The decrease of bulk-superconductivity close to the second critical field in the Ginzburg-Landau model

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Abstract

We study solutions of the Ginzburg-Landau equations describing superconductors in a magnetic field, just below the "second critical field" H_{c_2}. We thus bridge between situations described in [SS2] and [P1]. We prove estimates on the energy, among which one by an algebraic trick inspired by the Bogomoln'yi trick for self-duality. We thus show how, for energy-minimizers, superconductivity decreases in average in the bulk of the sample when the applied field increases to H_{c_2}.

I Introduction

Superconductivity is modelled by the 2D Ginzburg-Landau free energy

$$J(u,A) = \frac{1}{2}\int_\Omega |\nabla_A u|^2 + |h-h_{\text{ex}}|^2 + \frac{\kappa^2}{2}(1-|u|^2)^2. \tag{I.1}$$

We are interested in studying critical points of this energy when the applied field h_{ex} gets close (from below) to the "second critical field" H_{c_2}.

Let us first explain the notations. Ω is a smooth, bounded, simply connected domain of $\mathbb{R}^2$, corresponding to the section of an infinite cylindrical body. J is a function of u, the "order parameter", complex-valued function, and of the "vector-potential $A : \Omega \mapsto \mathbb{R}^2$. u

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indicates the local state of the material, $|u|^2 \leq 1$ being the local density of superconducting electrons. Roughly speaking, where $|u| \sim 1$ it is the "superconducting phase", while where $|u| \sim 0$, it is the "normal phase". A is the potential associated to the magnetic field $h = \operatorname{curl} A = \partial_1 A_2 - \partial_2 A_1$ (real-valued function) that exists in the sample. ∇_A denotes the covariant derivation $\nabla - iA$: it is an abelian gauge theory, and everything is invariant under gauge-transformations : $u \to ue^{i\Phi}$, $A \to A + \nabla\Phi$. A configuration is really an orbit of gauge-equivalent couples (u, A).

The parameter h_{ex} is the intensity of the applied magnetic field (assumed to be uniform, parallel to the cylinder axis). Finally κ is the Ginzburg-Landau parameter, it is the ratio of two characteristic lengthes of the material.
The equations associated to this functional are the Ginzburg-Landau equations

$$-\nabla_A^2 u = \kappa^2 u(1 - |u|^2) \quad \text{in } \Omega \tag{I.2}$$

$$-\nabla^\perp h = < iu, \nabla_A u > \quad \text{in } \Omega \tag{I.3}$$

$$h = h_{\mathrm{ex}} \quad \text{on } \partial\Omega \tag{I.4}$$

$$(\nabla u - iAu) \cdot \nu = 0 \quad \text{on } \partial\Omega, \tag{I.5}$$

where $\nabla^\perp$ denotes $(-\partial_2, \partial_1)$ and $< .,. >$ is the scalar product in $\mathbb{C}$ identified with $\mathbb{R}^2$.

When type-II superconductors are submitted to a magnetic field, they exhibit phase transitions for certain critical fields, denoted H_{c_1}, H_{c_2}, and H_{c_3}. When $h_{\mathrm{ex}} \leq H_{c_1}$, the sample is in the superconducting phase everywhere and repels the magnetic field (it is called the Meissner effect). At H_{c_1}, there is a phase transition where vortices appear. Vortices are zeros of the order parameter u around which u has a non-zero winding number. (For a mathematical description of vortices in Ginzburg-Landau without magnetic field, see [BBH] and subsequent works.) As h_{ex} increases, vortices get more and more numerous and tend to arrange in a triangular lattice, called "Abrikosov lattice". When $H_{c_2} \leq h_{\mathrm{ex}} \leq H_{c_3}$, the material is in the normal phase everywhere except on a layer near the boundary where superconductivity persists, while for $h_{\mathrm{ex}} \geq H_{c_3}$, it is normal everywhere ($u \equiv 0$). For a more thorough physical presentation, one may see [DeG, SST, T].

We are interested in the "London limit" $\kappa \to +\infty$. We will also write $\varepsilon = \frac{1}{\kappa}$. ε is the lengthscale of a vortex. Letting $\varepsilon \to 0$ corresponds to having vortices that are small compared to the scale of the sample.

Mathematically, a lot of results have been proved on this functional. Let us start with the situation around the third critical field H_{c_3}. First, observe that there is always a trivial normal solution ($u \equiv 0, h \equiv h_{\mathrm{ex}}$), and that its energy is $\frac{1}{4}|\Omega|\kappa^2$. When the applied field h_{ex} is decreased to H_{c_3}, there is a bifurcation from that normal solution to a branch of solutions with superconductivity on the boundary. This superconductivity actually first appears at H_{c_3} near the point of maximal curvature of the boundary.

The story goes back to Saint James and DeGennes [SdG] and later Chapman [C] who studied the bifurcation in the half-space, based on formal analysis. Rigorously, it was proved by Giorgi and Phillips [GP] that $H_{c_3} = O(\kappa^2)$ and that above H_{c_3} the only solution is the normal one. Then, in the particular case of a disc-domain, Bauman, Phillips and

Tang [BPT] considered radially symmetric solutions bifurcating from eigenfunctions. In a general domain, a formula relating H_{c_3} to the curvature of the boundary, as well as result showing that eigenfunctions concentrate around the points of maximal curvature of the boundary, were first given by Bernoff and Sternberg in [BS], through a formal analysis, by Del-Pino-Felmer-Sternberg [DFS], and simultaneously by Lu and Pan in [LP3], based on the linear analysis of [LP1, LP2]. Finally, Helffer and Pan obtained in [HP] the most accurate result, using the analysis of [HM], that is that superconductivity first appears at H_{c_3} near the point of maximum curvature of the boundary and that

$$H_{c_3} \sim_{\kappa\to\infty} \frac{\kappa^2}{\beta_0} + \left(\frac{C_1}{\beta_0^{3/2}} k_{max}\right)\kappa, \tag{I.6}$$

where β_0 is the lowest eigenvalue of a Schrödinger operator with magnetic field in the half-plane, and k_{max} is the maximum of the curvature on the boundary of Ω.

Let us now turn to the situation further below H_{c_3}. Recently, Pan proved in [P1] a very nice result describing global minimizers of the energy between H_{c_2} and H_{c_3}. He showed that H_{c_2} can be defined as the infimum of h_{ex} such that global minimizers of J do not have *bulk-superconductivity* but only surface-superconductivity, and that

$$H_{c_2} \sim_{\kappa\to\infty} \kappa^2. \tag{I.7}$$

Following his notations, we define b by

$$h_{\text{ex}} = (b + o(1))\kappa^2, \tag{I.8}$$

and will also denote by J_D the Ginzburg-Landau functional restricted to a subdomain D of Ω.

