Gamma-convergence of gradient flows with applications to Ginzburg-Landau, *Comm. Pure Appl. Math, 57*, No 12, (2004), 1627-1672.

[SS2] E. Sandier, S. Serfaty, Limiting Vorticities for the Ginzburg-Landau Equations, *Duke Math J., 117*, No 3, (2003), 403-446.

[SS3] E. Sandier S. Serfaty, A product-estimate for Ginzburg-Landau and corollaries, *J. Func. Anal., 211,* No 1, (2004), 219-244.

[SS4] E. Sandier, S. Serfaty, *Vortices in the Magnetic Ginzburg-Landau Model*, Progress in Nonlinear Differential Equations and their Applications, vol 70, Birkhauser, 2007.

[SS5] E. Sandier, S. Serfaty, A Rigorous Derivation of a Free-Boundary Problem Arising in Superconductivity, *Annales Scientifiques de l'ENS, 4e Ser, 33,* (2000), 561-592.

[Sa] N. Sato, A Simple Proof of Convergence of the Allen-Cahn Equation to Brakke's Motion by Mean Curvature, *Indiana Univ. Math. J, 57*, (2008), No 4, 1743-1751.

[Sc] R. Schätzle, private communication.

[Se1] S. Serfaty, Stability in 2D Ginzburg-Landau passes to the limit, *Indiana Univ. Math. J.*, 54, No. 1, (2005), 199-222

[Se2] S. Serfaty, Vortex collisions and energy-dissipation rates in the Ginzburg-Landau heat flow, part I: Study of the perturbed Ginzburg-Landau equation, *Journal Eur. Math Society , 9*, No 2, (2007), 177-217. part II: The dynamics, *Journal Eur. Math Society, 9, No 3,* (2007), 383-426.

[Vi] C. Villani, *Optimal transport. Old and new*, Springer, 2009.

[WT] M. G. Westdickenberg, Y. Tonegawa, Higher multiplicity in the one-dimensional Allen-Cahn action functional. *Indiana Univ. Math. J. 56* (2007), no. 6, 2935–2989.

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