Lorentz Space Estimates for the Ginzburg-Landau Energy

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May 1, 2007

Abstract

In this paper we prove novel lower bounds for the Ginzburg-Landau energy with or without magnetic field. These bounds rely on an improvement of the "vortex balls construction" estimates by extracting a new positive term in the energy lower bounds. This extra term can be conveniently estimated through a Lorentz space norm, on which it thus provides an upper bound. The Lorentz space $L^{2,\infty}$ we use is critical with respect to the expected vortex profiles and can serve to estimate the total number of vortices and get improved convergence results.

1 Introduction

1.1 Motivation

In this paper we consider the Ginzburg-Landau "free energy"

$$F_\varepsilon(u, A) = \frac{1}{2} \int_\Omega |\nabla_A u|^2 + |\operatorname{curl} A|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2}. \tag{1.1}$$

Here Ω is a bounded regular two dimensional domain of $\mathbb{R}^2$, u is a complex-valued function, and $A \in \mathbb{R}^2$ is a vector field in Ω. This functional is the free energy of the model of superconductivity developed by Ginzburg and Landau. In the model, A is the vector-potential of the magnetic field, the function $h := \operatorname{curl} A = \partial_1 A_2 - \partial_2 A_1$ is the induced magnetic field, and the complex-valued function u is the "order parameter" indicating

*Supported by NSF CAREER grant # DMS0239121 and a Sloan Foundation Fellowship

†Supported by an NSF Graduate Research Fellowship

the local state of the material (normal or superconducting): $|u|^2$ is the local density of superconducting electrons. The notation ∇_A refers to the covariant gradient, which acts according to $\nabla_A u = (\nabla - iA)u$.

We are interested in the regime of small ε: ε corresponds to a material constant, and small ε implies type-II superconductivity. In this regime, u (because it is complex-valued) can have zeroes with a nonzero topological degree. These defects are called the *vortices* of u and are the crucial objects of interest.

By setting $A \equiv 0$ we are led to studying the simpler Ginzburg-Landau energy "without magnetic field":

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2}. \tag{1.2}$$

All our results will thus apply to this energy as well, by setting $A \equiv 0$.

These functionals, and in particular the vortices arising in their minimizers or critical points, have been studied intensively in the mathematics literature. We refer in particular to the books [1] for E_ε and [8] for the functional with magnetic field. The interested reader can find there more information on the physical and mathematical background.

We are interested in proving lower bounds on F_ε, and in particular estimates which relate $F_\varepsilon(u, A)$ and $\|\nabla_A u\|_{L^{2,\infty}}$, the norm of $\nabla_A u$ in the Lorentz space $L^{2,\infty}$. Noticeably, Lorentz spaces were already used in the context of the Ginzburg-Landau energy by Lin and Rivière in [5]. Their goal there was to study energy critical points in 3 dimensions, but what they used was interpolation ideas and the duality between Lorentz spaces $L^{2,1}$ and $L^{2,\infty}$.

The Ginzburg-Landau energy is generally unbounded as $\varepsilon \to 0$; it blows up roughly like $\pi n |\log \varepsilon|$, where n is the number (or total degree) of vortices. Our investigation of estimates for $\|\nabla_A u\|_{L^{2,\infty}}$ is thus part of a quest for intrinsic quantities in $\nabla_A u$ which do not blow up as $\varepsilon \to 0$, but rather remain of the order of n.

1.2 Heuristics for idealized vortices

Let us now try to explain the interest and relevance of the Lorentz space $L^{2,\infty}$ for this problem. The space $L^{2,\infty}$, also known as "weak-L^2", is a functional space which is just "slightly larger" than the Lebesgue space L^2. One simple way of defining the $L^{2,\infty}$ norm is by

$$\|f\|_{L^{2,\infty}} = \sup_{|E|<\infty} |E|^{-\frac{1}{2}} \int_E |f(x)|\, dx, \tag{1.3}$$

where $|E|$ denotes the Lebesgue measure of E. An equivalent way is through the super-level sets of f:

$$\|f\|_{L^{2,\infty}} = \sup_{t>0} t\lambda_f(t)^{\frac{1}{2}}, \tag{1.4}$$

where $\lambda_f(t) = |\{x \in \Omega \mid |f(x)| > t\}|$. For more information on Lorentz spaces we refer for example to [2, 9]. A simple application of the Cauchy-Schwarz inequality in (1.3) allows to check that if f is in L^2 then it is in $L^{2,\infty}$ with $\|f\|_{L^{2,\infty}} \le \|f\|_{L^2}$.