He proved the following

Theorem (Pan [P1]) *Let (u, A) be a minimizer of J. For $1 < b < \frac{1}{\beta_0}$, there exist positive numbers E_b and κ_b such that for $\kappa > \kappa_b$,*

$$J(u, A) \sim_{\kappa\to\infty} \frac{|\Omega|}{4}\kappa^2 - \kappa E_b |\partial\Omega| + o(\kappa), \tag{I.9}$$

where $|\Omega|$ denotes the volume of Ω and $|\partial\Omega|$ denotes the length of $\partial\Omega$. For any closed subdomain D of $\overline{\Omega}$, for $\kappa > \kappa_D$,

$$J_D(u, A) \sim_{\kappa\to\infty} \frac{|D\cap\Omega|}{4}\kappa^2 - \kappa E_b |D \cap \partial\Omega| + o(\kappa). \tag{I.10}$$

Moreover, $\frac{1}{\kappa}|\nabla_A u|$ and $|u|$ exponentially decay in the interior of Ω, in the sense that for all $\alpha > 0$, for $\kappa > \kappa(\alpha)$,

$$\int_\Omega \left(|u|^2 + \frac{1}{\kappa^2}|\nabla_A u|^2\right) exp(\alpha\kappa dist(x, \partial\Omega))\, dx \le \frac{O(1)}{\kappa}.$$

He also proved results for the case $b = 1$. Let us point out that slightly stronger exponential-decay results have been proved by Almog in [Al1] replacing the large-kappa limit by the large-domain limit.

Thus, when $h_{\rm ex}$ is decreased and crosses H_{c_3}, superconductivity first nucleates at the points of maximal curvature of the boundary and u is a small perturbation of the normal solution 0. As $h_{\rm ex}$ further decreases, a uniform superconducting sheath of scale $\varepsilon = \frac{1}{\kappa}$ rapidly forms on the entire boundary of the sample while the bulk remains normal as shown in the previous theorem. Superconductivity increases on the boundary as $b \to 1$.

On the other hand, the situation is also well understood for small applied fields : the superconducting state below H_{c_1} has been studied in [S1, S3, SS1], the value of H_{c_1} being asymptotic to $C(\Omega)\log\kappa$ as proved in [S1, SS1, SS5]. Above H_{c_1}, we showed vortices appear first near the center of the domain [S1, S2], and a vortex region where the density of vortices is uniform and proportional to $h_{\rm ex}$, surrounded by a purely superconducting region, forms and inflates (see [SS3]). As soon as $h_{\rm ex} \gg \log\kappa$, the vortex region covers up the whole sample, and we proved the following

Theorem (Sandier-Serfaty [SS2]) *Assume $h_{\rm ex}$ is any function of κ such that $\log\kappa \ll h_{\rm ex} \ll \kappa^2$ as $\kappa \to \infty$. If (u, A) is a corresponding minimizer of J then*

$$J(u, A) \sim_{\kappa\to\infty} \frac{1}{2}|\Omega| h_{\rm ex} \log \frac{\kappa}{\sqrt{h_{\rm ex}}} \tag{I.11}$$

where $|\Omega|$ denotes the volume of Ω; and if D is any closed subdomain of $\overline{\Omega}$, then

$$J_D(u, A) \sim_{\kappa\to\infty} \frac{1}{2}|D| h_{\rm ex} \log \frac{\kappa}{\sqrt{h_{\rm ex}}}. \tag{I.12}$$

Moreover, the density of vortices converges in some sense to the uniform density $h_{\rm ex}$.

In this regime $h_{\rm ex} \ll \kappa^2$, the superconducting phase surrounding the vortices still dominates, in the sense that, from estimate (I.11), $\int_\Omega (1 - |u|^2)^2 = o(1)$. Essentially, one can think of the vortices as of degree one and placed regularly, for example on a periodic lattice, one per cell of size $\frac{1}{\sqrt{h_{\rm ex}}}$, which remains much larger than their characteristic size ε (as long as $h_{\rm ex} \ll \kappa^2$).

The question is thus to bridge the gap between the situations of these two theorems (that of $h_{\rm ex} \ll \kappa^2$, i.e. $b = 0$, and that above H_{c_2} i.e. $b > 1$) in the only range of applied fields which remained unstudied : $b \in [0, 1]$. How do the vortices disappear and how does the bulk superconductivity disappear? Essentially two scenarii could be suggested. One is that as $h_{\rm ex}$ increases, the distance between the vortices decreases and before it becomes smaller than their size $O(\varepsilon)$, the vortices merge into one "giant vortex" of large degree. The other scenario is that $\max|u|$ decreases in the bulk, while the vortex array structure remains unchanged, until $|u|$ is close to 0 in the bulk, and superconductivity only remains on the boundary, as described by Pan. It is considered by physicists that it is the second scenario rather than the first which occurs, at least in this limit $\kappa \to \infty$, and the results we prove confirm this. However, giant vortices do occur (and are observed) for smaller κ.

We start with a general lower bound result, proved through a very simple argument. Introducing the operator $\mathcal{D}_A = \partial_1 + i\partial_2 - i(A_1 + iA_2)$, we have the identity

$$|\mathcal{D}_A u|^2 = |\nabla_A u|^2 - \operatorname{curl}(iu, \nabla_A u) - |u|^2 h. \tag{I.13}$$

This operator is the one that was used by Bogomoln'yi (see [JT]) to exhibit the self-duality of the Ginzburg-Landau equations for $\kappa = \frac{1}{\sqrt{2}}$. By a purely algebraic manipulation quite similar to the trick of Bogomoln'yi (the same kind of manipulation was also behind the results of [M] and [GP]), we deduce from (I.13) the following *nontrivial* lower bound.

Proposition 1 *Let (u, A) be any solution of the Ginzburg-Landau system (I.2)—(I.5)), then, for all $R_\kappa \gg \varepsilon = \frac{1}{\kappa}$ and all balls B_{R_κ} in Ω,*

$$\begin{aligned}
\frac{J_{B_R}(u,A)}{|B_R|} &\geq \frac{\kappa^2}{4} + o(\kappa^2) \qquad \text{if } b \geq 1; \\
&= \left(\frac{b}{2} - \frac{b^2}{4}\right)\kappa^2 + \frac{1}{2|B_R|}\int_{B_R} |\mathcal{D}_A u|^2 + \frac{\kappa^2}{2}(1 - b - |u|^2)^2 + |h - h_{\mathrm{ex}}|^2 + o(\kappa^2) \\
&\geq \left(\frac{b}{2} - \frac{b^2}{4}\right)\kappa^2 + o(\kappa^2) \quad \text{if } b \leq 1.
\end{aligned} \tag{I.14}$$

Observe that this estimate is true for *any solution* of the equations, not necessarily minimizing or stable. It is in fact true for any configuration that satisfies the a priori estimates $\|u\|_{L^\infty(\overline{\Omega})} \leq 1$, $\|\nabla_A u\|_{L^\infty(\overline{\Omega})} \leq C\kappa$, $\|h - h_{\mathrm{ex}}\|_{L^\infty(\overline{\Omega})} \leq C\kappa$ (see the proof).

