

A product-estimate for Ginzburg-Landau and corollaries

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Abstract

We prove a new inequality for the Jacobian (or vorticity) associated to the Ginzburg-Landau energy in any dimension. It allows to retrieve existing lower bounds on the energy, to extend them to the case of unbounded vorticity, and to get a few other corollaries. It also provides a new estimate on the time-variation for time-dependent families, which has applications for the study of Ginzburg-Landau dynamics.

I Main result

We are interested in proving lower bounds on the Ginzburg-Landau energy in any dimension:

$$(I.1) \quad E_\varepsilon(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 + \frac{1}{2\varepsilon^2} (1 - |u|^2)^2,$$

where Ω is a smooth bounded domain of \mathbb{R}^n , with $n \geq 2$, and u is complex-valued. This energy is a simple version (without magnetic field) of the Ginzburg-Landau energy of superconductivity. It also appears in other models from physics, for superfluidity, nonlinear

optics, Bose-Einstein condensates, and the complex-valued function u , called order parameter, plays the role of a pseudo wave-function. The zero-set of u is a crucial object. Indeed, since u is complex-valued, it can have a nonzero degree around its zeroes, they are then called *topological defects*, typically of codimension 2. In dimension $n = 2$, one thus expects point defects, called *vortices*, in dimension $n = 3$, *line vortices*. These codimension 2 sets can be clearly extracted at the limit $\varepsilon \rightarrow 0$, and lower bounds on the Ginzburg-Landau energy serve to relate the energy to the topology of these defects, or to the *vorticity* (understood as in fluid mechanics). The first result bounding below the Ginzburg-Landau energy by the degrees of the vortices was obtained by Bethuel-Brezis-Hélein in [BBH] for the $n = 2$ case and a bounded number of vortices, and also in [BMR]. Then, the works of Sandier [Sa1] and Jerrard [J1], allowed to generalize these lower bounds to possibly unbounded numbers of vortices, thanks to a suitable growing-ball procedure. Then, lower bounds in dimensions 3 and higher were addressed in [Ri, LR, Sa2, JS1, BBO, ABO].

In this paper, we present an optimal (or sharp) lower bound, with a rather simple proof. It is, to our knowledge, the first product-type lower bound on Ginzburg-Landau, a slight improvement of the existing lower bounds (which it contains), but which allows to get some new results as well. Our initial motivation was to obtain optimal estimates and additional regularity for time-dependent Ginzburg-Landau (see Theorem 3 in Section III), for which the product nature of the estimate turns out to be crucial; but our result encompasses both the static and dynamic cases. We use those estimates in a forthcoming work on Ginzburg-Landau dynamics [SS5].

The proof, presented in Section IV, relies on the same ingredients as the other proofs of lower bounds, i.e. on the ball construction method of [Sa1, J1], but the main new idea is to use a deformation of the metric, and thus a construction with *growing ellipses* instead of balls. Ellipses allow the freedom necessary to “separate” the directions. (Observe also that the trace of a radial line-vortex on a plane which is not perpendicular to its axis is an “elliptic vortex”.)

Following [JS1], for any sufficiently regular complex-valued u , the current of u is defined as the 1-form

$$(I.2) \quad ju = (iu, du) = \sum_{k=1}^n (iu, \partial_k u) dx_k,$$

where (\cdot, \cdot) denotes the scalar product in \mathbb{C} identified with \mathbb{R}^2 i.e $(a, b) = \frac{\bar{a}b + a\bar{b}}{2}$. It is related to the Jacobian determinants Ju of u through

$$(I.3) \quad Ju = \frac{1}{2}d(ju) = \frac{1}{2}d(iu, du),$$

where

$$Ju = \sum_{j < k} (i\partial_j u, \partial_k u) dx_j \wedge dx_k.$$

Thus Ju acts on couples of vectors fields $(X, Y) \in (\mathbb{R}^n)^2$ with the standard rule that $dx_i \wedge dx_j(X, Y) = \frac{1}{2}(X_i Y_j - Y_i X_j)$. It can also be seen as an $(n - 2)$ -dimensional current acting on $(n - 2)$ -forms by the relation

$$Ju(\phi) = \frac{1}{2} \int_{\Omega} Ju \wedge \phi dx.$$

The Jacobian carries the topological information on the zero-set of u , or the vorticity. $|J|$ will denote the total variation of the current, $\|\cdot\|$ the total mass of a measure, and measure-valued 2-forms means forms whose coefficients are in the space of bounded Radon measures on Ω .

In all the paper, $M(\varepsilon)$ will be any function of ε satisfying

$$(I.4) \quad \forall \alpha > 0, \quad \lim_{\varepsilon \rightarrow 0} \varepsilon^\alpha M(\varepsilon) = 0, \quad \lim_{\varepsilon \rightarrow 0} \frac{|\log \varepsilon|}{M(\varepsilon)^\alpha} = 0, \quad \text{and} \quad \log M(\varepsilon) = o(|\log \varepsilon|) \text{ as } \varepsilon \rightarrow 0.$$

For example $M(\varepsilon) = \exp(\sqrt{|\log \varepsilon|})$ satisfies this.

Our main result is the following.

Theorem 1. *Let u_ε be a family of $H^1(\Omega, \mathbb{C})$ such that*

$$(I.5) \quad E_\varepsilon(u_\varepsilon) \leq N_\varepsilon |\log \varepsilon| \ll M(\varepsilon),$$

with $M(\varepsilon)$ as in (I.4). Then, up to extraction

$$\frac{Ju_\varepsilon}{N_\varepsilon} \rightharpoonup J \quad \text{in } (C_c^{0,\gamma}(\Omega))', \forall \gamma > 0,$$

where J is a measure-valued 2-form. If N_ε is bounded independently of ε then the limit of $\frac{1}{\pi} Ju_\varepsilon$ is in addition a rectifiable integer-multiplicity current. Moreover, for all continuous vector-fields X and Y compactly supported in Ω ,

$$\frac{|X \cdot \nabla u_\varepsilon|}{\sqrt{N_\varepsilon |\log \varepsilon|}}, \quad \frac{|Y \cdot \nabla u_\varepsilon|}{\sqrt{N_\varepsilon |\log \varepsilon|}}$$

are bounded in L^2 and if we let ν_X, ν_Y be their defect measures, we have

$$(I.6) \quad \|\nu_X\|^{\frac{1}{2}} \|\nu_Y\|^{\frac{1}{2}} \geq \left| \int_{\Omega} J(X, Y) \right|.$$

Corollary 1. *Under the same hypotheses, we deduce*

$$(I.7) \quad \liminf_{\varepsilon \rightarrow 0} \frac{1}{N_\varepsilon |\log \varepsilon|} \left(\int_{\Omega} |X \cdot \nabla u_\varepsilon|^2 \int_{\Omega} |Y \cdot \nabla u_\varepsilon|^2 \right)^{\frac{1}{2}} \geq \left| \int_{\Omega} J(X, Y) \right|.$$

Remark 1. The compactness of Ju_ε was proved in [JS1], together with the rectifiability of the limit, in the case of N_ε independent of ε . It has also been proved lately in [ABO]. The compactness was proved in case $n = 2$ in a weaker form in [SS3], [ASS]. We include a proof in the general case close to that of [ASS], for the convenience of the reader.

Remark 2. We have considered $N_\varepsilon \ll \frac{M(\varepsilon)}{|\log \varepsilon|}$ for the sake of generality, but the result is most interesting for $N_\varepsilon \leq C|\log \varepsilon|$. Indeed, for larger order of N_ε , the relevant order of energy to consider in order to obtain a nontrivial limit is N_ε^2 rather than $N_\varepsilon|\log \varepsilon|$, as we have shown for example in [SS4] Theorem 3, in which case a relevant lower bound is immediate (see Theorem 2).

II Application to static Ginzburg-Landau

II.1 Case $n = 2$

In the case $n = 2$, one may identify the 2-form Ju with a distribution. Then taking $X = f(x)e_1$ and $Y = g(x)e_2$ where (e_1, e_2) is a constant orthonormal frame and f, g are $C_c^0(\Omega)$ functions, we obtain, by taking the supremum over f and g such that $|f| \leq 1$ and $|g| \leq 1$, the following corollary.

Corollary 2. ($n = 2$) Under the hypotheses of Theorem 1, up to extraction $\frac{Ju_\varepsilon}{N_\varepsilon} \rightharpoonup J$ where J is a measure and

$$(II.1) \quad \liminf_{\varepsilon \rightarrow 0} \frac{1}{N_\varepsilon |\log \varepsilon|} \left(\int_{\Omega} |\partial_1 u_\varepsilon|^2 \int_{\Omega} |\partial_2 u_\varepsilon|^2 \right)^{\frac{1}{2}} \geq |J|(\Omega),$$

where $|J|(\Omega)$ denotes the mass of J , i.e. $\sup_{X \in C_c^0(\Omega), Y \in C_c^0(\Omega), |X| \leq 1, |Y| \leq 1} \int_{\Omega} J(X, Y)$.

Observe that the case $N_\varepsilon \leq C$ corresponds to the case of a bounded vorticity, case in which J (limit of Ju_ε) is a finite sum of the form $J = \pi \sum_{i=1}^k d_i \delta_{a_i}$ where $d_i \in \mathbb{Z}$ and $a_i \in \Omega$ (d_i is the topological degree of the vortex at a_i) and one obtains

Corollary 3. ($n = 2$) Under the hypothesis $E_\varepsilon(u_\varepsilon) \leq C|\log \varepsilon|$, after extraction $Ju_\varepsilon \rightharpoonup J = \pi \sum_{i=1}^k d_i \delta_{a_i}$ with $d_i \in \mathbb{Z}$ and $a_i \in \Omega$, and we have

$$(II.2) \quad \liminf_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} \left(\int_{\Omega} |\partial_1 u_\varepsilon|^2 \int_{\Omega} |\partial_2 u_\varepsilon|^2 \right)^{\frac{1}{2}} \geq \pi \sum_{i=1}^k |d_i|.$$

Applying the arithmetico-geometric inequality, one has

$$\left(\int_{\Omega} |\partial_1 u_\varepsilon|^2 \int_{\Omega} |\partial_2 u_\varepsilon|^2 \right)^{\frac{1}{2}} \leq \frac{1}{2} \int_{\Omega} |\partial_1 u_\varepsilon|^2 + |\partial_2 u_\varepsilon|^2 = \frac{1}{2} \int_{\Omega} |\nabla u_\varepsilon|^2,$$

from which Corollary 3 allows to retrieve the result of [BBH, JS1]. Observe that $E_\varepsilon(u_\varepsilon)$ is itself bounded below by $\frac{1}{2} \int_\Omega |\nabla u_\varepsilon|^2$ so that this really provides lower bounds on the total Ginzburg-Landau energy. Corollary 3 also implies

Corollary 4. ($n = 2$) *If $Ju_\varepsilon \rightharpoonup J = \pi \sum_{i=1}^k d_i \delta_{a_i}$ with for every i , $d_i = \pm 1$ and $\frac{1}{2} \int_\Omega |\nabla u_\varepsilon|^2 \leq \pi k |\log \varepsilon| (1 + o(1))$ as $\varepsilon \rightarrow 0$, then for every unit vector $\frac{\partial}{\partial x_1}$,*

$$(II.3) \quad \int_{\Omega} |\partial_{x_1} u_\varepsilon|^2 = \pi k |\log \varepsilon| (1 + o(1)) \quad \text{as } \varepsilon \rightarrow 0,$$

and for all vector fields $X, Y \in C_c^0(\Omega)$,

$$(II.4) \quad \lim_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} \int_{\Omega} (X \cdot \nabla u_\varepsilon, Y \cdot \nabla u_\varepsilon) = \pi \sum_{i=1}^k X(a_i) \cdot Y(a_i).$$

In other words, if a vortex of degree ± 1 carries exactly the minimum amount of energy $\pi |\log \varepsilon|$ then the projection of its gradient on any coordinate carries exactly half of the amount of the energy, i.e. an isotropic behavior is preferred.