Let us now consider vortices of a complex-valued function u in the context of Ginzburg-Landau. In the regime of small ε, u can have zeroes, but because of the strong penalization of the term $\int_\Omega (1-|u|^2)^2$, $|u|$ can be small only in (small) regions of characteristic size ε.

Then around a zero at a point x_0, u has a degree defined as the topological degree of $u/|u|$ as a map from a circle to $\mathbb{S}^1$, or in other words

$$d = \frac{1}{2\pi}\int_{\partial B(x_0,r)} \frac{\partial}{\partial\tau}\left(\frac{u}{|u|}\right) \in \mathbb{Z}, \tag{1.5}$$

where r is sufficiently small. One can describe the situation very roughly as follows: $|u|$ is small in a ball of radius $C\varepsilon$, and $|u| \approx 1$ outside of this ball, say in an annulus $B(x_0,R)\backslash B(x_0,C\varepsilon)$. The size of R is meant to account for possible neighboring zeroes. In this annulus, the model case is that of a radial vortex of degree d, i.e

$$u(r,\theta) = f(r)e^{id\theta}, \tag{1.6}$$

where (r,θ) are the polar coordinates centered at x_0, and f is a real-valued function, close to 1 in $B(x_0,R)\backslash B(x_0,C\varepsilon)$. When computing the L^2 norm of ∇u, we find that $|\nabla u| \approx \frac{|d|}{r}$ in the annulus and thus, using polar coordinates,

$$\begin{aligned} \|\nabla u\|^2_{L^2(B(x_0,R))} &\geq \int_{B(0,R)\backslash B(0,C\varepsilon)} \left|\frac{d}{r}\right|^2 = \int_{C\varepsilon}^R \frac{2\pi d^2}{r}\,dr \\ &\geq 2\pi d^2 \log\frac{R}{C\varepsilon}. \end{aligned} \tag{1.7}$$

This tells us that the (square of the) L^2 norm of ∇u blows up like $2\pi d^2|\log\varepsilon|$ as $\varepsilon\to 0$. This is a crucial fact in the analysis of Ginzburg-Landau, much used since [1]. Jerrard [3] and Sandier [6] showed that this picture is actually accurate even for arbitrary configurations: without assuming that the vortex profile is radial, the inequality (1.7) still holds (the radial profile is actually the one that is minimal for the L^2 norm). Moreover, any configuration with an arbitrary number of vortices can be understood as many such annuli, possibly at very close distance to each other, glued together. Good lower bounds like (1.7) can be added up together by keeping annuli with the same conformal type. This was the basis of the "vortex-balls construction" that they formulated and which was used extensively to understand Ginzburg-Landau minimizers, in particular in [8].

On the other hand, let us calculate (roughly) the $L^{2,\infty}$ norm of ∇u for the above vortex. We recall that $|\nabla u| \approx \frac{|d|}{r}$ in the annulus $B(x_0,R)\backslash B(x_0,C\varepsilon)$. Using the definition (1.4), we have $|\nabla u| > t$ if and only if $r < |d|/t$. Thus

$$\lambda_{|\nabla u|}(t) \approx \pi d^2/t^2,$$

and we find

$$\|\nabla u\|_{L^{2,\infty}(B(x_0,R)\backslash B(x_0,C\varepsilon))} \approx \sqrt{\pi}|d|. \tag{1.8}$$

So in contrast, the $L^{2,\infty}$ norm of ∇u *does not blow up* as $\varepsilon \to 0$. One can see that this space is critical in the sense that $1/|x|$ (barely) fails to be in L^2 or in $L^{2,q}$ for any $q < \infty$ (its norm blows up logarithmically in all cases) but is in $L^{2,\infty}$ and in all L^p for $p < 2$.

Moreover, from this formula (1.8), it is expected that the $L^{2,\infty}$ norm can serve to estimate the total degree $\sum |d_i|$ of all the vortices of a configuration. This is convenient since the total degree $\sum |d_i|$ is generally obtained via a "ball construction" that is nonunique. On the other hand $\|\nabla u\|_{L^{2,\infty}}$ provides a unique and intrinsic quantity useful to evaluate the number of vortices.