Let us now turn to energy-minimizers. We denote by $\min J_{B_R}$ the minimum of the energy-functional on a ball B_R i.e.

$$\min J_{B_R} = \min_{(u,A)} \frac{1}{2}\int_{B_R} |\nabla_A u|^2 + |\operatorname{curl} A - h_{\mathrm{ex}}|^2 + \frac{\kappa^2}{2}(1 - |u|^2)^2.$$

We also denote by $(\overline{u}, \overline{A})$ a minimizer for this problem.

Theorem 1 *Let $0 \leq b \leq 1$. There exists a continuous increasing function f from $[0,1]$ to $[0, \frac{1}{4}]$, such that, as $\kappa \to \infty$, for (u, A) any minimizer of J, for all $R_\kappa \gg \varepsilon = \frac{1}{\kappa}$, and all balls B_{R_κ} in Ω,*

$$\frac{J_{B_R}(u,A)}{\kappa^2 |B_R|} \sim \frac{\min J_{B_R}}{\kappa^2|B_R|} \longrightarrow f(b) \tag{I.15}$$

$$\frac{1}{|B_R|}\int_{B_R} |u|^4 \sim \frac{1}{|B_R|}\int_{B_R} |\overline{u}|^4 \longrightarrow 1 - 4f(b) \tag{I.16}$$

$$|u|^4 \rightharpoonup 1 - 4f(b) \quad \text{in } L^\infty \text{ weak - *} \tag{I.17}$$

$$(I.18)\qquad \frac{1-4f(b)}{1-b} - o(1) \leq \frac{1}{|B_R|}\int_{B_R} |u|^2 \leq \sqrt{1-4f(b)} + o(1),$$

and the following estimates hold : there exists universal constants $0 < \alpha < 1$ *and* $c > 0$ *such that*

$$(I.19)\qquad \frac{b}{2} - \frac{b^2}{4} \leq f(b) \leq \min\left(\frac{b}{4}(\log\frac{1}{b} + c), \frac{1-\alpha(1-b)^2}{4}\right) \leq \frac{1}{4},$$

hence

$$\alpha(1-b)^2 \leq 1 - 4f(b) \leq (1-b)^2.$$

Corollary 1 *For all* D *closed subdomain of* $\overline{\Omega}$,

$$J_D(u,A) \sim_{\kappa\to\infty} |D| f(b)\kappa^2.$$

Corollary 2 $f(0) = 0$ *and* $f(1) = \frac{1}{4}$, *therefore,*

$$\text{for } b = 0,\ \forall R_\kappa \gg \varepsilon, \qquad \lim_{\kappa\to\infty} \frac{J_{B_R}(u,A)}{\kappa^2 |B_R|} = 0 \qquad \text{(known from [SS2])}$$

$$\text{for } b \geq 1,\ \forall R_\kappa \gg \varepsilon, \qquad \lim_{\kappa\to\infty} \frac{J_{B_R}(u,A)}{\kappa^2 |B_R|} = \frac{1}{4} \qquad \text{(known from [P1] and Proposition 1).}$$

We thus show that the loss of superconductivity happens through a decrease of the average of $|u|^4$ like $(1-b)^2$ in Ω, and that the energy-repartition remains uniform. Those two facts go in the direction of the second scenario.

We have given asymptotic estimates of the minimal energy which extend that of (I.11). We have proved that the energy is uniformly spread over Ω and that a minimizer almost minimizes locally the energy, at any scale $\gg \varepsilon$ (ε being the characteristic scale of variation of u). At scales $O(\varepsilon)$ this ceases to be true, minimizers in regions of smaller sizes start to depend greatly on the region-size, as seen in [AD]. We have also shown that for global minimizers, some superconductivity remains in the bulk, as long as $b < 1$, since from (I.18) the average of $|u|^2$ remains larger than $\alpha(1-b)$.

This theorem relies on upper and lower bounds for the energy, in the spirit of Gamma-convergence. It seems difficult to give a more explicit expression, or a finer estimate on $f(b)$. The lower bound is given by Proposition 1. The upper bound is obtained by constructing test-configurations. They are chosen to be periodic with respect to a square lattice of size $\sqrt{\frac{2\pi}{b}}\varepsilon$, with a vortex of degree 1 in each cell. In view of (I.14), a minimizer (u,A) should be an almost minimizer of $\frac{1}{2}\int_\Omega |\mathcal{D}_A u|^2 + |h - b\kappa^2|^2 + \frac{\kappa^2}{2}(1-b-|u|^2)^2$. We choose our test-configuration to satisfy $|u| \leq C\sqrt{1-b}$ and also $\mathcal{D}_A u = 0$ (following somewhat the construction of [JT] of vortex solutions in the self-dual case). This configuration has of course no reason to be optimal (nor is the square-shape), but gives the right order of energy.

We get as a corollary of Proposition 1 and Theorem 1 that, for energy-minimizers,

$$\limsup_{\kappa\to\infty} \frac{1}{2\kappa^2|B_R|}\int_{B_R} |\mathcal{D}_A u|^2 + \frac{\kappa^2}{2}(1-b-|u|^2)^2 + |h-h_{\text{ex}}|^2 \le f(b) - \frac{b}{2} + \frac{b^2}{4} \le \frac{1-\alpha}{4}(1-b)^2$$

and that

$$\text{(I.20)} \qquad \frac{1}{R^2}\int_{B_R} (1-b-|u|^2)^2 \le (1-\alpha)(1-b)^2$$

from which one can deduce (I.18). It is also tempting, in view of (I.14), to think that $|u|^2 \le C(1-b)$ in the bulk.

There remains many open questions on the behavior of minimizers, which all seem quite delicate.

First of all, we conjecture that, next to an interior point of Ω, a minimizing solution should converge, after blow-up at the scale ε, to a unique limiting profile in $\mathbb{R}^2$. A much more difficult task would be to show that this limiting profile is periodic. For a study of periodic solutions of Ginzburg-Landau, see [Du], [C] and [Al2].

We have not mentioned vortices of the minimizers. It is difficult to describe them and even define them : $|u|$ becomes uniformly small, so one can no longer define the vortices as the regions where $|u|$ is small. Nevertheless, there should be vortices (they appear in our upper bound construction), with a total degree $2\pi h_{\text{ex}}$ on the boundary of Ω. Heuristically, using the second Ginzburg-Landau equation

$$-\frac{\nabla^\perp h}{\rho^2} + h = \nabla\varphi$$

where we write $u = \rho e^{i\varphi}$ in polar coordinates. Taking the curl of this equation, we are led to

$$\operatorname{div}\left(\frac{\nabla h}{\rho^2}\right) + h = \pi \sum_i d_i \delta_{a_i}$$

where the a_i are the zeros (or vortices) of u, and d_i their degrees or winding number. Since $h \to h_{\text{ex}}$ strongly, we should have, at least formally

$$2\pi \sum_i d_i \delta_{a_i} \sim h_{\text{ex}}$$

(as we had for $h_{\text{ex}} \ll \kappa^2$). However, it seems difficult to give a rigorous meaning to this statement. We can prove that on any subdomain D of volume $R^2 \gg \frac{1}{\kappa^2}$ such that $|u| > c > 0$ independently of κ on ∂D, and such that the perimeter of D is less than $O(R)$, the total degree of u on ∂D is equivalent to $h_{\text{ex}}|D|$. But the existence of such a D is not proved.