Proof. We may isolate the a_i 's in disjoint balls $B(a_i, r)$ of small radius r . In each of them, we have, according to Corollary 3,

$$(II.5) \quad \frac{1}{2|\log \varepsilon|} \int_{B(a_i, r)} |\nabla u_\varepsilon|^2 \geq \frac{1}{|\log \varepsilon|} \left(\int_{B(a_i, r)} |\partial_1 u_\varepsilon|^2 \int_{B(a_i, r)} |\partial_2 u_\varepsilon|^2 \right)^{\frac{1}{2}} \geq \pi + o(1).$$

On the other hand $\frac{1}{2} \int_\Omega |\nabla u_\varepsilon|^2 \leq \pi k |\log \varepsilon| (1 + o(1))$, so we must have, for each i ,

$$(II.6) \quad \frac{1}{2|\log \varepsilon|} \int_{B(a_i, r)} |\nabla u_\varepsilon|^2 = \frac{1}{|\log \varepsilon|} \left(\int_{B(a_i, r)} |\partial_1 u_\varepsilon|^2 \int_{B(a_i, r)} |\partial_2 u_\varepsilon|^2 \right)^{\frac{1}{2}} + o(1) = \pi + o(1).$$

We deduce that

$$\frac{1}{|\log \varepsilon|} \left(\left(\int_{B(a_i, r)} |\partial_1 u_\varepsilon|^2 \right)^{\frac{1}{2}} - \left(\int_{B(a_i, r)} |\partial_2 u_\varepsilon|^2 \right)^{\frac{1}{2}} \right)^2 = o(1)$$

and thus

$$\frac{1}{|\log \varepsilon|} \int_{B(a_i, r)} |\partial_1 u_\varepsilon|^2 = \frac{1}{|\log \varepsilon|} \int_{B(a_i, r)} |\partial_2 u_\varepsilon|^2 + o(1),$$

and finally that

$$\frac{1}{|\log \varepsilon|} \int_{\cup_{i=1}^k B(a_i, r)} |\partial_1 u_\varepsilon|^2 = \pi k + o(1) \quad \frac{1}{|\log \varepsilon|} \int_{\cup_{i=1}^k B(a_i, r)} |\partial_2 u_\varepsilon|^2 = \pi k + o(1)$$

and since the sum of the two is less than $\frac{1}{|\log \varepsilon|} \int_\Omega |\nabla u_\varepsilon|^2 = 2\pi k + o(1)$, we must have

$$\frac{1}{|\log \varepsilon|} \int_{\cup_{i=1}^k B(a_i, r)} |\partial_1 u_\varepsilon|^2 = \frac{1}{|\log \varepsilon|} \int_\Omega |\partial_1 u_\varepsilon|^2 + o(1) = \pi k + o(1),$$

which proves (II.3). This also implies that

$$(II.7) \quad \frac{1}{|\log \varepsilon|} \int_{\Omega \setminus \cup_i B(a_i, r)} |\partial_1 u_\varepsilon|^2 = o(1)$$

and that for each i , and each unit norm vector e ,

$$(II.8) \quad \frac{1}{|\log \varepsilon|} \int_{B(a_i, r)} |e \cdot \nabla u_\varepsilon|^2 = \pi + o(1).$$

If $X \in C^0(\Omega)$, we may assume by taking r small enough, that X is a constant vector equal to $X(a_i)$ in each $B(a_i, r)$. Then (II.7) and (II.8) imply that

$$\frac{1}{|\log \varepsilon|} \int_\Omega |X \cdot \nabla u_\varepsilon|^2 = \frac{1}{|\log \varepsilon|} \sum_{i=1}^k \int_{B(a_i, r)} |X \cdot \nabla u_\varepsilon|^2 + o(1) = \sum_{i=1}^k \pi |X(a_i)|^2 + o(1).$$

We can then polarize this result (applying it to $X - Y$ and $X + Y$ successively) to obtain (II.4). \square

Remark 3. *These estimates (hence Theorem 1) are sharp, for example for a radial vortex of degree ± 1 .*

II.2 Case $n = 3$

Let us now turn to the dimension 3. First, when $N_\varepsilon = O(1)$, it is known from [JS1] (and we reprove it here) that J seen as a 1-current is rectifiable without boundary, with $\frac{J}{\pi}$ integer-multiplicity. In other words, $\frac{J}{\pi}$ is the sum of integer-multiple ‘‘Dirac-masses’’ supported on some rectifiable curves (the ‘‘vortex-lines’’). Applying Corollary 1 to X and Y perpendicular to each other and such that $|X| \leq 1$ and $|Y| \leq 1$ and taking the supremum over such C_c^0 vector fields, one obtains (with the use of the arithmetico-geometric inequality)

Corollary 5. ($n = 3$) Under the hypothesis $E_\varepsilon(u_\varepsilon) \leq C|\log \varepsilon|$, up to extraction $Ju_\varepsilon \rightharpoonup J$ (with $\frac{J}{\pi}$ rectifiable and integer multiplicity), and we have

$$\liminf_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} \int_{\Omega} |\nabla u_\varepsilon|^2 \geq 2|J|(\Omega),$$

This lower bound was obtained in [JS1] and strengthened that of [Sa2].

Remark 4. Recently, Bourgain, Brezis and Mironescu have proved in [BBM] that in dimension $n = 3$, the limiting J is in fact in the smaller space $(W_0^{1,3}(\Omega))'$.

II.3 General case

Theorem 2. Let u_ε be a family such that $E_\varepsilon(u_\varepsilon) \leq N_\varepsilon |\log \varepsilon| \ll M(\varepsilon)$, and $\|u_\varepsilon\|_{L^\infty(\Omega)} \leq C$, then up to extraction,

$$\frac{Ju_\varepsilon}{N_\varepsilon} \rightharpoonup J \quad \text{measure-valued 2-form, in } (C_c^{0,\gamma}(\Omega))', \gamma > 0,$$

$$\frac{ju_\varepsilon}{\sqrt{N_\varepsilon |\log \varepsilon|}} \rightharpoonup j \quad \text{weakly in } L^2(\Omega),$$

and

$$(II.9) \quad \liminf_{\varepsilon \rightarrow 0} \frac{1}{N_\varepsilon |\log \varepsilon|} \int_{\Omega} |\nabla u_\varepsilon|^2 \geq 2|J|(\Omega) + \int_{\Omega} |j|^2,$$

where ju_ε was defined in (I.2).

Remark 5. If $\sqrt{N_\varepsilon |\log \varepsilon|} = N_\varepsilon$ i.e. if $N_\varepsilon = |\log \varepsilon|$, then J and j are related via $J = \frac{1}{2} dj$ (from the weak L^2 convergence of ju_ε).

Also, a similar result can be obtained without the assumption $\|u_\varepsilon\|_{L^\infty(\Omega)} \leq C$ (but with a weaker convergence of ju_ε .)

This theorem is the lower-bound part of the Γ -convergence result on Ginzburg-Landau energy, and includes the case of unbounded vorticity. In dimension 2, we retrieve the result of [JS2] which was similar to the result of [SS3] when setting the magnetic fields equal to zero. We see that $2|J|$ plays the role of the defect measure of $\frac{ju_\varepsilon}{\sqrt{N_\varepsilon |\log \varepsilon|}}$.

In order to complete the Γ -convergence, one would need to do a construction, i.e. prove that for every limiting j and J , there exists a sequence u_ε such that $\frac{Ju_\varepsilon}{N_\varepsilon} \rightharpoonup J$ and $\frac{ju_\varepsilon}{\sqrt{N_\varepsilon |\log \varepsilon|}} \rightharpoonup j$, and with an energy of order $2|J|(\Omega) + \int_{\Omega} |j|^2$. This is much more delicate. Alberti, Baldo and Orlandi have obtained a result corresponding to this for bounded N_ε , see [ABO].

If $N_\varepsilon \gg |\log \varepsilon|$, then the right order of energy to consider is N_ε^2 and the immediate lower bound

$$\liminf_{\varepsilon \rightarrow 0} \frac{1}{N_\varepsilon^2} \int_{\Omega} |\nabla u_\varepsilon|^2 \geq \int_{\Omega} |j|^2$$

is sharp and give the right principal order of energy.

Proof of Theorem 2. The first assertion follows directly from Theorem 1.

We prove the second assertion. Choose e_1, e_2, \dots, e_n an orthonormal (moving) frame that may depend on $x \in \Omega$, and $f, g \in C_c^0(\Omega)$ with $|f| \leq 1$ and $|g| \leq 1$. Then, let $X_1 = fe_1$, $X_2 = ge_2$, $X_3 = e_3, \dots, X_n = e_n$. The inequality

$$(II.10) \quad |\nabla u_\varepsilon|^2 \geq \sum_{i=1}^n |X_i \cdot \nabla u_\varepsilon|^2$$

holds. Since $|X_i \cdot ju_\varepsilon| \leq |X_i \cdot \nabla u_\varepsilon| |u_\varepsilon|$, we have

$$|X_i \cdot ju_\varepsilon| - |X_i \cdot \nabla u_\varepsilon| \leq (|u_\varepsilon| - 1) |X_i \cdot \nabla u_\varepsilon|.$$

Thanks to the bound on $E_\varepsilon(u_\varepsilon)$ and $|u_\varepsilon| \leq C$, we infer directly that $\frac{ju_\varepsilon}{\sqrt{N_\varepsilon |\log \varepsilon|}}$ is bounded in $L^2(\Omega)$, hence weakly compact, and that

$$\frac{(|X_i \cdot ju_\varepsilon| - |X_i \cdot \nabla u_\varepsilon|)_+}{\sqrt{N_\varepsilon |\log \varepsilon|}} \rightarrow 0$$

as $\varepsilon \rightarrow 0$ in $L^1(\Omega)$. It follows that denoting by ϕ_{X_i} the weak L^2 limit of

$$\frac{|X_i \cdot \nabla u_\varepsilon|}{\sqrt{N_\varepsilon |\log \varepsilon|}},$$

we have $|X_i \cdot j| \leq \phi_{X_i}$ almost everywhere, where j is the weak limit of the normalized currents.