Because of these remarks and because of the paper [5], it could be expected that Lorentz spaces are a suitable functional setting in which to study Ginzburg-Landau vortices. One may point out that there are other spaces that would be critical for the profile $1/|x|$, such as Besov spaces; however, it seems difficult to find an effective way of using them in connection with the Ginzburg-Landau energy.

The main goal of our results is to give a rigorous basis to the above observations. The connection with the Lorentz norm of ∇u is made through the "vortex-balls construction" of Jerrard and Sandier, as formulated in [8]. Our estimates will in fact provide an improvement of these lower bounds by adding an extra positive term in the lower bounds, which is then related to the Lorentz norm. Just as in the ball construction method, one of the interests of the result is that it is valid under very few assumptions: only a very weak upper bound on the energy, even when u has a large number of vortices, unbounded as $\varepsilon \to 0$. This creates serious technical difficulties but is important since such situations occur for energy minimizers when there is a large applied magnetic field, as proved in [8].

1.3 Main results

Let us point out that the estimates we prove are not on the Lorentz norm of ∇u but rather on that of $\nabla_A u$. The reason is that the energy F_ε is *gauge-invariant* : it satisfies $F_\varepsilon(u, A) = F_\varepsilon(ue^{i\Phi}, A + \nabla\Phi)$ for any smooth function Φ. Thus the quantity $|\nabla u|$ is not a gauge-invariant quantity, hence not an intrinsic physical quantity. This is why it is replaced by the gauge-invariant "covariant derivative" $|\nabla_A u|$.

Our method consists in proving the following improvement of the "ball construction" lower bounds (see [8], Chapter 4):

Theorem 1 (Improved lower bounds)**.** *Let $\alpha \in (0, 1)$. There exists $\varepsilon_0 > 0$ (depending on α) such that for $\varepsilon \le \varepsilon_0$ and u, A both C^1 such that $F_\varepsilon(|u|, \Omega) \le \varepsilon^{\alpha-1}$, the following hold.*

For any $1 > r > C\varepsilon^{\alpha/2}$, where C is a universal constant, there exists a finite, disjoint collection of closed balls, denoted by $\mathcal{B}$, with the following properties.

1. The sum of the radii of the balls in the collection is r.

2. Defining $\Omega_\varepsilon = \{x \in \Omega \,|\, \mathrm{dist}(x, \partial\Omega) > \varepsilon\}$, we have
$\{x \in \Omega_\varepsilon \mid |u(x) - 1| \ge \delta\} \subset V := \Omega_\varepsilon \cap (\cup_{B\in\mathcal{B}} B)$, where $\delta = \varepsilon^{\alpha/4}$.

3. We have

$$\begin{aligned}&\frac{1}{2}\int_V |\nabla_A u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2 + r^2(\operatorname{curl} A)^2\\&\geq \pi D\left(\log\frac{r}{\varepsilon D} - C\right) + \frac{1}{18}\int_V |\nabla_{A+G} u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2,\end{aligned}\tag{1.9}$$

where G is some explicitly constructed vector field, d_B denotes $\deg(u, \partial B)$ *if $B \subset \Omega_\varepsilon$ and* 0 *otherwise,*

$$D = \sum_{\substack{B\in\mathcal{B}\\ B\subset\Omega_\varepsilon}} |d_B|$$

is assumed to be nonzero, and C is universal.

The improvement with respect to Theorem 4.1 in [8] is the addition of the extra term $\frac{1}{18}\int |\nabla_{A+G}u|^2$. The term G is a vector-field constructed in the course of the ball construction, which essentially compensates for the expected behavior of $\nabla_A u$ in the vortices. One can take it to be $\tau d/r$ in every annulus of the ball construction where u has a constant degree d, τ denotes the unit tangent vector to each circle centered at x_0, the center of the annulus, and $r = |x - x_0|$. By extending G to be zero outside of the union of balls V, we easily deduce:

Corollary 1.1. *Let (u, A) be as above, then*

$$\int_\Omega |\nabla_A u - iGu|^2 \leq C\left(F_\varepsilon(u,A) - \pi D \log\left(\frac{r}{\varepsilon D} - C\right)\right) \tag{1.10}$$

where G is the explicitly constructed vector field of Theorem 1, and C a universal constant.