To conclude, it would be very nice, but certainly difficult, to prove a bifurcation at H_{c_2} from the surface-superconductivity solution to one of the known periodic-like vortex solution.

II The algebraic trick

From now on, we denote $h = \operatorname{curl} A$ and $u = \rho e^{i\varphi}$ in polar coordinates. Then

$$|\nabla_A u|^2 = |\nabla\rho|^2 + \rho^2|\nabla\varphi - A|^2.$$

We are interested in this section in studying families of solutions of Ginzburg-Landau, or configurations which satisfy the following a priori estimates:

Lemma II.1 *If (u, A) is a solution of Ginzburg-Landau, we have*

$$\|h - h_{\text{ex}}\|_{C^1(\overline{\Omega})} \le C\kappa, \qquad \|h - h_{\text{ex}}\|_{C^2(\overline{\Omega})} \le C\kappa^2. \tag{II.1}$$

$$\|\nabla_A u\|_{L^\infty(\overline{\Omega})} \le C\kappa \quad \|\nabla\rho\|_{L^\infty(\overline{\Omega})} \le C\kappa \tag{II.2}$$

$$e_\kappa(u, A) := |\nabla_A u|^2 + |h - h_{\text{ex}}|^2 + \frac{\kappa^2}{2}(1 - \rho^2)^2 \le C\kappa^2. \tag{II.3}$$

These estimates are proved in [HP] Proposition 4.3, see also [P1] Lemma 7.1. They rely on a blow-up at scale $\varepsilon = \frac{1}{\kappa}$, which leads to equations at scale 1, for which all the quantities are uniformly bounded.

Proof of Proposition 1 : As already mentioned, it relies on the Bogomoln'yi identity on the operator $\mathcal{D}_A = \partial_1 + i\partial_2 - i(A_1 + iA_2)$. One can check that, in polar coordinates,

$$|\mathcal{D}_A u|^2 = |\rho(\nabla\varphi - A) - \nabla^\perp\rho|^2. \tag{II.4}$$

Expanding the square on the right-hand side, one gets the crucial identity

$$|\mathcal{D}_A u|^2 = |\nabla_A u|^2 - \operatorname{curl} j - \rho^2 h, \tag{II.5}$$

where j is the superconducting current $< iu, \nabla_A u >$. Inserting (II.5) in J, we are led to

$$J_{B_R}(u, A) = \frac{1}{2}\int_{B_R} |\mathcal{D}_A u|^2 + \operatorname{curl} j + \rho^2 h + |h - h_{\text{ex}}|^2 + \frac{\kappa^2}{2}(1 - \rho^2)^2. \tag{II.6}$$

Moreover, using the fact that $|j| \le |\nabla_A u| \le C\kappa$ with (II.2), we have

$$\left|\int_{B_R} \operatorname{curl} j\right| = \left|\int_{\partial B_R} j \cdot \tau\right| \le \int_{\partial B_R} |j| \le O(R\kappa). \tag{II.7}$$

Also $h = h_{\text{ex}} + O(\kappa) = b\kappa^2 + o(\kappa^2)$ in view of (II.1). Combining these facts with (II.6) yields

$$\begin{aligned} J_{B_R}(u, A) &= \frac{1}{2}\int_{B_R} |\mathcal{D}_A u|^2 + \rho^2 b\kappa^2 + \frac{\kappa^2}{2}(1 - \rho^2)^2 + |h - h_{\text{ex}}|^2 + O(R\kappa) + o(R^2\kappa^2) \\ &= \frac{1}{2}\int_{B_R} |\mathcal{D}_A u|^2 + \kappa^2\left(\frac{1}{2} + \rho^2(b - 1) + \frac{\rho^4}{2}\right) + |h - h_{\text{ex}}|^2 + O(R\kappa) + o(R^2\kappa^2). \end{aligned} \tag{II.8}$$

If $b \geq 1$, this immediately implies that

$$\frac{J_{B_R}(u,A)}{\kappa^2|B_R|} \geq \frac{1}{4} + o(1).$$

(Thus, we see why the value $b = 1$ plays a particular role.)
If $b \leq 1$, we observe that $\rho^2(b-1) + \frac{\rho^4}{2} = \frac{1}{2}(\rho^2 - (1-b))^2 - \frac{1}{2}(1-b)^2$ and obtain

$$J_{B_R}(u,A) = |B_R|\frac{\kappa^2}{4}(1-(1-b)^2) + \frac{1}{2}\int_{B_R} |\mathcal{D}_A u|^2 + \frac{\kappa^2}{2}(1-b-\rho^2)^2 + |h-h_{\text{ex}}|^2 + O(R\kappa) + o(R^2\kappa^2). \tag{II.9}$$

We conclude that (I.14) holds. $\square$

III Energy localisation and convergence

We are now interested in families of global minimizers of J. The following lemma allows to localize all energy comparisons.

Lemma III.1 *Let R_κ be such that $R_\kappa \gg \varepsilon$. Then, for (u,A) minimizer of J, and any ball B_R of radius R_κ in Ω,*

$$\frac{J_{B_R}(u,A)}{\kappa^2|B_R|} = \frac{\min J_{B_R}}{\kappa^2|B_R|} + o(1).$$

Proof : One inequality is obvious :

$$J_{B_R}(u,A) \geq \min J_{B_R}.$$

The converse relies on a comparison argument. Let $(\tilde{u},\tilde{A})$ be a minimizer of J_{B_R}. We construct a test-configuration in Ω which coincides with (u,A) in $\Omega\backslash B_R$, and with $(\tilde{u},\tilde{A})$ in $B_{R-3\varepsilon}$.

Let χ be a $C^\infty(\Omega)$ function such that

$$\begin{cases} \chi(x) = 1 & \text{in } \Omega\backslash B_R \\ \chi(x) = 0 & \text{in } B_{R-\frac{\varepsilon}{2}}\backslash B_{R-\frac{5}{2}\varepsilon} \\ \chi(x) = 1 & \text{in } B_{R-3\varepsilon} \\ |\nabla\chi| \leq \dfrac{C}{\varepsilon} & \\ \displaystyle\int_\Omega |\nabla\chi|^2 \leq O(\frac{R}{\varepsilon}). & \end{cases} \tag{III.1}$$

We define $(\overline{u},\overline{A})$ by

$$\begin{aligned} (\overline{u},\overline{A}) &= (\chi u, A) \text{ in } \Omega\backslash B_{R-\varepsilon} \\ (\overline{u},\overline{A}) &= (\chi\tilde{u},\tilde{A}) \text{ in } B_{R-2\varepsilon}. \end{aligned}$$

There remains to extend $(\overline{u}, \overline{A})$ in $B_{R-\varepsilon} \backslash B_{R-2\varepsilon}$. We take $\overline{u} = 0$ there and may extend $\overline{A}$ in such a way that

$$\|\mathrm{curl}\,\overline{A} - h_{\mathrm{ex}}\|_{L^\infty(\Omega)} \le C\kappa, \tag{III.2}$$