Denoting by ν_{X_1} and ν_{X_2} the defect measures of

$$\frac{|X_1 \cdot \nabla u_\varepsilon|}{\sqrt{N_\varepsilon |\log \varepsilon|}}, \quad \frac{|X_2 \cdot \nabla u_\varepsilon|}{\sqrt{N_\varepsilon |\log \varepsilon|}}$$

respectively, it follows from (II.10) and the very definition of a defect measure that

$$\liminf_{\varepsilon \rightarrow 0} \frac{1}{N_\varepsilon |\log \varepsilon|} \int_{\Omega} |\nabla u_\varepsilon|^2 \geq \|\nu_{X_1}\| + \|\nu_{X_2}\| + \int_{\Omega} |\phi_{X_1}|^2 + |\phi_{X_2}|^2 + \dots + |\phi_{X_n}|^2,$$

thus using Theorem 1 and the above, we are led to

$$(II.11) \quad \begin{aligned} \liminf_{\varepsilon \rightarrow 0} \frac{1}{N_\varepsilon |\log \varepsilon|} \int_{\Omega} |\nabla u_\varepsilon|^2 &\geq 2 \left| \int_{\Omega} J(X_1, X_2) \right| + \int_{\Omega} |X_1 \cdot j|^2 + |X_2 \cdot j|^2 + \dots + |X_n \cdot j|^2 \\ &\geq 2 \left| \int_{\Omega} fgJ(e_1, e_2) \right| + \int_{\Omega} |j|^2 + \int_{\Omega} (|f|^2 - 1) |j \cdot e_1|^2 + (|g|^2 - 1) |j \cdot e_2|^2. \end{aligned}$$

Taking the supremum over all such frames e_1, \dots, e_n and all compactly supported $|f| \leq 1$, $|g| \leq 1$ proves the proposition. \square

II.4 Application to Ginzburg-Landau with magnetic field

In any dimension $n \geq 2$, one may consider the Ginzburg-Landau energies with magnetic field

$$(II.12) \quad G_\varepsilon(u, A) = \frac{1}{2} \int_{\Omega} |d_A u|^2 + |h - h_{\text{ex}} p|^2 + \frac{1}{2\varepsilon^2} (1 - |u|^2)^2$$

where A is a real-valued 1-form on Ω (the magnetic potential), $d_A = d - iA$, $h = \star dA$ (the magnetic field), \star being the Hodge transform, and p is a given $(n-2)$ -form. Here h_{ex} is a real number (depending on ε), such that $\lim_{\varepsilon \rightarrow 0} \frac{h_{\text{ex}}}{|\log \varepsilon|} = \lambda < \infty$. G_ε is a gauge-invariant version of E_ε , the one introduced as a model for superconductivity (for $n = 2$ and 3) by Ginzburg and Landau (for more details, we refer to [T] and [SS3] for example), with h_{ex} then corresponding to the intensity of an applied magnetic field. The gauge transformations are

$$\begin{cases} u \rightarrow ue^{i\Phi} \\ A \rightarrow A + d\Phi. \end{cases}$$

We define the gauge-invariant version of the Jacobian

$$J(u, A) = \frac{1}{2} d((iu, d_A u) + A).$$

We have the following variant of Theorem 2:

Corollary 6. *Let $(u_\varepsilon, A_\varepsilon)$ be such that $G_\varepsilon(u_\varepsilon, A_\varepsilon) \leq Ch_{\text{ex}} |\log \varepsilon|$ and $h_\varepsilon = \star dA_\varepsilon$. Then, up to extraction the rescaled Jacobians $\frac{J(u_\varepsilon, A_\varepsilon)}{h_{\text{ex}}}$ weakly converge to J , measure-valued 2-form, in $(C_c^{0,\gamma}(\Omega))'$, $\frac{(iu_\varepsilon, d_{A_\varepsilon} u_\varepsilon)}{h_{\text{ex}}} \rightharpoonup j$ in $L^2(\Omega)$, $\frac{h_\varepsilon}{h_{\text{ex}}} \rightharpoonup h$ weakly in $L^2(\Omega)$, and*

$$(II.13) \quad \liminf_{\varepsilon \rightarrow 0} \frac{G_\varepsilon(u_\varepsilon, A_\varepsilon)}{h_{\text{ex}} |\log \varepsilon|} \geq |J|(\Omega) + \frac{\lambda}{2} \int_{\Omega} |j|^2 + |h - p|^2.$$

Remark 6. *If in addition, the relation $-\star dh_\varepsilon = (iu_\varepsilon, d_{A_\varepsilon} u_\varepsilon)$ is satisfied (which is the case when minimizing the energy with respect to A) then we also have $-\star dh = j$.*

In dimension $n = 2$, this result is the lower bound part of the result of [SS3]. *Proof:* Choosing the Coulomb gauge $d^\star A = 0$, $A \cdot n = 0$ on $\partial\Omega$, we obtain from the energy upper bound a bound on $\frac{A_\varepsilon}{h_{\text{ex}}}$ in $H^1(\Omega)$. Thus, $\frac{A_\varepsilon}{h_{\text{ex}}}$ is compact in $L^2(\Omega)$ and there is no defect measure of L^2 convergence of $\frac{(iu_\varepsilon, d_{A_\varepsilon} u_\varepsilon)}{h_{\text{ex}}}$ associated to A , hence the only defect measure is that of $(iu_\varepsilon, du_\varepsilon)$, and is J . The rest can be proved as in Theorem 2 and [SS3]. \square

III Application to Ginzburg-Landau dynamics

In this section, we wish to consider families u_ε which depend both on space and time. For that purpose, we take the first coordinate to be time and work in $n + 1$ dimensions where n is the number of space dimensions. In that framework we have

$$ju = (iu, \partial_t u)dt + (iu, d_{sp}u),$$

where $d_{sp}u$ denotes the differential with respect to the space coordinates only. When considering the total Jacobian Ju , we can split it again between the time and space coordinates and write

$$Ju = \sum_{i=1}^n V_i dt \wedge dx_i + \frac{1}{2}d_{sp}(iu, d_{sp}u)$$

where $\frac{1}{2}d_{sp}(iu, d_{sp}u)$ is the space-only Jacobian, corresponding to the vorticity, that we will denote by μ . We will also write $V = \sum_{i=1}^n V_i dt \wedge dx_i$, and identify at times V with a vector-field. V corresponds to the velocity part of the Jacobian. We thus have

$$(III.1) \quad Ju = V + \mu.$$

Writing that the form Ju is closed, i.e. $d(Ju) = 0$, we have

$$(III.2) \quad d_t \mu + dV = 0,$$

where d_t denotes the differential with respect to the time variable only (indeed $d_{sp}\mu = 0$ because μ is a space-closed form). Equation (III.2) expresses that μ is transported via V . In dimension $n = 2$, μ can be identified with a function (or distribution) and V with a vector-field $(V_1, V_2) = \frac{1}{2}(\partial_t(iu, \partial_1 u) - \partial_1(iu, \partial_t u), (\partial_t(iu, \partial_2 u) - \partial_2(iu, \partial_t u))$, and (III.2) rewrites

$$(III.3) \quad \partial_t \mu - \text{curl } V = 0.$$

In dimension $n = 3$, μ and V can be identified with vector-fields and (III.2) rewrites again (III.3) with the extra relation

$$\text{div } \mu = 0$$

(coming from the fact that μ is a space-closed form). Theorem 1 applies similarly to this case. We define the following norm on measure-valued 2-forms on Ω :

$$(III.4) \quad \|\mu\|_1 = \sup_{\zeta \text{ smooth } (n-2)\text{-form on } \Omega, \zeta=0 \text{ on } \partial\Omega, |d\zeta| \leq 1} \left| \int_{\Omega} \mu \wedge \zeta \right|,$$

i.e. the norm in the dual of Lipschitz forms (it is very similar to the flat norm, though possibly smaller). In dimension $n = 2$, ζ is simply a function and, for measures of the type $\mu = \sum_i d_i \delta_{a_i}$ and $\mu' = \sum_i d'_i \delta_{b_i}$ with $d_i, d'_i \in \mathbb{Z}$, $\|\mu - \mu'\|_1$ corresponds to the minimal connection between the a_i 's and the b_i 's as introduced by Brezis-Coron-Lieb in [BCL].

With the perspective of studying solutions of time-dependent Ginzburg-Landau equations, we will make the extra assumption that the energy of $E_\varepsilon(u_\varepsilon)$ remains uniformly bounded in time by $N_\varepsilon|\log \varepsilon|$. The idea of the following result is simply to apply Theorem 1 to the orthogonal vector-fields $Y = f \frac{\partial}{\partial t}$ and $X = (0, X')$ where X' is some vector-field on Ω that we denote X in the following, and to observe that $J(X, Y)$ reduces to $fV \cdot X'$ (where V is identified with a vector). $\mathcal{M}(\Omega)$ will denote the space of forms whose coefficients are bounded Radon measures on Ω .

Theorem 3. *Let $u_\varepsilon(t, x)$ be defined over $[0, T] \times \Omega$ (with $\Omega \subset \mathbb{R}^n$) and be such that*

$$(III.5) \quad \begin{cases} \forall t \in [0, T], & E_\varepsilon(u_\varepsilon) = \frac{1}{2} \int_\Omega |\nabla u_\varepsilon|^2 + \frac{1}{2\varepsilon^2} (1 - |u_\varepsilon|^2)^2 \leq N_\varepsilon |\log \varepsilon| \ll M(\varepsilon) \\ \int_{[0, T] \times \Omega} |\partial_t u_\varepsilon|^2 \leq N_\varepsilon |\log \varepsilon| \ll M(\varepsilon). \end{cases}$$

Then, V_ε and μ_ε being defined as in (III.1), there exist $\mu \in L^\infty([0, T], \mathcal{M}(\Omega))$, and $V \in L^2([0, T], \mathcal{M}(\Omega))$ such that, after extraction,

$$\begin{aligned} \frac{\mu_\varepsilon}{N_\varepsilon} &\rightharpoonup \mu && \text{in } (C_c^{0,\gamma}([0, T] \times \Omega))', \quad \forall \gamma > 0, \\ \frac{V_\varepsilon}{N_\varepsilon} &\rightharpoonup V && \text{in } (C_c^{0,\gamma}([0, T] \times \Omega))', \quad \forall \gamma > 0, \end{aligned}$$

with

$$(III.6) \quad d_t \mu + dV = 0.$$

This implies that $\mu(t)$ is $C^{0, \frac{1}{2}}$ in time for the $\|\cdot\|_1$ -norm, and that for all $t \in [0, T]$, we have

$$\frac{\mu_\varepsilon(t)}{N_\varepsilon} \rightharpoonup \mu(t) \quad \text{in } (C_c^{0,\gamma}(\Omega))', \quad \forall \gamma > 0.$$