The right-hand side of this inequality can be considered as the "energy-excess", difference between the total energy and the expected vortex energy provided by the ball construction lower bounds. Thus we control $\int_\Omega |\nabla_A u - iGu|^2$ by the energy-excess. This fact is used repeatedly in the sequel paper [10] to better understand the behavior of $\nabla_A u$ for minimizers and almost minimizers of the Ginzburg-Landau energy with applied magnetic field.

One can also note that such a control (1.10) has a similar flavor to a result of Jerrard-Spirn [4] where they control the difference (in a weaker norm but with better control) of the Jacobian of u to a measure of the form $\sum d_i \delta_{a_i}$ by the energy-excess.

Once Theorem 1 is proved, we turn to obtaining an $L^{2,\infty}$ estimate from which G has disappeared. In order to do so, we can bound below $\|\nabla_{A+G}u\|_{L^2}$ by $\|\nabla_{A+G}u\|_{L^{2,\infty}}$; the more delicate task is then to control $\|G\|_{L^{2,\infty}}$ in a way that only depends on the final data of the theorem, that is on the degrees of the final balls constructed above and on the energy. This task is complicated by the possible presence of large numbers of vortices very close to each other, and compensations of vortices of large positive degrees with vortices of large negative degrees. To overcome this, G is not defined exactly as previously said, but in a modified way, and $\|G\|_{L^{2,\infty}}$ is controlled not only through the degrees but also through the total energy.

We then arrive at the following main result :

Theorem 2 (Lorentz norm bound)**.** *Assume the hypotheses and results of Theorem 1. Then there exists a universal constant C such that*

$$\frac{1}{2}\int_V |\nabla_A u|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2} + r^2(\mathrm{curl}\, A)^2 + \pi \sum |d_B|^2$$
$$\geq C\,\|\nabla_A u\|^2_{L^{2,\infty}(V)} + \pi \sum |d_B| \left(\log \frac{r}{\varepsilon \sum |d_B|} - C\right), \quad (1.11)$$

where the sums are taken over all the balls B in the final collection $\mathcal{B}$ that are included in Ω_ε.

This theorem bounds below the energy contained in the union of balls V in terms of the $L^{2,\infty}$ norm on V. It is a simple matter to extend these estimates to all of Ω, and deduce a control of the $L^{2,\infty}$ norm of $\nabla_A u$ by the energy-excess, plus the term $\sum |d_B|^2$. This is the content of the following corollary.

Corollary 1.2. *Assuming the hypotheses and results of Theorem 1, there exists a universal constant C such that*

$$\|\nabla_A u\|^2_{L^{2,\infty}(\Omega)} \leq C\left(F_\varepsilon(u,A) - \pi \sum |d_B| \log \frac{r}{\varepsilon \sum |d_B|} + \sum |d_B|^2\right), \quad (1.12)$$

where the sums are taken over all the balls B in the final collection $\mathcal{B}$ that are included in Ω_ε.

These estimates can indeed help to bound from above $\|\nabla_A u\|^2_{L^{2,\infty}(\Omega)}$ by the total number of vortices, provided we can control the energy-excess by that number of vortices. This can in turn serve to obtain stronger convergence results when a weak limit of $\nabla_A u$ is known. For example, if one considers the energy E_ε (which we recall amounts to setting $A \equiv 0$), it is known from Bethuel-Brezis-Hélein [1] that $\pi \sum |d_B|\,|\log \varepsilon| = \pi n\,|\log \varepsilon|$ is the leading order of the energy (at least for minimizers) and that the next order term is a term of order 1, called the "renormalized energy" W, that accounts for the interaction between the vortices. The upper bound of Corollary 1.2 roughly tells us that

$$\|\nabla u\|^2_{L^{2,\infty}(\Omega)} \leq C(W + \sum |d_B|^2 + \sum |d_B| \log \sum |d_B|).$$

It is expected that the total cost of interaction of the vortices in W is of order of n^2, where $n = \sum |d_B|$ is the total vorticity mass (here n can blow up as $\varepsilon \to 0$). Thus, we obtain a bound of the form

$$\|\nabla u\|^2_{L^{2,\infty}(\Omega)} \leq Cn^2,$$

which indeed bounds the $L^{2,\infty}$ norm of ∇u by an order of n, the total vorticity mass, as expected in the heuristic calculations of Section 1.2.