(indeed, this is true for A and $\tilde{A}$.) Then, (u, A) being a minimizer of J, we have

$$\begin{aligned}\text{(III.3)}\quad 0 \ge J(u,A) - J(\overline{u}, \overline{A}) &= \int_{B_R} e_\kappa(u,A) - e_\kappa(\overline{u}, \overline{A}) \\ &= \int_{B_R \backslash B_{R-\varepsilon}} + \int_{B_{R-\varepsilon} \backslash B_{R-2\varepsilon}} + \int_{B_{R-2\varepsilon} \backslash B_{R-3\varepsilon}} + \int_{B_{R-3\varepsilon}} e_\kappa(u,A) - e_\kappa(\overline{u}, \overline{A}).\end{aligned}$$

Then,

$$\begin{aligned}\text{(III.4)}\quad & \left| \int_{B_R \backslash B_{R-\varepsilon}} e_\kappa(u,A) - e_\kappa(\overline{u}, \overline{A}) \right| \\ &= \frac{1}{2} \left| \int_{B_R \backslash B_{R-\varepsilon}} |\nabla \rho|^2 + \rho^2 |\nabla \varphi - A|^2 + \frac{\kappa^2}{2}(1-\rho^2)^2 \right. \\ &\quad \left. - \int_{B_R \backslash B_{R-\varepsilon}} |\nabla(\chi\rho)|^2 + \chi^2 \rho^2 |\nabla\varphi - A|^2 + \frac{\kappa^2}{2}(1 - \rho^2\chi^2)^2 \right| \\ &= \frac{1}{2} \left| \int_{B_R \backslash B_{R-\varepsilon}} (1-\chi^2)|\nabla_A u|^2 + \frac{\kappa^2}{2}\left((1-\rho^2)^2 - (1-\rho^2\chi^2)^2\right) - \int_{B_R \backslash B_{R-\varepsilon}} \rho^2 |\nabla \chi|^2 \right| \\ &\le O(\frac{R}{\varepsilon}),\end{aligned}$$

where we have used (II.3) and (III.1). Similarly, exchanging the roles of (u, A) and $(\tilde{u}, \tilde{A})$, we find

$$\left| \int_{B_{R-2\varepsilon} \backslash B_{R-3\varepsilon}} e_\kappa(u,A) - e_\kappa(\overline{u}, \overline{A}) \right| \le O(\frac{R}{\varepsilon}). \tag{III.5}$$

In $B_{R-\varepsilon} \backslash B_{R-2\varepsilon}$, $\overline{u} = 0$, so with (II.3) again and (III.2),

$$\left| \int_{B_{R-\varepsilon} \backslash B_{R-2\varepsilon}} e_\kappa(u,A) - e_\kappa(\overline{u}, \overline{A}) \right| \le O(\frac{R}{\varepsilon}) + \frac{1}{2} \int_{B_{R-\varepsilon} \backslash B_{R-2\varepsilon}} |\mathrm{curl}\,\overline{A} - h_{\mathrm{ex}}|^2 \le O(\frac{R}{\varepsilon}). \tag{III.6}$$

By (II.3) again, we have

$$\int_{B_{R-3\varepsilon}} e_\kappa(u,A) = \int_{B_R} e_\kappa(u,A) + O(\frac{R}{\varepsilon}) \tag{III.7}$$

$$\int_{B_{R-3\varepsilon}} e_\kappa(\tilde{u},\tilde{A}) = \int_{B_R} e_\kappa(\tilde{u},\tilde{A}) + O(\frac{R}{\varepsilon}). \tag{III.8}$$

But, since $(\tilde{u}, \tilde{A})$ minimizes J_{B_R}, we have

$$\int_{B_R} e_\kappa(u,A) \geq \int_{B_R} e_\kappa(\tilde{u}, \tilde{A}). \tag{III.9}$$

Combining this with (III.7) and (III.8), and using the fact that $(\overline{u}, \overline{A})$ is equal to $(\tilde{u}, \tilde{A})$ in $B_{R-3\varepsilon}$, we deduce that

$$\int_{B_{R-3\varepsilon}} e_\kappa(u,A) - e_\kappa(\overline{u}, \overline{A}) \geq O(\frac{R}{\varepsilon}).$$

Combining this with (III.3)—(III.7), we get

$$\left| \int_{B_R} e_\kappa(u,A) - e_\kappa(\overline{u}, \overline{A}) \right| \leq O(\frac{R}{\varepsilon}),$$

i.e.

$$J_{B_R}(u,A) = J_{B_R}(\tilde{u}, \tilde{A}) + O(R\kappa)$$

which leads to the result. □

What we did with balls in the previous lemma can be done with squares K_R of size R.

Lemma III.2 *For all $b \geq 0$, and $R_\kappa \geq R'_\kappa \gg \varepsilon$,*

$$\frac{\min J_{K_R}}{\kappa^2 |K_R|} = \frac{\min J_{K_{R'}}}{\kappa^2 |K_{R'}|} + o(1), \tag{III.10}$$

hence $\frac{\min J_{K_R}}{\kappa^2 |K_R|}$ does not depend on $R \gg \varepsilon$ (up to a $o(1)$).

Proof : Let us denote by $[.]$ the integer part of a real number. Assume first that $R' \ll R$. K_R can be split into at least $[R^2/R'^2]$ disjoint squares of size R'. Thus, for (u,A) a minimizer of J_{K_R},

$$\begin{aligned} J_{K_R}(u,A) &\geq \left[\frac{R^2}{(R')^2} \right] J_{K_{R'}}(u,A) \\ &\geq \left[\frac{R^2}{(R')^2} \right] \min J_{K'_R}. \end{aligned}$$

We deduce that

$$\frac{\min J_{K_R}}{\kappa^2 |K_R|} \geq \frac{\min J_{K_{R'}}}{\kappa^2 |K_{R'}|}(1 + o(1)).$$

Conversely, let us split K_R into $[R^2/(R')^2] + o(1)$ squares of size R' with a layer of size 3ε between them. Using the pasting procedure of Lemma III.1, we can construct a test-configuration (u,A) in K_R that agrees with the minimizer of $J_{K_{R'}}$ in each subsquare of size R', and such that

$$J_{K_R}(u,A) \leq ([R^2/(R')^2] + o(1)) \left(\min J_{K_{R'}} + C\frac{R'}{\varepsilon} \right).$$

We can check that the error terms are negligible and deduce that

$$\frac{\min J_{K_R}}{\kappa^2|K_R|} \leq \frac{\min J_{K_{R'}}}{\kappa^2|K_{R'}|}(1+o(1)).$$

Since for all R, $\frac{\min J_{K_R}}{\kappa^2|K_R|} \leq \frac{1}{4} \leq O(1)$ (by comparison with the normal solution), we deduce that (III.10) holds. If R and R' are of the same order, one may introduce R'' such that $R' \gg R'' \gg \varepsilon$. From the above, one deduces that

$$\frac{\min J_{K'_R}}{\kappa^2|K'_R|} = \frac{\min J_{K_{R''}}}{\kappa^2|K_{R''}|} + o(1)$$

and the same with R' replaced by R, from which it follows that

$$\frac{\min J_{K_R}}{\kappa^2|K_R|} = \frac{\min J_{K'_R}}{\kappa^2|K'_R|} + o(1).$$