Moreover, for any $X \in C_c^0([0, T] \times \Omega, \mathbb{R}^n)$ and $f \in C_c^0([0, T] \times \Omega)$, denoting by ν_X and ν_T the defect measures of L^2 convergence of

$$\frac{|X \cdot \nabla u_\varepsilon|}{\sqrt{N_\varepsilon |\log \varepsilon|}}, \quad \frac{f |\partial_t u_\varepsilon|}{\sqrt{N_\varepsilon |\log \varepsilon|}},$$

we have

$$(III.7) \quad \liminf_{\varepsilon \rightarrow 0} \|\nu_X\|^{\frac{1}{2}} \|\nu_T\|^{\frac{1}{2}} \geq \left| \int_{\Omega \times [0, T]} V \cdot f X \right|.$$

This last relation immediately implies that

$$(III.8) \quad \liminf_{\varepsilon \rightarrow 0} \frac{1}{N_\varepsilon |\log \varepsilon|} \left(\int_{\Omega \times [0, T]} |X \cdot \nabla u_\varepsilon|^2 \int_{\Omega \times [0, T]} f^2 |\partial_t u_\varepsilon|^2 \right)^{\frac{1}{2}} \geq \left| \int_{\Omega \times [0, T]} V \cdot f X \right|.$$

Proof. As we mentioned, Theorem 1 directly implies the convergence to the measure-valued 2-forms μ and V , and the lower bound (III.7). The fact that $\mu \in L^\infty([0, T], \mathcal{M}(\Omega))$ comes from applying for example Theorem 1 in space variables only at each time. (III.6) follows by passing to the limit in (III.2). There remains to prove the additional regularity on μ and V . First, notice that (III.7) and (III.8) can be extended (by density) to f and X which are continuous and not compactly supported, as long as V is seen as a measure on $]0, T[\times \Omega$ (which does not weigh on the boundary).

Inserting the a priori estimates (III.5) into (III.8), we are led to

$$(III.9) \quad \left| \int_{]0, T[\times \Omega} V \cdot X \right|^2 \leq C \int_0^T \|X\|_{L^\infty(\Omega)}^2 dt.$$

This proves by duality that V is L^2 in time with values in $\mathcal{M}(\Omega)$. Moreover, for every $C_c^0([0, T] \times \Omega)$ vector-field X such that $\|X\|_{L^\infty(\Omega)} \leq 1$, we have

$$\left| \int_{[t_1, t_2] \times \Omega} V \cdot X \right| \leq \|V\|_{L^2([0, T], \mathcal{M}(\Omega))} \sqrt{t_2 - t_1}.$$

Returning to the formulation in differential forms, this means that for every $(n-1)$ -form X such that $|X| \leq 1$, we have

$$\left| \int_{[t_1, t_2] \times \Omega} V \wedge X \right| \leq \|V\|_{L^2([0, T], \mathcal{M}(\Omega))} \sqrt{t_2 - t_1}.$$

Let us approximate V in $L^2([0, T], \mathcal{M}(\Omega))$ by some smooth V_α , and μ by some smooth μ_α such that (III.2) holds. Considering ζ a smooth compactly supported $(n-2)$ -form on Ω (i.e. independent of time) such that $|d\zeta| \leq 1$, we have

$$(III.10) \quad \left| \int_{[t_1, t_2] \times \Omega} V_\alpha \wedge d\zeta \right| \leq C \sqrt{t_2 - t_1}.$$

But, in view of (III.2) and the fact that $d_t\zeta = 0$, we have

$$\begin{aligned}
\int_{[t_1, t_2] \times \Omega} V_\alpha \wedge d\zeta &= - \int_{[t_1, t_2] \times \Omega} dV_\alpha \wedge \zeta \\
&= \int_{[t_1, t_2] \times \Omega} d_t \mu_\alpha \wedge \zeta \\
&= \int_{[t_1, t_2] \times \Omega} d_t (\mu_\alpha \wedge \zeta) \\
(III.11) \qquad &= \int_{\Omega} \mu_\alpha(t_2) \wedge \zeta - \int_{\Omega} \mu_\alpha(t_1) \wedge \zeta.
\end{aligned}$$

Consequently, (III.10) implies that

$$\left| \int_{\Omega} (\mu_\alpha(t_2) - \mu_\alpha(t_1)) \wedge \zeta \right| \leq \|V_\alpha\|_{L^2([0, T], \mathcal{M}(\Omega))} \sqrt{t_2 - t_1},$$

that is $\|\mu_\alpha(t_2) - \mu_\alpha(t_1)\|_1 \leq \|V_\alpha\|_{L^2([0, T], \mathcal{M}(\Omega))} \sqrt{t_2 - t_1}$. By passing to the limit $\alpha \rightarrow 0$ we deduce that $\mu(t)$ is Hölder continuous in time (of exponent $\frac{1}{2}$) for the 1-norm, with

$$(III.12) \qquad \|\mu\|_{C^{0, \frac{1}{2}}([0, T], (\mathcal{M}(\Omega), \|\cdot\|_1))} \leq \|V\|_{L^2([0, T], \mathcal{M}(\Omega))},$$

and that (III.11) holds for V and μ . This regularity is also true for the flat norm, with a similar proof.

Let us now choose a time $t_0 \in [0, T[$. Since we know that for all $t \in [0, T]$, $E_\varepsilon(u_\varepsilon(t)) \leq CN_\varepsilon |\log \varepsilon|$, applying Theorem 1, we know that $\frac{1}{N_\varepsilon} \mu_\varepsilon(t_0)$ is also compact in $(C_c^{0, \gamma}(\Omega))'$, $\forall \gamma > 0$. Let ν denote its weak limit (after extraction). Let us consider $\overline{u_\varepsilon}$ defined in $] - T, T]$ by $\overline{u_\varepsilon} = u_\varepsilon(t_0)$ for $t < t_0$ and $\overline{u_\varepsilon} = u_\varepsilon$ for $t \geq t_0$. Let us denote by $\overline{\mu_\varepsilon}$, the associated vorticity. It is clear that $\overline{\mu_\varepsilon} = \mu_\varepsilon(t_0)$ for $t < t_0$ and $\overline{\mu_\varepsilon} = \mu_\varepsilon$ for $t \geq t_0$. One can easily check that $\overline{u_\varepsilon}$ satisfies the hypotheses of Theorem 3, thus we deduce that $\frac{1}{N_\varepsilon} \overline{\mu_\varepsilon}$ converges weakly in $(C_c^{0, \gamma}([-T, T] \times \Omega))'$ (after extraction) to some limiting measure $\overline{\mu}$, continuous in time for the 1-norm. By using test-functions, we see that necessarily $\overline{\mu} = \nu$ a.e. in $] - T, t_0[$, and $\overline{\mu} = \mu$ a.e. in $]t_0, T]$. But μ and $\overline{\mu}$ are both continuous in time, hence we must have, by continuity at the time t_0 , $\nu = \mu(t_0)$. We deduce that the only possible limit of extracted sequences of $\frac{1}{N_\varepsilon} \mu_\varepsilon(t_0)$ is $\mu(t_0)$, and thus that $\frac{1}{N_\varepsilon} \mu_\varepsilon(t_0)$ converges in $(C_c^{0, \gamma}(\Omega))'$, $\forall \gamma > 0$ to $\mu(t_0)$, for all $t_0 \in [0, T[$. For the time T , the same argument can be applied by extending u_ε to $[0, 2T]$. □

In the case of a bounded number of vortices (i.e. $N_\varepsilon = O(1)$) in two space dimensions, we retrieve as a corollary the following result stated in [J2], Proposition 3. For a treatment of the case with magnetic field, see [SS5].

Corollary 7. Assume $E_\varepsilon(u_\varepsilon) \leq C|\log \varepsilon|$ and $\int_{[0,T] \times \Omega} |\partial_t u_\varepsilon|^2 \leq C|\log \varepsilon|$, that there exists a finite collection of continuous points $a_i(t)$ and integers $d_i = \pm 1$ independent of time, such that $\mu_\varepsilon \rightharpoonup \mu(t) = \pi \sum_i d_i \delta_{a_i(t)}$, and that $\int_\Omega |\nabla u_\varepsilon|^2 \leq 2\pi \sum_i |\log \varepsilon| (1 + o(1))$ for all t . Then for all interval $[t_1, t_2] \subset [0, T]$ on which the a_i remain distinct, we have

$$(III.13) \quad \liminf_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} \int_{\Omega \times [t_1, t_2]} |\partial_t u_\varepsilon|^2 \geq \pi \sum_i \int_{t_1}^{t_2} |\partial_t a_i|^2 dt.$$

The existence of a fixed number of such continuous $a_i(t)$ is true for example if one knows that the energy $E_\varepsilon(u_\varepsilon(t))$ decreases in time (using the continuity of $\mu(t)$ for the minimal connection stated in Theorem 3).

Proof. Since we assume that the a_i 's remain distinct, and there is only a finite number of them, we can find open balls B_i such that each B_i contains only one $a_i(t)$ on the time interval $[t, t + \delta]$, δ small. Applying (III.8) with (III.11) (which we saw is valid even for non compactly supported test-functions), we have, for every $\zeta \in C_c^1(B_i)$,

$$(III.14) \quad \liminf_{\varepsilon \rightarrow 0} \left(\frac{1}{|\log \varepsilon|^2} \int_{B_i \times [t, t+\delta]} |\nabla^\perp \zeta \cdot \nabla u_\varepsilon|^2 \int_{B_i \times [t, t+\delta]} |\partial_t u_\varepsilon|^2 \right) \geq \pi^2 |d_i(\zeta(a_i(t+\delta)) - \zeta(a_i(t)))|^2$$

In view of the hypothesis, we may use Corollary 4, more specifically (II.4), to say that for all $t \in [t_1, t_2]$, $\lim_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} \int_{B_i} |\nabla^\perp \zeta \cdot \nabla u_\varepsilon|^2(t) = \pi |\nabla \zeta(a_i(t))|^2$, and taking the supremum over the $\zeta \in C_c^1(B_i)$ such that $|\nabla \zeta| \leq 1$, (III.14) reduces to

$$\liminf_{\varepsilon \rightarrow 0} \left(\frac{\pi \delta}{|\log \varepsilon|} \int_{B_i \times [t, t+\delta]} |\partial_t u_\varepsilon|^2 \right) \geq \pi^2 |a_i(t+\delta) - a_i(t)|^2.$$

We deduce that for every subdivision (t_k) of $[t_1, t_2]$,

$$\pi \sum_{i,k} \frac{|a_i(t_{k+1}) - a_i(t_k)|^2}{|t_{k+1} - t_k|} \leq \liminf_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} \int_{\Omega \times [t_1, t_2]} |\partial_t u_\varepsilon|^2.$$