In the simplest case where we know that $E_\varepsilon(u_\varepsilon) \le \pi n |\log \varepsilon| + C$, which happens for energy minimizers when n is bounded, as proved in [1], we then deduce that $\|\nabla u\|_{L^{2,\infty}} \le C$. To be more precise, for the minimizers of E_ε found in [1], we have

Proposition 1.3 (Application to minimizers of E_ε with Dirichlet boundary conditions)**.** *Let Ω be starshaped and u_ε minimize E_ε under the constraint $u_\varepsilon = g$ on $\partial\Omega$, where g is a fixed $\mathbb{S}^1$-valued map of degree $d > 0$ on the boundary of Ω, as studied in [1]. Then there exists a universal constant C such that*

$$\|\nabla u_\varepsilon\|^2_{L^{2,\infty}(\Omega)} \le C(\min_{\Omega^d} W + d(\log d + 1)) + o_\varepsilon(1).$$

Moreover, as $\varepsilon \to 0$,

$$\nabla u_\varepsilon \rightharpoonup \nabla u_\star \quad \text{weakly-}* \text{ in } \ L^{2,\infty}(\Omega),$$

where $u_\star$ is the S^1-valued "canonical harmonic map" of [1] to which converges u in C^k_{loc} outside of a set of d vortex points.

Note that the renormalized energy W depends on g (hence on d), and the $d \log d$ is not optimal here; rather, it should be d. It is more delicate to obtain this kind of improvement to the estimate; this is one of the things done in [10] in the context of the energy with applied magnetic field. Also the convergence of ∇u_ε *cannot be strengthened*, convergence in $L^{2,\infty}$ strong does not hold, as illustrated by the following model case: let V_ε be the vector field $\frac{(x-p_\varepsilon)^\perp}{|x-p_\varepsilon|^2}$ and $V = \frac{(x-p)^\perp}{|x-p|^2}$ with $p_\varepsilon \to p$ as $\varepsilon \to 0$. Then $2\sqrt{\pi} \le \|V_\varepsilon - V\|_{L^{2,\infty}} \le 4\sqrt{\pi}$, while clearly $V_\varepsilon \rightharpoonup V$ weakly-$*$ in $L^{2,\infty}$.

We have focused on proving upper bounds on $\|\nabla_A u\|_{L^{2,\infty}}$ in terms of its L^2 norm and Ginzburg-Landau energy. It is not difficult to obtain some adapted, though not optimal, lower bounds. For example, we can prove the following:

Proposition 1.4. *Let $f \in L^\infty(\Omega)$ be such that $\|f\|_{L^\infty(\Omega)} \le \frac{C}{\varepsilon}$ for some $\varepsilon < 1$. Then*

$$\|f\|^2_{L^{2,\infty}(\Omega)} \ge \frac{1}{2|\log \varepsilon|} \int_\Omega |f|^2 - \frac{C^2 |\Omega|}{2|\log \varepsilon|}. \tag{1.13}$$

This proposition is a direct consequence of the definition of the $L^{2,\infty}$ norm. Its short proof is presented in Section 6.1.

For critical points of the Ginzburg-Landau energy, it is known that the gradient bound $\|\nabla_A u\|_{L^\infty(\Omega)} \le \frac{C}{\varepsilon}$ holds. Thus applying Proposition 1.4 to $f = \nabla_A u$, we find

$$\|\nabla_A u\|^2_{L^{2,\infty}(\Omega)} \ge \frac{1}{2|\log \varepsilon|} \int_\Omega |\nabla_A u|^2 - o(1).$$

Knowing some lower bounds (provided by the ball construction) of the type $\int_\Omega |\nabla_A u|^2 \ge 2\pi n |\log \varepsilon|$, where n is the total degree of the vortices, we find lower bounds of the type $\|\nabla_A u\|^2_{L^{2,\infty}(\Omega)} \ge \pi n$, also relating the $L^{2,\infty}$ norm of $\nabla_A u$ to the total number of vortices.

In [10], which is the sequel of this paper, the ideas and main results of this paper are extended to the case of the full Ginzburg-Landau energy with an applied magnetic field, getting better estimates on $\|\nabla_A u\|_{L^{2,\infty}(\Omega)}$ in terms of the number of vortices. These results lead to a somewhat stronger (than previously known results) convergence of $\nabla_A u$ and of the Jacobian determinants of u when certain energy conditions are fulfilled.