$\square$

Lemma III.3 *For all $R_\kappa \gg \varepsilon$, $\frac{\min J_{B_R}}{\kappa^2|B_R|}$ has a limit as $\kappa \to \infty$, which depends only on b. We denote it $f(b)$. f is continuous, increasing in $[0,1]$.*

Consider $R_\kappa \gg \varepsilon$ and (u, A) a minimizer of J_{B_R}. We denote for a moment by J_{κ,B_R} the functional for κ defined on B_R (b being fixed, $h_{\text{ex}} = b\kappa^2$). Let $\lambda < 1$, and define in $B_{\frac{R}{\lambda}}$,

$$v(x) = u(\lambda x) \qquad B(x) = \lambda A(\lambda x).$$

Then, by change of variables, we have

$$\text{(III.11)}\quad \min J_{\kappa,B_R} = J_{\kappa,B_R}(u,A) = \frac{1}{2}\int_{B_{R/\lambda}} |\nabla_B v|^2 + \frac{1}{\lambda^2}|\operatorname{curl} B - \kappa^2\lambda^2 b|^2 + \frac{\kappa^2\lambda^2}{2}(1-|v|^2)^2.$$

Since $\frac{1}{\lambda} > 1$, this implies that

$$\begin{aligned}
\frac{\min J_{\kappa,B_R}}{\kappa^2 R^2} &\geq \frac{1}{2\kappa^2R^2}\int_{B_{R/\lambda}} |\nabla_B v|^2 + \frac{1}{\lambda^2}|\operatorname{curl} B - \kappa^2\lambda^2 b|^2 + \frac{\kappa^2\lambda^2}{2}(1-|v|^2)^2 \\
&\geq \frac{J_{\kappa\lambda, B_{R/\lambda}}(v,B)}{\kappa^2R^2} + \frac{1}{2\kappa^2R^2}\left(\frac{1}{\lambda^2}-1\right)\int_{B_{R/\lambda}} |\operatorname{curl} B - \kappa^2\lambda^2 b|^2 \\
&\geq \frac{\min J_{\kappa\lambda, B_{R/\lambda}}}{(\kappa\lambda)^2(R/\lambda)^2} + \frac{1}{2\kappa^2R^2}(1-\lambda^2)\int_{B_R} |\operatorname{curl} A - b\kappa^2|^2.
\end{aligned}$$

But from Lemma III.2, we have

$$\frac{\min J_{\kappa\lambda, B_{R/\lambda}}}{(\kappa\lambda)^2(R/\lambda)^2} = \frac{\min J_{\kappa\lambda, B_R}}{(\kappa\lambda)^2R^2} + o(1).$$

Hence, we deduce that for all $\lambda < 1$,

$$\text{(III.12)} \qquad \frac{\min J_{\kappa,B_R}}{\kappa^2 R^2} \geq \frac{\min J_{\kappa\lambda,B_R}}{(\kappa\lambda)^2 R^2} + o(1) + (1-\lambda^2)\frac{1}{2\kappa^2 R^2}\int_{B_R} |\operatorname{curl} A - b\kappa^2|^2.$$

Hence $\dfrac{\min J_{\kappa,B_R}}{\kappa^2 R^2}$ is monotonic (up to $o(1)$) with respect to κ, and must have a limit as $\kappa \to \infty$, which depends only on b. We denote it by $f(b)$. Letting then κ tend to infinity in (III.12), yields

$$f(b) \geq f(b) + \limsup_{\kappa\to\infty}(1-\lambda^2)\frac{1}{2\kappa^2 R^2}\int_{B_R} |\operatorname{curl} A - b\kappa^2|^2,$$

thus we also deduce that

$$\text{(III.13)} \qquad \frac{1}{2\kappa^2 R^2}\int_{B_R} |h - h_{\mathrm{ex}}|^2 = o(1).$$

This means that, for energy-minimizers, the term $\int_\Omega |h - h_{\mathrm{ex}}|^2$ is negligible in the energy. This will be helpful later.

We now prove that f is monotonic. Let still $\lambda < 1$ and let J_{b,B_R} now denote the functional restricted to B_R for the value $b\kappa^2$ of the applied field. Let us consider the same v and B defined previously. By definition,

$$J_{\lambda^2 b, B_{R/\lambda}}(v,B) = \frac{1}{2}\int_{B_{R/\lambda}} |\nabla_B v|^2 + |\operatorname{curl} B - \lambda^2 b\kappa^2|^2 + \frac{\kappa^2}{2}(1-|v|^2)^2.$$

Using (III.11),

$$\begin{aligned} J_{\lambda^2 b, B_{R/\lambda}}(v,B) &= \frac{1}{\lambda^2}J_{b,B_R}(u,A) - (\frac{1}{\lambda^2}-1)\frac{1}{2}\int_{B_{R/\lambda}} |\nabla_B v|^2 - (\frac{1}{\lambda^4}-1)\frac{1}{2}\int_{B_{R/\lambda}} |\operatorname{curl} B - \lambda^2 b\kappa^2|^2 \\ \text{(III.14)} \qquad &= \frac{1}{\lambda^2}J_{b,B_R}(u,A) - (\frac{1}{\lambda^2}-1)\frac{1}{2}\int_{B_R} |\nabla_A u|^2 - o(1), \end{aligned}$$

where we have used (III.13). Therefore,

$$\frac{\min J_{\lambda^2 b, B_{R/\lambda}}}{\kappa^2 (R/\lambda)^2} \leq \frac{J_{b,B_R}(u,A)}{\lambda^2\kappa^2(R/\lambda)^2}.$$

In view of the previous results, the left-hand side of this inequality converges to $f(\lambda^2 b)$ while the right-hand side converges to $f(b)$. We deduce that for all $\lambda < 1$,

$$f(\lambda^2 b) \leq f(b),$$

thus f is nondecreasing. One can even deduce from (III.14) that f is increasing, because $\liminf \frac{1}{\kappa^2 R^2}\int_{B_R} |\nabla_A u|^2 > 0$. Taking now $\lambda \geq 1$, we get as in (III.14) that

$$f(\lambda^2 b) \leq f(b) - \psi(\lambda)$$

where $\psi(\lambda) \to 0$ as $\lambda \to 1$. This implies that f is continuous. $\square$

In view of the result of Proposition II.1, we have, for $b \leq 1$,

$$\frac{b}{2} - \frac{b^2}{4} \leq f(b) \leq \frac{1}{4}. \tag{III.15}$$

We will prove the upper bound on f in the next section. Leaving it aside, let us now complete the proof of the theorem.