This implies that $a_i(t) \in H^1([t_1, t_2])$ and

$$\pi \sum_i \int_{t_1}^{t_2} |\partial_t a_i|^2 \leq \liminf_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} \int_{\Omega \times [t_1, t_2]} |\partial_t u_\varepsilon|^2.$$

The argument goes as follows : first we deduce that $a_i(t)$ is absolutely continuous (i.e. $\forall \varepsilon > 0, \exists \delta > 0, \sum_k |t_{k+1} - t_k| < \delta \Rightarrow \sum_k |a_i(t_{k+1}) - a_i(t_k)| < \varepsilon$), then it has a derivative almost everywhere, and finally this derivative is L^2 . \square

Remark 7. *This estimate (hence that of Theorem 3) is optimal as can be seen for example for the case of a radial vortex translating at a constant velocity.*

IV Proof of Theorem 1

IV.1 Idea of the proof

By using a slicing argument and approximation, the proof of Theorem 1 reduces to the case of a two-dimensional domain Ω and constant vectors X and Y . The lower bounds introduced in [BBH, J1, Sa1] and a Jacobian estimate (see [JS1, SS3, ASS]) yield the known result that under the assumption $E_\varepsilon(u_\varepsilon) \leq N_\varepsilon |\log \varepsilon|$, the normalized Jacobian determinants $J(u_\varepsilon)/N_\varepsilon$ converge as $\varepsilon \rightarrow 0$ to a measure-valued 2-form J and that

$$(IV.1) \quad \liminf_{\varepsilon \rightarrow 0} \frac{1}{N_\varepsilon |\log \varepsilon|} \int_{\Omega} |\nabla u_\varepsilon|^2 \geq |J|(\Omega).$$

Theorem 1 follows by noticing that the proof of this lower bound remains valid if one chooses a different metric in Ω . For instance given two linearly independent vectors X, Y one may choose a metric g_λ for which $g_\lambda(X, X) = \lambda$, $g_\lambda(Y, Y) = 1/\lambda$ and $g_\lambda(X, Y) = 0$. Then (IV.1) becomes

$$(IV.2) \quad \liminf_{\varepsilon \rightarrow 0} \frac{1}{N_\varepsilon |\log \varepsilon|} \int_{\Omega} \left(\frac{1}{\lambda} |X \cdot \nabla u_\varepsilon|^2 + \lambda |Y \cdot \nabla u_\varepsilon|^2 \right) \frac{dx dy}{|X \wedge Y|} \geq |J|(\Omega),$$

while $|J||X \wedge Y| = |J(X, Y)|$. Another way of stating this is that we can apply the usual Euclidean lower bounds to the map $v_\varepsilon(x, y) = u_\varepsilon(xX + yY)$. Minimizing the left-hand side with respect to λ for each ε yields the desired product estimate

$$\liminf_{\varepsilon \rightarrow 0} \frac{1}{N_\varepsilon |\log \varepsilon|} \left(\int_{\Omega} |X \cdot \nabla u_\varepsilon|^2 \int_{\Omega} |Y \cdot \nabla u_\varepsilon|^2 \right)^{\frac{1}{2}} \geq \left| \int_{\Omega} J(X, Y) \right|.$$

We now investigate the details.

IV.2 Modified vortex-balls

Here we restate the vortex ball construction of [Sa1] for a constant metric g in \mathbb{R}^2 . We denote by $\text{per } A$ the Euclidean perimeter of a set A and $\text{per}_g A$ its perimeter with respect to a metric g . Similarly we let

$$D_g(u, A) = \frac{1}{2} \int_A |\nabla u|_g^2,$$

where the integral is taken with respect to the surface element associated to g and $|\nabla u|_g^2 = g^{ij}(\partial_i u, \partial_j u)$. Finally we define the radius of a compact set K to be the infimum over all finite coverings of K by disjoint balls B_1, \dots, B_n of $r_1 + \dots + r_n$, where r_i is the radius of B_i . We write $r(K)$ for the radius with respect to the Euclidean metric and $r_g(K)$ for the radius with respect to a metric g , and recall that the radius is controlled by the perimeter.

Proposition IV.1. *Assume Ω is a domain in \mathbb{R}^2 and ω is a compact subset of \mathbb{R}^2 . Then for any $\alpha > 0$, any constant metric g and any $t \geq 1$, there exists a family B_1, \dots, B_n of disjoint balls for the metric g , of radii r_1, \dots, r_n such that*

$$(IV.3) \quad \sum_i r_i \leq t(r_g(\omega) + \alpha)$$

and for any unit vector field $u : \Omega \setminus \omega \rightarrow S^1$ and any $1 \leq i \leq n$ such that $B_i \subset \Omega$

$$(IV.4) \quad D_g(u, B_i \setminus \Omega) \geq \pi |d_i| \log t,$$

where $d_i = \deg(u, \partial B_i)$.

For the proof, it suffices to apply the standard Euclidean lower bound of [Sa1] to $v(x, y) = u(xX + yY)$, where X, Y is an orthonormal frame for g . We denote by g_0 the standard metric on \mathbb{R}^2 . We recall that $M(\varepsilon)$ is such that

$$(IV.5) \quad \forall \alpha > 0, \quad \lim_{\varepsilon \rightarrow 0} \varepsilon^\alpha M(\varepsilon) = 0, \quad \lim_{\varepsilon \rightarrow 0} \frac{|\log \varepsilon|}{M(\varepsilon)^\alpha} = 0 \quad \text{and} \quad \log M(\varepsilon) = o(|\log \varepsilon|) \text{ as } \varepsilon \rightarrow 0.$$

A consequence of the previous proposition is:

Proposition IV.2. *Let $\Omega \subset \mathbb{R}^2$ a bounded domain and $\lambda > 0$. We assume that g is a constant metric such that $\lambda^{-1}g_0 \leq g \leq \lambda g_0$, and that $E_\varepsilon(u_\varepsilon) < KM(\varepsilon)$ for some $0 < \varepsilon < 1$. Then there exists disjoint balls (depending on ε) B_1, \dots, B_n for the metric g with $B_i = B_g(a_i, r_i)$ such that letting $\tilde{\Omega} = \{x \in \Omega \mid \text{dist}(x, \partial\Omega) > \varepsilon\}$*

1. $\sum_i r_i \leq \lambda/M(\varepsilon)$.
2. For any $x \in \tilde{\Omega} \setminus \cup_i B_i$, $||u_\varepsilon(x)| - 1| \leq 2/M(\varepsilon)$.
3. If $B_i \subset \tilde{\Omega}$,

$$(IV.6) \quad D_g(u_\varepsilon, B_i) \geq \pi |d_i| |\log \varepsilon| (1 - o(1)),$$

where $d_i = \deg(u_\varepsilon, \partial B_i)$. The $o(1)$ appearing in the lower bound is a function that goes to zero with ε and which depends only on K . Moreover, letting

$$(IV.7) \quad \mu_\varepsilon = \pi \sum_{\{i|a_i \in \tilde{\Omega}\}} d_i \delta_{a_i},$$

we have

$$(IV.8) \quad \|\star J u_\varepsilon - \mu_\varepsilon\|_{(C_c^{0,1}(\Omega))'} \leq C \lambda^2 \frac{E_\varepsilon(u_\varepsilon)}{M(\varepsilon)},$$

where C depends only on K and \star denotes the Hodge operator with respect to the Euclidean metric.

For the case $g = g_0$, the result in this form was proved in [JS1]. The proof below adapts arguments in [SS1, ASS] where a slightly weaker result was proved. Throughout the proof C denotes a constant depending only on K .

Proof of 1), 2), 3). The co-area formula implies the existence of a t such that $1/M(\varepsilon) < t < 2/M(\varepsilon)$ such that — writing $\omega_t = \{|u_\varepsilon| - 1| \geq t\}$ — the estimate $\text{per}_\Omega \omega_t < C\varepsilon M(\varepsilon)^2$ holds, where per_Ω is the Euclidean perimeter in Ω . We may also assume that t is a regular value of $|u_\varepsilon|$, thus ω_t has regular boundary. Using the upper bound on the energy we may also control the area of ω_t by a $C\varepsilon^2 M(\varepsilon)^2$. This control implies that for some $s \in (0, \varepsilon)$ the length of $\{x \in \omega_t \mid \text{dist}(x, \partial\Omega) = s\}$ is less than $C\varepsilon M(\varepsilon)^2$. Let $\tilde{\Omega} = \{x \in \omega_t \mid \text{dist}(x, \partial\Omega) > s\}$. Then if ε is small enough, $\text{per}(\omega_t \cap \tilde{\Omega}) < C\varepsilon M(\varepsilon)^2$, thus $\omega_t \cap \tilde{\Omega}$ may be included in a union of disjoint Euclidean balls whose union we call ω and such that $\text{per} \omega < C\varepsilon M(\varepsilon)^2$. We have $|u| \geq 1 - 2M(\varepsilon)^{-1}$ in $\tilde{\Omega} \setminus \omega$.

Let g be a metric such that $\lambda^{-1}g_0 \leq g \leq \lambda g_0$. Then $\text{per}_g \omega \leq \lambda C\varepsilon M(\varepsilon)^2$ and thus, using (IV.5), if ε is small enough depending on K , we find $2\text{per}_g \omega < \lambda/M(\varepsilon)$. We may apply Proposition IV.1 in $\tilde{\Omega}$ to $v = u/|u|$ with $\alpha = \text{per}_g \omega$ and t such that $2t\text{per}_g \omega = \lambda/M(\varepsilon)$ (hence $t \geq 1$) to find a family of disjoint balls for the metric g , denoted B_1, \dots, B_n with $B_i = B_g(a_i, r_i)$, such that $\sum_i r_i \leq \lambda/M(\varepsilon)$ and for every i such that $B_i \subset \tilde{\Omega}$,

$$D_g(v, B_i \setminus \Omega) \geq \pi |d_i| \log t,$$

where $d_i = \deg(u, \partial B_i)$. It follows from $\text{per}_g \omega \leq \lambda C\varepsilon M(\varepsilon)^2$ that

$$t \geq C \frac{1}{\varepsilon M(\varepsilon)^3},$$

and then from (IV.5) that

$$\frac{D_g(u, B_i \setminus \Omega)}{|\log \varepsilon|} \geq \pi |d_i| |\log \varepsilon| (1 - o(1)),$$

where $o(1)$ depends only on K . Items 1, 2, 3 of the proposition follow. \square

Proof of (IV.8). We proceed with the proof of (IV.8) as in [SS4], Lemma II.1 and II.2. First, we consider $\chi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ as follows

$$\begin{cases} \chi(x) = x & \text{if } |x - 1| \geq \frac{1}{2} \\ \chi(x) = 1 & \text{if } |x - 1| \leq 1 - M(\varepsilon)^{-1} \\ \chi & \text{is continuous and piecewise affine.} \end{cases}$$

We then define

$$\tilde{u}_\varepsilon = \chi(|u_\varepsilon|) \frac{u_\varepsilon}{|u_\varepsilon|}.$$

It is easy to check that $\|u_\varepsilon - \tilde{u}_\varepsilon\|_{L^\infty(\Omega)} \leq C/M(\varepsilon)$ and to deduce that, defining ju_ε and $j\tilde{u}_\varepsilon$ as in (I.2),

$$\|ju_\varepsilon - j\tilde{u}_\varepsilon\|_{L^2(\Omega)}^2 \leq CM(\varepsilon)^{-2} E_\varepsilon(u_\varepsilon),$$

where $|\alpha dx + \beta dy|^2 = \alpha^2 + \beta^2$. It follows that for any smooth compactly supported function ξ ,

$$\left| \int_{\Omega} (Ju_\varepsilon - J\tilde{u}_\varepsilon) \xi \right| = \frac{1}{2} \left| \int_{\Omega} (ju_\varepsilon - j\tilde{u}_\varepsilon) \wedge d\xi \right| \leq CM(\varepsilon)^{-1} \sqrt{E_\varepsilon(u_\varepsilon)} \|\xi\|_{C^{0,1}(\Omega)}.$$

and therefore

$$(IV.9) \quad \|\star Ju_\varepsilon - \star J\tilde{u}_\varepsilon\|_{(C_c^{0,1})'} \leq C \frac{\sqrt{E_\varepsilon(u_\varepsilon)}}{M(\varepsilon)}.$$

Now we wish to estimate $J\tilde{u}_\varepsilon - \mu_\varepsilon$, with μ_ε defined in (IV.7). Let ξ be a smooth compactly supported function. Since $|\tilde{u}_\varepsilon| = 1$ outside of $\tilde{\Omega} \cap (\cup_i B_i)$ we have $J\tilde{u}_\varepsilon = 0$ there. Therefore

$$(IV.10) \quad \int_{\Omega} \xi J\tilde{u}_\varepsilon = \int_{\Omega \setminus \tilde{\Omega}} \xi J\tilde{u}_\varepsilon + \sum_{B_i \not\subset \tilde{\Omega}} \int_{B_i \cap \tilde{\Omega}} \xi J\tilde{u}_\varepsilon + \sum_{B_i \subset \tilde{\Omega}} \int_{B_i} \xi J\tilde{u}_\varepsilon = I_1 + I_2 + I_3.$$

Since ξ vanishes on $\partial\Omega$ and from the definition of $\tilde{\Omega}$ we find $|\xi(x)| < \varepsilon \|\xi\|_{C^{0,1}(\Omega)}$ for any $x \in \Omega \setminus \tilde{\Omega}$. It is easy to check that $|J\tilde{u}_\varepsilon| < C|\nabla u_\varepsilon|^2$ thus

$$(IV.11) \quad I_1 \leq C\varepsilon E_\varepsilon(u_\varepsilon) \|\xi\|_{C^{0,1}(\Omega)}.$$

The second integral is taken care of in a similar way. From the definition of $\tilde{\Omega}$ and since the Euclidean radius of any ball is less than $\lambda^2 M(\varepsilon)^{-1}$ it follows that if $B_i \not\subset \tilde{\Omega}$ and $x \in \Omega \cap B_i$ then $|\xi(x)| < \|\xi\|_{C^{0,1}} \lambda^2 / M(\varepsilon)$. It follows that

$$(IV.12) \quad I_2 \leq C\lambda^2 \frac{E_\varepsilon(u_\varepsilon)}{M(\varepsilon)} \|\xi\|_{C^{0,1}(\Omega)}.$$

To deal with the third integral we define $\bar{\xi}$ to be equal to $\xi(a_i)$ on B_i for any $B_i = B_g(a_i, r_i) \subset \tilde{\Omega}$ and $\bar{\xi} = 0$ elsewhere. Then letting A be the union of the B_i 's which are included in $\tilde{\Omega}$, we have $|\xi - \bar{\xi}| \leq \lambda^2 \|\xi\|_{C^{0,1}} / M(\varepsilon)$ on A while

$$\int_A \bar{\xi} J\tilde{u}_\varepsilon = \sum_{B_i \subset \tilde{\Omega}} \xi(a_i) \int_{B_i} J\tilde{u}_\varepsilon = \sum_{B_i \subset \tilde{\Omega}} \pi d_i \xi(a_i) = \int \xi d\mu_\varepsilon,$$

where we have used the fact that $|\tilde{u}_\varepsilon| = 1$ on ∂B_i . Therefore

$$(IV.13) \quad \left| I_3 - \int \xi d\mu_\varepsilon \right| \leq C\lambda^2 \frac{E_\varepsilon(u_\varepsilon)}{M(\varepsilon)} \|\xi\|_{C^{0,1}(\Omega)}.$$

It follows from (IV.9), (IV.10), (IV.11), (IV.12) and (IV.13) that for any compactly supported smooth ξ

$$\left| \int_{\Omega} \xi Ju_{\varepsilon} - \int \xi d\mu_{\varepsilon} \right| \leq C\lambda^2 \frac{E_{\varepsilon}(u_{\varepsilon})}{M(\varepsilon)} \|\xi\|_{C^{0,1}}.$$

and the proposition is proved. \square

IV.3 Convergence of the Jacobians

The results in this section are proved in [JS1] (see also [ABO]), the proof is included here for the convenience of the reader. It uses Proposition IV.2 together with the nice interpolation argument of [JS1].

Proposition IV.3. *Let Ω be a bounded domain in \mathbb{R}^n and $\{u_{\varepsilon}\}_{\varepsilon>0}$ be a family in $H^1(\Omega, \mathbb{C})$ such that*

$$(IV.14) \quad E_{\varepsilon}(u_{\varepsilon}) \leq N_{\varepsilon} |\log \varepsilon| \ll M(\varepsilon).$$

Then the normalized Jacobians $N_{\varepsilon}^{-1} Ju_{\varepsilon}$ converge subsequentially in the dual of $C_c^{0,\gamma}(\Omega)$ to a measure valued two-form J , for any $\gamma > 0$. When N_{ε} is independent of ε the limit of $\pi^{-1} Ju_{\varepsilon}$ is in addition a integer multiplicity rectifiable current.

Moreover, given constant vectors X, Y , a function η with compact support in Ω and $\lambda > 0$, there exists sets A_{ε} with measures tending to 0 such that

$$(IV.15) \quad \liminf_{\varepsilon \rightarrow 0} \frac{1}{2N_{\varepsilon} |\log \varepsilon|} \int_{A_{\varepsilon}} \lambda^{-1} |\eta X \cdot \nabla u_{\varepsilon}|^2 + \lambda |\eta Y \cdot \nabla u_{\varepsilon}|^2 \geq \left| \int_{\Omega} J(\eta X, \eta Y) \right|.$$

Proof of compactness. We begin by proving compactness of the Jacobians, by slicing the current Ju_{ε} as in [JS1].

Let $(v, w, \sigma) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2}$ denote coordinates in \mathbb{R}^n . Let, σ being given, $\Omega_{\sigma} = \{(v, w, \sigma) \in \Omega\}$. We let $J_{\varepsilon} = Ju_{\varepsilon}(\partial_v, \partial_w)$, and write $J_{\varepsilon, \sigma}$ for its restriction to Ω_{σ} . Finally we let

$$(IV.16) \quad e_{\varepsilon}(\sigma) = \frac{1}{2} \int_{\Omega_{\sigma}} |\nabla u_{\varepsilon}|^2 + \frac{1}{2\varepsilon^2} (1 - |u_{\varepsilon}|^2)^2.$$

We claim that for any $\sigma \in \mathbb{R}^{n-2}$ there exists a measure $\mu_{\varepsilon, \sigma}$ in Ω_{σ} such that

$$(IV.17) \quad \|\mu_{\varepsilon, \sigma}\| \leq C \frac{e_{\varepsilon}(\sigma)}{|\log \varepsilon|}, \quad \|J_{\varepsilon, \sigma} - \mu_{\varepsilon, \sigma}\|_{(C_c^{0,1})'} \leq CM(\varepsilon)^{-1/2} e_{\varepsilon}(\sigma),$$

where C is independent of ε, σ . The convergence of Ju_{ε} follows from (IV.17) as follows. Integrating w.r.t. σ and using the energy bound (IV.14) we find, letting μ_{ε} be the measure whose slices are $\{\mu_{\varepsilon, \sigma}\}_{\sigma}$ and $\nu_{\varepsilon} = N_{\varepsilon}^{-1}(J_{\varepsilon} - \mu_{\varepsilon})$,

$$(IV.18) \quad \|\nu_{\varepsilon}\|_{(C_c^{0,1})'} \leq C |\log \varepsilon| M(\varepsilon)^{-1/2}, \quad \|N_{\varepsilon}^{-1} \mu_{\varepsilon}\|_{(C^0)'} \leq C.$$

Besides, since $|Ju_\varepsilon| \leq C|\nabla u_\varepsilon|^2$, the bound (IV.14) yields

$$\|\nu_\varepsilon\|_{(C^0)'} \leq C|\log \varepsilon|.$$

For any $\gamma \in (0, 1)$ (see Lemma 3.3 of [JS1]) it holds that

$$\|\nu_\varepsilon\|_{(C_c^{0,\gamma})'} \leq C\|\nu_\varepsilon\|_{(C_c^0)'}^{1-\gamma} \|\nu_\varepsilon\|_{(C_c^{0,1})'}^\gamma,$$

and it follows from (IV.18) and (IV.5) that ν_ε goes to zero in $(C_c^{0,\gamma})'$ for any $0 < \gamma < 1$. The compactness of $N_\varepsilon^{-1}\mu_\varepsilon$ in $(C_c^{0,\gamma})'$ is true because of its boundedness in $(C^0)'$ and the compact embedding of $C^{0,\gamma}$ in C^0 (see [JS1]). It follows that $N_\varepsilon^{-1}J_\varepsilon$ subsequentially converges in $(C_c^{0,\gamma})'$ to the same limit as $N_\varepsilon^{-1}\mu_\varepsilon$, i.e. to a measure. But recall $J_\varepsilon = Ju_\varepsilon(\partial_v, \partial_w)$ so that by choosing different coordinates we get convergence for the other components and conclude that the normalized Jacobians $N_\varepsilon^{-1}Ju_\varepsilon$ subsequentially converge in $(C_c^{0,\gamma})'$ to a measure valued 2-form.

The proof of (IV.17) is straightforward. If $e_\varepsilon(\sigma) < M(\varepsilon)$, Proposition IV.2 applies and, $\mu_{\varepsilon,\sigma}$ being defined by (IV.7),

$$\|J_{\varepsilon,\sigma} - \mu_{\varepsilon,\sigma}\|_{(C_c^{0,1})'} \leq C\lambda^2 \frac{e_\varepsilon(\sigma)}{M(\varepsilon)},$$

where $C > 0$ is an absolute constant, while from (IV.6),

$$\|\mu_{\varepsilon,\sigma}\| |\log \varepsilon| \leq Ce_\varepsilon(\sigma).$$

Thus (IV.17) is verified.