End of the proof of Theorem 1 :
Taking the scalar product of the first Ginzburg-Landau equation (I.2) with u yields the standard equation for $\rho = |u|$:

$$-\Delta\rho + \rho|\nabla\varphi - A|^2 = \kappa^2\rho(1-\rho^2). \tag{III.16}$$

Then, we multiply it by ρ and integrate. We are led, after integration by parts (using (I.5)), to

$$\int_\Omega |\nabla\rho|^2 + \rho^2|\nabla\varphi - A|^2 = \int_\Omega \kappa^2\rho^2(1-\rho^2).$$

We deduce the following relation, true for any solution of Ginzburg-Landau :

$$J(u,A) = \frac{\kappa^2}{4}\int_\Omega (1-\rho^4) + \frac{1}{2}\int_\Omega |h - h_{\mathrm{ex}}|^2. \tag{III.17}$$

In view of (III.13), if (u,A) is an energy-minimizer, this becomes

$$J(u,A) = \frac{\kappa^2}{4}\int_\Omega (1-\rho^4) + o(\kappa^2).$$

If we integrate over B_R instead of Ω, and use (II.2) to handle the boundary term, we find, still for minimizers,

$$J_{B_R}(u,A) = \frac{\kappa^2}{4}\int_{B_R} (1-\rho^4) + O(\kappa R) + o(\kappa^2 R^2). \tag{III.18}$$

Applying (III.18) to u and $\overline{u}$ successively gives (I.16).
Then,

$$\frac{1}{|B_R|}\int_{B_R} |u|^4 \to 1 - 4f(b)$$

for all $R \gg \varepsilon$, which implies the weaker conclusion that $|u|^4 \rightharpoonup 1 - 4f(b)$ in L^∞ weak-*.
We also deduce from (I.14) combined (I.19) that (I.20) holds. Plugging (I.16) in (I.20), we obtain

$$\begin{aligned}\frac{1}{|B_R|}\int_{B_R} |u|^2 &\geq \frac{(1-b)^2 + 1 - 8f(b) + 2b - b^2}{2(1-b)} \\ &\geq \frac{1-4f(b)}{1-b} \geq \alpha(1-b),\end{aligned}$$

while $\frac{1}{|B_R|}\int_{B_R}|u|^2 \leq \sqrt{\frac{1}{|B_R|}\int_{B_R}|u|^4}$ comes from the Cauchy-Schwartz inequality. This completes the proof of the theorem. $\square$

IV Contruction of test-configurations

The upper bound of Theorem 1 relies on the construction of two test-configurations, one being more interesting when $b \to 1$, the other one when $b \to 0$. Let us start with the first one, which follows somehow the construction of vortex solutions of [JT] in the self-dual situation.

Proposition 2 *With the notations of the previous section, there exists a universal constant* $0 < \alpha < 1$ *such that*

$$f(b) \leq \frac{1-\alpha(1-b)_+^2}{4}. \tag{IV.1}$$

Proof : Assume $b \leq 1$ (otherwise the conclusion is trivial). We construct a test-configuration which is periodic with respect to a square lattice of size $\sqrt{\frac{2\pi}{b}}\varepsilon$. Let $K_{\sqrt{\frac{2\pi}{b}}\varepsilon}$ denote an elementary square of the lattice and let $K_{\sqrt{2\pi}}$ denote the square of size $\sqrt{2\pi}$ centered at the origin. We solve for

$$\begin{cases} \Delta \log\rho_0 + 1 = 2\pi\delta_0 & \text{in } K_{\sqrt{2\pi}} \\ \dfrac{\partial \log\rho_0}{\partial n} = 0 & \text{on } \partial K_{\sqrt{2\pi}} \\ \displaystyle\int_{K_{\sqrt{2\pi}}} \log\rho_0 = 0. & \end{cases} \tag{IV.2}$$

There exists a (unique) solution to this system because the volume of $K_{\sqrt{2\pi}}$ is 2π. Let (r,θ) be the polar coordinates in the plane. We observe that $\log\rho_0 - \log r$ is smooth in $K_{\sqrt{2\pi}}$, hence $\frac{\rho}{r}$ too, and thus $\rho_0(0) = 0$. We then define in $K_{\sqrt{\frac{2\pi}{b}}\varepsilon}$,

$$\rho(x) = C\rho_0\left(\frac{x\sqrt{b}}{\varepsilon}\right)$$

where C minimizes $\int_{K_{\sqrt{\frac{2\pi}{b}}\varepsilon}}(1-b-C^2\rho_0^2(\frac{x\sqrt{b}}{\varepsilon}))^2$ i.e. (after a little computation)

$$\rho(x) = \sqrt{1-b}\sqrt{\frac{\int_{K_{\sqrt{2\pi}}}\rho_0^2}{\int_{K_{\sqrt{2\pi}}}\rho_0^4}}\,\rho_0\left(\frac{x\sqrt{b}}{\varepsilon}\right). \tag{IV.3}$$

One can see that ρ is solution of

$$\begin{cases} \Delta \log\rho + \frac{b}{\varepsilon^2} = 2\pi\delta_0 & \text{in } K_{\sqrt{\frac{2\pi}{b}}\varepsilon} \\ \dfrac{\partial \log\rho}{\partial n} = 0 & \text{on } \partial K_{\sqrt{\frac{2\pi}{b}}\varepsilon}. \end{cases} \tag{IV.4}$$

ρ_0 is symmetric with respect to the axes of symmetry of the square and $\frac{\partial \rho_0}{\partial n} = 0$ on $\partial K_{\sqrt{2\pi}}$, thus we may extend ρ to any ball B_R ($R \gg \varepsilon$) by periodicity and get a C^1 function, which vanishes on a lattice Λ.
We then pick A to solve

$$\begin{cases} \operatorname{curl} A = b\kappa^2 & \text{in } B_R \\ \operatorname{div} A = 0 & \text{in } B_R. \end{cases}$$

and φ to satisfy

$$\nabla\varphi = \frac{\nabla^\perp \rho}{\rho} + A = \nabla^\perp \log \rho + A. \tag{IV.5}$$

To achieve this, we fix a point x_0 of $B_R \backslash \Lambda$ and define

$$\varphi(x) = \int_{x_0}^{x} \partial_n \log \rho + A \cdot \tau.$$

This definition does not depend on the path joining x_0 to x, modulo 2π. Indeed, if $\gamma = \partial\omega$ is a closed path in $B_R \backslash \Lambda$ with positive orientation, using (IV.4), we have

$$\int_\gamma \partial_n \log \rho + A \cdot \tau = \int_\omega \Delta \log \rho + \operatorname{curl} A = \int_\omega \Delta \log \rho + b\kappa^2 = 2\pi card(\omega \cap \Lambda) \in 2\pi\mathbb{Z}.$$

Hence $e^{i\varphi(x)}$ is well-defined in $B_R \backslash \Lambda$. We then take

$$u(x) = \rho(x) e^{i\varphi(x)}$$

which has a continuous extension in B_R because ρ vanishes on Λ. Once this test-configuration (u, A) is constructed, we evaluate its energy. In view of (II.9),

$$J_{B_R}(u, A) = |B_R|\kappa^2 \left(\frac{b}{2} - \frac{b^2}{4}\right) + \frac{1}{2}\int_{B_R} |\mathcal{D}_A u|^2 + \frac{\kappa^2}{2}(1 - b - \rho^2)^2 + O(R\kappa^2). \tag{IV.6}$$