In the case $e_\varepsilon(\sigma) > M(\varepsilon)$ we let $\mu_{\varepsilon,\sigma} = 0$. Then if ξ is a smooth compactly supported function, an integration by parts yields

$$\int_{\Omega_\sigma} \xi J_{\varepsilon,\sigma} = -\frac{1}{2} \int_{\Omega_\sigma} d\xi \wedge j_{\varepsilon,\sigma},$$

where $j_{\varepsilon,\sigma}$ is the current restricted to the slice and $d\xi$ is the differential of ξ in the slice also. The last integral may be bounded by $\|j_\varepsilon\|_{L^1} \|\xi\|_{C^{0,1}}$. There remains to prove that

$$(IV.19) \quad \|j_\varepsilon\|_{L^1(\Omega_\sigma)} \leq CM(\varepsilon)^{-1/2} e_\varepsilon(\sigma).$$

From the identity $j_\varepsilon = \rho^2 d\varphi$ where $u_\varepsilon = \rho e^{i\varphi}$ it follows easily that

$$|j_\varepsilon| \leq |u_\varepsilon|^2 - 1 |\nabla u_\varepsilon| + |\nabla u_\varepsilon|,$$

and then

$$\|j_\varepsilon\|_{L^1(\Omega_\sigma)} \leq C(\varepsilon e_\varepsilon(\sigma) + e_\varepsilon(\sigma)^{1/2}).$$

The bound (IV.19) follows by noting that if $e_\varepsilon(\sigma) > M(\varepsilon)$ then $e_\varepsilon(\sigma)^{1/2} < M(\varepsilon)^{-1/2} e_\varepsilon(\sigma)$. This concludes the proof of (IV.19), (IV.17) and the compactness of $N_\varepsilon^{-1}Ju_\varepsilon$. \square

Proof of the rectifiability. Rectifiability of the limit requires that N_ε be a constant, which we assume here. It is proved in [JS1] and uses a rectifiability criterion which has been investigated recently by several authors (see [W], [JS3] and also [AK]) which involves slices of currents. Let T be an $(n-2)$ -current in \mathbb{R}^n and let $(v, w, \sigma) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2}$ denote coordinates as above. The 0-dimensional currents $\{T_\sigma\}_\sigma$ are said to be the slices of T under the map $(v, w, \sigma) \rightarrow \sigma$ if T_σ has support in the plane $\{(v, w, \sigma)/v, w \in \mathbb{R}\}$ and for any smooth ξ ,

$$(IV.20) \quad \int_{\mathbb{R}^{n-2}} T_\sigma(\xi) = T(\xi d\sigma_1 \wedge \cdots \wedge d\sigma_{n-2}).$$

(see [AK]). Let $J_\varepsilon = Ju_\varepsilon(\partial_v, \partial_w)$. Then the restrictions $J_{\varepsilon, \sigma}$ of J_ε to Ω_σ are the slices of the current Ju_ε in the above sense, where k -forms are freely identified with $(n-k)$ -dimensional currents.

An $(n-2)$ -current T in \mathbb{R}^n is then rectifiable (see [W], [JS3] and also [AK]) if and only if almost every slice under projections on any coordinate plane is a rectifiable 0-dimensional current. If the slices are in addition integer-multiplicity then so is T . Letting J be the limit of Ju_ε , we must then identify its slices. To this aim, let ξ be a smooth function compactly supported in Ω and

$$f_{\varepsilon, \xi}(\sigma) = \int_{\Omega_\sigma} J_{\varepsilon, \sigma} \xi.$$

The function $f_{\varepsilon, \xi}$ is bounded in $BV_{\text{loc}}(\mathbb{R}^{n-2}, \mathbb{R})$ independently of ε . Indeed, following [AK], for any smooth compactly supported $\psi : \mathbb{R}^{n-2} \rightarrow \mathbb{R}$, and using the identity $dJu_\varepsilon = 0$, we have

$$\int_{\mathbb{R}^{n-2}} f_{\varepsilon, \xi}(\sigma) \partial_{\sigma_i} \psi = \int_{\Omega} \xi d\psi \wedge Ju_\varepsilon \wedge (\star d\sigma_i) = - \int_{\Omega} \psi \partial_{\sigma_i} \xi Ju_\varepsilon \wedge d\sigma,$$

where \star denotes the Hodge operator with respect to the $n-2$ variables $\sigma_1, \dots, \sigma_{n-2}$ and $d\sigma = d\sigma_1 \wedge \cdots \wedge d\sigma_{n-2}$. It follows that

$$\left| \int_{\mathbb{R}^{n-2}} f_{\varepsilon, \xi}(\sigma) \partial_{\sigma_i} \psi \right| \leq C \|\psi\|_{C^0} \|Ju_\varepsilon\|_{(C_c^{0,1})'} \|\xi\|_{C^2}.$$

Since $\|Ju_\varepsilon\|_{(C_c^{0,1})'}$ is bounded independently of ε (see (IV.8) and (IV.6)), the result follows. Therefore by compact embedding, $f_{\varepsilon, \xi}$ converges subsequentially as $\varepsilon \rightarrow 0$ in $L^1(\mathbb{R}^{n-2})$ and almost everywhere. This is true for any ξ , thus using a diagonal argument, we may extract a subsequence such that $f_{\varepsilon, \xi}$ converges for a.e. σ and any ξ in a countable dense subset A of $C_c^2(\Omega)$ to some $f_\xi(\sigma)$. Let us identify this limit.

Defining $e_\varepsilon(\sigma), \mu_{\varepsilon, \sigma}$ as in (IV.16), (IV.17), and since $\{|\log \varepsilon|^{-1} e_\varepsilon(\sigma)\}_\varepsilon$ is bounded in $L^1(\mathbb{R}^{n-2})$, for a.e. σ , there exists a subsequence $\varepsilon' \rightarrow 0$ — depending on σ — such that $|\log \varepsilon'|^{-1} e_{\varepsilon'}(\sigma)$ is bounded, which implies using (IV.17) that $\{\mu_{\varepsilon', \sigma}\}_{\varepsilon'}$ is bounded in $(C^0)'$.

A subsequence then converges to a weak limit μ_σ and

$$\int_{\Omega_\sigma} \xi d\mu_\sigma = \lim_{\varepsilon' \rightarrow 0} \int_{\Omega_\sigma} \xi d\mu_{\varepsilon', \sigma} = \lim_{\varepsilon' \rightarrow 0} \int_{\Omega_\sigma} \xi J_{\varepsilon', \sigma} = f_\xi(\sigma),$$

for a.e. σ and $\xi \in A$. Now $\pi^{-1}\mu_\sigma$ is the limit of a linear combination of a bounded number (depending on σ) of Dirac masses with integer coefficients, and therefore is such a combination itself. Moreover

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^{n-2}} \int_{\Omega_\sigma} J_{\varepsilon, \sigma} \xi = \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^{n-2}} f_{\varepsilon, \xi}(\sigma) = \int_{\mathbb{R}^{n-2}} f_\xi(\sigma) = \int_{\mathbb{R}^{n-2}} \int_{\Omega_\sigma} \xi d\mu_\sigma,$$

which proves that the slices of $\pi^{-1}J$ under the map $(v, w, \sigma) \rightarrow \sigma$ are the measures $\{\mu_\sigma\}_\sigma$ which are integer multiplicity rectifiable, for a.e. σ . We deduce the rectifiability and integer-multiplicity of J from Theorem 8.1 of [AK]. Note that J is not necessarily a normal current but as noted in [DL], the conclusion of Theorem 8.1 in [AK] remains valid if J is a *local* normal current, i.e. if its boundary has locally finite mass. Here we have the stronger property that the boundary of the current J *vanishes* locally in Ω , i.e. for any $n-1$ -form α compactly supported in Ω

$$\int_{\Omega} J \wedge d\alpha = 0.$$

□

Proof of the lower bound. The lower bound (IV.15) is trivial if X and Y are colinear, thus we assume they are not and we choose a system of coordinates (v, w, σ) such that the span of (X, Y) is the plane $\{\sigma = 0\}$. Then we define $\mu_\varepsilon, J_\varepsilon, \mu_{\varepsilon, \sigma}, J_{\varepsilon, \sigma}$ as above. On a slice Ω_σ , we let g be the metric such that $g(X, X) = g(Y, Y) = 1$ and $g(X, Y) = 0$. Then Proposition IV.2 implies that for any σ such that $e_\varepsilon(\sigma) < M(\varepsilon)$ there exists a collection of balls $\{B_i\}_i$ for the metric g in Ω_σ satisfying the properties there described. Then for any smooth η compactly supported in Ω , it follows from (IV.6) that

$$(IV.21) \quad \frac{1}{2|\log \varepsilon|} \int_{B_i} \lambda^{-1} |\eta X \cdot \nabla u_\varepsilon|^2 + \lambda |\eta Y \cdot \nabla u_\varepsilon|^2 \frac{dv dw}{|X \wedge Y|} \geq \pi |d_i| \left(\min_{B_i} \eta^2 - o(1) \right).$$