But $|\mathcal{D}_A u|^2 = |\rho(\nabla\varphi - A) - \nabla^\perp \rho|^2 = 0$ by construction, cf. (IV.5). Moreover, from (IV.3),

$$\begin{aligned} \int_{K_{\sqrt{\frac{2\pi}{b}}\varepsilon}} (1 - b - \rho^2)^2 &= (1-b)^2 \int_{K_{\sqrt{\frac{2\pi}{b}}\varepsilon}} \left(1 - \frac{\int_{K_{\sqrt{2\pi}}} \rho_0^2}{\int_{K_{\sqrt{2\pi}}} \rho_0^4} \rho_0^2 (\frac{x\sqrt{b}}{\varepsilon})\right) dx \\ &= (1-b)^2 \left(\frac{2\pi}{b}\varepsilon^2 - \frac{\varepsilon^2}{b} \frac{\left(\int_{K_{\sqrt{2\pi}}} \rho_0^2\right)^2}{\int_{K_{\sqrt{2\pi}}} \rho_0^4}\right). \end{aligned} \tag{IV.7}$$

Let us write

$$\alpha = \frac{\left(\displaystyle\int_{K_{\sqrt{2\pi}}} \rho_0^2\right)^2}{2\pi \displaystyle\int_{K_{\sqrt{2\pi}}} \rho_0^4}.$$

Since ρ_0 is not a constant function (see (IV.2)), we have a strict Cauchy-Schwartz inequality

$$\left(\int_{K_{\sqrt{2\pi}}} \rho_0^2\right)^2 < 2\pi \int_{K_{\sqrt{2\pi}}} \rho_0^4,$$

and hence $0 < \alpha < 1$. Then, from (IV.7),

$$\int_{B_R} (1 - b - \rho^2)^2 = |B_R|(1-b)^2(1-\alpha) + o(R^2). \tag{IV.8}$$

Combining (IV.6) and (IV.8), we are led to

$$J_{B_R}(u, A) = |B_R|\kappa^2 \left(\frac{b}{2} - \frac{b^2}{4} + \frac{(1-b)^2(1-\alpha)}{4}\right) + o(\kappa^2 R^2).$$

We conclude that

$$f(b) \le \limsup_{\kappa\to\infty} \frac{\min J_{B_R}}{\kappa^2|B_R|} \le \limsup_{\kappa\to\infty} \frac{J_{B_R}(u,A)}{\kappa^2|B_R|} \le \frac{1-\alpha(1-b)^2}{4}.$$

$\square$

Proposition 3 *There exists a universal constant c such that for $b \le 1$,*

$$f(b) \le \frac{b}{4}\left(\log\frac{1}{b} + c\right). \tag{IV.9}$$

Proof: This estimate is stronger than (IV.1) when $b \to 0$, and corresponds to a regime in which the distance between vortices is rather large compared to their core size ε, i.e. is close to the regime described in [SS2]. In order to prove this estimate, we just adjust the construction of a test-function that we did in [SS2].

This test-function is again periodic with respect to a square lattice of size $\sqrt{\frac{2\pi}{b}}\varepsilon$. Let us consider an elementary square $K_{\sqrt{\frac{2\pi}{b}}\varepsilon}$ centered at the origin, and B_ε the ball of radius ε centered at the origin, which is included in $K_{\sqrt{\frac{2\pi}{b}}\varepsilon}$ for all $b \le 1$. The centers of the squares of the lattice will be denoted a_i. We take a $\rho \le 1$ which satisfies

$$\begin{cases} \rho \equiv 1 \text{ in } K_{\sqrt{\frac{2\pi}{b}}\varepsilon} \backslash B_\varepsilon \\ \rho \equiv 0 \text{ in } B_{\varepsilon/2} \\ \displaystyle\int_{K_{\sqrt{\frac{2\pi}{b}}\varepsilon}} |\nabla\rho|^2 + \frac{\kappa^2}{2}(1-\rho^2)^2 \le C. \end{cases} \tag{IV.10}$$

Then, we take h such that

$$\begin{cases} -\Delta h + h = \frac{8}{\varepsilon^2}\mathbf{1}_{B_{\varepsilon/2}} & \text{in } K_{\sqrt{\frac{2\pi}{b}}\varepsilon} \\ \dfrac{\partial h}{\partial n} = 0 & \text{on } \partial K_{\sqrt{\frac{2\pi}{b}}\varepsilon}. \end{cases} \tag{IV.11}$$

where $\mathbf{1}$ denotes a characteristic function. We extend ρ and h by periodicity to B_R ($R \gg \varepsilon$) and pick A such that $\operatorname{curl} A = h$ and $\operatorname{div} A = 0$. Then we take φ such that

$$\nabla\varphi = -\nabla^\perp h + A, \tag{IV.12}$$

i.e. by choosing a point x_0 in $B_R \backslash \cup_i B_{\varepsilon/2}(a_i)$ and setting

$$\varphi(x) = \int_{x_0}^{x} -\frac{\partial h}{\partial n} + A \cdot \tau.$$

This integral does not depend on the path joining x_0 to x in $B_R \backslash \cup_i B_{\varepsilon/2}(a_i)$, modulo 2π. Thus can be seen from (IV.11). Thus $e^{i\varphi}$ is well-defined in $B_R \backslash \cup_i B_{\varepsilon/2}(a_i)$, and

$$u(x) = \rho(x) e^{i\varphi(x)}$$

has a meaning on all of B_R (since $\rho \equiv 0$ in $\cup_i B_{\varepsilon/2}(a_i)$). Exactly as in [SS2], one shows that

$$\frac{1}{2}\int_{K_{\sqrt{\frac{2\pi}{b}}\varepsilon}} |\nabla h|^2 + |h - h_{\mathrm{ex}}|^2 \leq \pi \log \frac{\sqrt{\frac{1}{b}}\varepsilon}{b} + C = \frac{\pi}{2}\log\frac{1}{b} + C. \tag{IV.13}$$

There remains to evaluate the energy of (u, A) per square. From (IV.12), we have $\rho^2|\nabla\varphi - A|^2 \leq |\nabla h|^2$, hence

$$\begin{aligned} J_K(u, A) &= \int_{K_{\sqrt{\frac{2\pi}{b}}\varepsilon}} |\nabla\rho|^2 + \rho^2|\nabla\varphi - A|^2 + |h - h_{\mathrm{ex}}|^2 + \frac{\kappa^2}{2}(1-\rho^2)^2 \\ &\leq \frac{\pi}{2}\log\frac{1}{b} + c. \end{aligned}$$

Multiplying this estimate by the number of squares in B_R, $\frac{|B_R| b}{2\pi\varepsilon^2}$, we find

$$J_{B_R}(u, A) \leq |B_R|\kappa^2 \left(\frac{b}{4}\log\frac{1}{b} + cb\right).$$

We then conclude, as in the previous proposition, that (IV.9) holds. □

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