Besides, writing $B_i = B_g(a_i, r_i)$, we have $\min_{B_i} \eta^2 \geq \eta^2(a_i) - C\lambda r_i \|\eta\|_{C^{0,1}}$. Also $\sum_i r_i \leq \lambda/M(\varepsilon)$. Plugging in (IV.21) and summing over i , we have

$$(IV.22) \quad \frac{1}{2|\log \varepsilon|} \int_{\cup_i B_i} \lambda^{-1} |\eta X \cdot \nabla u_\varepsilon|^2 + \lambda |\eta Y \cdot \nabla u_\varepsilon|^2 \frac{dv dw}{|X \wedge Y|} \geq \left| \int_{\Omega_\sigma} (\eta^2 - o(1)) d\mu_{\varepsilon, \sigma} \right|,$$

where $o(1)$ is a quantity that tends to 0 when $\varepsilon \rightarrow 0$ independently of σ . This is in fact true for every σ because $\mu_{\varepsilon, \sigma}$ was set to be 0 if $e_\varepsilon(\sigma) > M(\varepsilon)$. Integrating (IV.22) w.r.t. σ , we find

$$\frac{1}{2|\log \varepsilon|} \int_{A_\varepsilon} \lambda^{-1} |\eta X \cdot \nabla u_\varepsilon|^2 + \lambda |\eta Y \cdot \nabla u_\varepsilon|^2 \geq \left| \int_{\Omega} (\eta^2 - o(1)) d\mu_\varepsilon \right|,$$

where $A_\varepsilon = \cup_\sigma \cup_i B_i(\varepsilon, \sigma)$. In particular the Lebesgue measure of A_ε is bounded above by $C(\Omega, \lambda)M(\varepsilon)^{-1}$ and therefore goes to 0 when $\varepsilon \rightarrow 0$. Dividing the above inequality by N_ε we find

$$\liminf_{\varepsilon \rightarrow 0} \frac{1}{2N_\varepsilon |\log \varepsilon|} \int_{A_\varepsilon} \lambda^{-1} |\eta X \cdot \nabla u_\varepsilon|^2 + \lambda |\eta Y \cdot \nabla u_\varepsilon|^2 \geq |X \wedge Y| \left| \int_{\Omega} \eta^2 J(\partial_v, \partial_w) \right|,$$

where J is the limit of $N_\varepsilon^{-1} J u_\varepsilon$. The proposition is proved by noting that

$$J(\eta X, \eta Y) = |X \wedge Y| \eta^2 J(\partial_v, \partial_w).$$

□

IV.4 Proof of Theorem 1, completed

Let X, Y be continuous vector fields compactly supported in Ω . It follows from (I.5) that

$$(IV.23) \quad j_{\varepsilon, X} = \frac{|X \cdot \nabla u_\varepsilon|}{\sqrt{N_\varepsilon |\log \varepsilon|}}, \quad j_{\varepsilon, Y} = \frac{|Y \cdot \nabla u_\varepsilon|}{\sqrt{N_\varepsilon |\log \varepsilon|}}$$

are bounded in L^2 and therefore converge weakly subsequentially. We fix a convergent subsequence and let j_X, j_Y denote the weak L^2 limits. Then

$$(IV.24) \quad |j_{\varepsilon, X}|^2 \rightharpoonup |j_X|^2 + \nu_X, \quad |j_{\varepsilon, Y}|^2 \rightharpoonup |j_Y|^2 + \nu_Y,$$

weakly as measures, where ν_X and ν_Y are positive Radon measures, called the defect measures of the sequences.

We are going to approximate X and Y by constant vector fields. Let K denote the union of the supports of X and Y . Choose $\alpha > 0$ smaller than the distance of K to $\partial\Omega$. Let $\mathcal{B} = \{B_1, \dots, B_n\}$ be a covering of K by balls of radius α . Then there exists a partition of unity $\eta_1^2, \dots, \eta_n^2, \eta_K^2$ subordinate to $\mathcal{B} \cup \{\mathbb{R}^n \setminus K\}$, where for every $1 \leq k \leq n$ the function η_k^2 has compact support in Ω and for every $x \in K$

$$(IV.25) \quad \sum_{k=1}^n \eta_k^2(x) = 1.$$

We let X_k, Y_k denote the average value of X, Y on B_k . Then

$$(IV.26) \quad \delta(\alpha) = \sup_{\substack{1 \leq k \leq n \\ x \in B_k}} \{|X_k - X(x)|, |Y_k - Y(x)|\} \xrightarrow{\alpha \rightarrow 0} 0.$$

and

$$(IV.27) \quad |\eta_k(X - X_k)| \leq \delta(\alpha), \quad |\eta_k(Y - Y_k)| \leq \delta(\alpha).$$

We use Proposition IV.3 for every k to find sets $A_{\varepsilon,k}$ of measure tending to 0 such that

$$\liminf_{\varepsilon \rightarrow 0} \frac{1}{2N_\varepsilon |\log \varepsilon|} \int_{A_{\varepsilon,k}} \lambda^{-1} |\eta_k X_k \cdot \nabla u_\varepsilon|^2 + \lambda |\eta_k Y_k \cdot \nabla u_\varepsilon|^2 \geq \left| \int_{\Omega} J(\eta_k X_k, \eta_k Y_k) \right|,$$

for every $1 \leq k \leq n$. Using (IV.27) we find

$$\liminf_{\varepsilon \rightarrow 0} \frac{1}{2N_\varepsilon |\log \varepsilon|} \int_{A_{\varepsilon,k}} \lambda^{-1} |\eta_k X \cdot \nabla u_\varepsilon|^2 + \lambda |\eta_k Y \cdot \nabla u_\varepsilon|^2 \geq \left| \int_{\Omega} J(\eta_k X, \eta_k Y) \right| - C\delta(\alpha).$$

Letting $A_\varepsilon = \cup_k A_{\varepsilon,k}$ and summing over k yields, in view of (IV.25),

$$(IV.28) \quad \liminf_{\varepsilon \rightarrow 0} \frac{1}{2N_\varepsilon |\log \varepsilon|} \int_{A_\varepsilon} \lambda^{-1} |X \cdot \nabla u_\varepsilon|^2 + \lambda |Y \cdot \nabla u_\varepsilon|^2 \geq \left| \int_{\Omega} J(X, Y) \right| - C\delta(\alpha).$$

We claim that

$$(IV.29) \quad \frac{1}{2} (\lambda^{-1} \|\nu_X\| + \lambda \|\nu_Y\|) \geq \liminf_{\varepsilon \rightarrow 0} \frac{1}{2N_\varepsilon |\log \varepsilon|} \int_{A_\varepsilon} \lambda^{-1} |X \cdot \nabla u_\varepsilon|^2 + \lambda |Y \cdot \nabla u_\varepsilon|^2.$$

Let us see how Theorem 1 follows. Using (IV.28) and (IV.29) and letting α tend to 0 we find

$$\frac{1}{2} (\lambda^{-1} \|\nu_X\| + \lambda \|\nu_Y\|) \geq \left| \int_{\Omega} J(X, Y) \right|.$$

Minimizing the left-hand side w.r.t λ yields the conclusion.

Inequality (IV.29) is quite standard. Choose a subsequence $\varepsilon_n \rightarrow 0$ such that the measure of $A_n = \cup_{m \geq n} A_{\varepsilon_m}$ goes to 0 when $n \rightarrow +\infty$. For any n we have $\|\nu_X\| \geq \nu_X(A_n)$ and similarly for ν_Y . But from (IV.24),

$$\nu_X(A_n) + \int_{A_n} |j_X|^2 = \liminf_{m \rightarrow +\infty} \int_{A_n} |j_{\varepsilon_m, X}|^2 \geq \liminf_{m \rightarrow +\infty} \int_{A_{\varepsilon_m}} |j_{\varepsilon_m, X}|^2.$$

Letting n go to $+\infty$, since the measure of A_n goes to 0, we find

$$\|\nu_X\| \geq \liminf_{m \rightarrow +\infty} \int_{A_{\varepsilon_m}} |j_{\varepsilon_m, X}|^2,$$

and a similar inequality holds for ν_Y . A linear combination of these two inequalities, in view of (IV.23), yields (IV.29).

References

- [ABO] G. Alberti, S. Baldo and G. Orlandi, Variational convergence for functionals of Ginzburg-Landau type, preprint.
- [AK] L. Ambrosio and B. Kirchheim, Currents in metric spaces, *Acta Math.* 185, No.1, (2000), 1-80.
- [ASS] A. Aftalion, E. Sandier and S. Serfaty, Pinning Phenomena in the Ginzburg-Landau Model of Superconductivity, *J. Math. Pures Appl.*, 80, No 3, (2001), 339-372.
- [BBH] F. Bethuel, H. Brezis and F. Hélein, *Ginzburg-Landau Vortices*, Birkhäuser, (1994).
- [BBO] F. Bethuel, H. Brezis and G. Orlandi, Asymptotics for the Ginzburg-Landau equation in arbitrary dimensions, *J. Func. Anal.*, 186, (2001), 432-520.
- [BBM] J. Bourgain, H. Brezis, and P. Mironescu, $H^{\frac{1}{2}}$ maps with values into the circle: minimal connections, lifting, and the Ginzburg-Landau equations, forthcoming.
- [BCL] H. Brezis, J. M. Coron, and E. Lieb, Harmonic maps with defects, *Comm. Math. Phys.*, 107, (1986), 649-705.
- [BMR] H. Brezis, F. Merle and T. Rivière, Quantization effects for $-\Delta u = u(1 - |u|^2)$ in R^2 , *Arch. Rat. Mech. Anal.*, 126, (1994), no. 1, 35-58.
- [DL] C. De Lellis, Some remarks on the distributional Jacobian, preprint.
- [J1] R. Jerrard, Lower Bounds for Generalized Ginzburg-Landau Functionals, *SIAM Journal Math. Anal.*, 30, No 4, (1999), 721-746.
- [J2] R. Jerrard, Vortex dynamics for the Ginzburg-Landau wave equation, *Calc. Var. PDE*, 9, (1999), 1-30.
- [JS1] R.L. Jerrard and H.M. Soner, The Jacobian and the Ginzburg-Landau functional, *Calc. Var.*, 14, (2002), No 2, 151-191.
- [JS2] R.L. Jerrard and H.M. Soner, Limiting behavior of the Ginzburg-Landau energy, *J. Funct. Analysis*, 192, (2002), No 2, 524-561.
- [JS3] R.L. Jerrard and H.M. Soner, Functions of bounded n -variation, forthcoming.
- [LR] F.H. Lin and T. Rivière, Complex Ginzburg-Landau equations in high dimensions and codimension two area minimizing currents, *J. Eur. Math. Soc.*, 1, (1999), 237-311.
- [Ri] T. Rivière, Line vortices in the $\mathbb{U}(1)$ -Higgs model, *ESAIM: COCV*, 1, (95/96), 77-167.
- [Sa1] E. Sandier, Lower Bounds for the Energy of Unit Vector Fields and Applications, *J. Funct. Analysis*, 152, No 2, (1998), 379-403.

- [Sa2] E. Sandier, Ginzburg-Landau minimizers from \mathbb{R}^{n+1} to \mathbb{R}^n and minimal connections, *Indiana Univ. Math. J.*, 50, (2001), No 4, 1807-1844.
- [SS1] E. Sandier and S. Serfaty, Global Minimizers for the Ginzburg-Landau Functional below the First Critical Magnetic Field, *Annales IHP, Analyse non linéaire.* 17, 1, (2000), 119-145.
- [SS2] E. Sandier and S. Serfaty, On the Energy of Type-II Superconductors in the Mixed Phase, *Reviews in Math. Phys.*, 12, No 9, (2000), 1219-1257.
- [SS3] E. Sandier and S. Serfaty, A Rigorous Derivation of a Free-Boundary Problem Arising in Superconductivity, *Annales Sci. Ecole Normale Sup.*, 4e série, 33, (2000), 561-592.
- [SS4] E. Sandier and S. Serfaty, Limiting Vorticities for the Ginzburg-Landau Equations, to appear in *Duke Math. J.*
- [SS5] E. Sandier and S. Serfaty, Gamma-Convergence of Gradient-Flows and Application to Ginzburg-Landau, forthcoming.
- [T] M. Tinkham, *Introduction to Superconductivity*, 2d edition, McGraw-Hill, (1996).
- [W] B.White, Rectifiability of flat chains, *Ann. of Math.* (2) 150, No.1, (1999). 165-184.