1.4 Plan

The paper is organized as follows: in Section 2, for the convenience of the reader, we give a review (with slight modifications) of the crucial definitions and ingredients for the vortex-balls construction following Chapter 4 of [8].

In Section 3 we present the main argument, with the introduction of the function G and the "trick" that allows us to gain an extra term in the lower bounds for the energy on annuli.

In Section 4 we show how this extra term incorporates into the estimates through the growing and merging of balls, and hence through the whole ball construction.

In Section 5 we deduce the proof of the main results.

In Section 6 we estimate the $L^{2,\infty}$ norm of G in order to pass from Theorem 1 to Theorem 2. This is the only section in which $L^{2,\infty}$ comes into play.

In Section 7 we show how the methods of this paper can be adapted to work with the version of the ball construction formulated by Jerrard in [3], at the expense of less control of $\|G\|_{L^{2,\infty}}$.

2 Reminders for the vortex balls construction

2.1 The ball growth method

In finding lower bounds for the Ginzburg-Landau energy of a configuration (u, A) it is most convenient to work on annuli, the deleted interior discs of which contain the set where u is near 0, and in particular the vortices. On each annulus, a lower bound is found in terms of a topological term (the degree of the vortex) and a conformal factor, which we define to be the logarithm of the ratio of the outer and inner radii of the annulus. Therefore, to create useful lower bounds we must be able to identify the set where u is near 0 and then create a family of annuli with large conformal type outside this set. The first component of the process uses energy methods to find a covering of the set by small, disjoint balls, and is addressed later. The second component is known as the general ball growth method and is presented in this section. Here we follow the construction of Chapter 4 from [8].

As a technical tool we will need the ability to merge two tangent or overlapping balls into a single ball that contains the original balls, and with the property that its radius is equal to the sum of the radii of the original balls. Our first lemma recalls how to do such a merging. We write $r(B)$ for the radius of a ball B.

Lemma 2.1. *Let B_1 and B_2 be closed balls in $\mathbb{R}^n$ such that $B_1 \cap B_2 \neq \varnothing$. Then there is a closed ball B such that $r(B) = r(B_1) + r(B_2)$ and $B_1 \cup B_2 \subset B$.*

Proof. If $B_1 = B(a_1, r_1)$ and $B_2 = B(a_2, r_2)$, then $B = B\left(\frac{r_1 a_1 + r_2 a_2}{r_1 + r_2}, r_1 + r_2\right)$ has the desired properties. □

The ball growth lemma now provides an algorithm for growing an initial collection of small balls into a final collection of large balls. Essentially, the balls in a collection are grown concentrically by increasing their radii by the same conformal factor. This is continued until a tangency occurs, at which point the previous lemma is used to merge the tangent balls. The process is then repeated in stages until the collection is of the desired size. The annuli of interest at each stage are formed by deleting the initial collection of balls from the final collection; the construction guarantees that all of the annuli in a stage have the same conformal type.

Given a finite collection of disjoint balls, $\mathcal{B}$, we define the radius of the collection, $r(\mathcal{B})$, to be the sum of the radii of the balls in the collection, i.e.

$$r(\mathcal{B}) = \sum_{B \in \mathcal{B}} r(B).$$

For any $\lambda > 0$ and any ball $B = B(a, r)$, we define $\lambda B = B(a, \lambda r)$. Extending this notation to collections of balls, we write $\lambda \mathcal{B} = \{\lambda B \mid B \in \mathcal{B}\}$. For an annulus $A = B(a, r_1) \backslash B(a, r_0)$, we define the conformal factor by $\tau = \log(r_1/r_0)$. We can now state the ball growth lemma, the proof of which can be found in Theorem 4.2 of [8].

Lemma 2.2 (Ball growth lemma)**.** *Let $\mathcal{B}_0$ be a finite collection of disjoint, closed balls. There exists a family $\{\mathcal{B}(t)\}_{t \in \mathbb{R}_+}$ of collections of disjoint, closed balls such that the following hold.*

1. $\mathcal{B}_0 = \mathcal{B}(0)$.

2. For $s \geq t \geq 0$,

$$\bigcup_{B \in \mathcal{B}(t)} B \subseteq \bigcup_{B \in \mathcal{B}(s)} B.$$

3. There exists a finite set $T \subset \mathbb{R}^+$ such that if $[t, s] \subset \mathbb{R}^+ \backslash T$, then $\mathcal{B}(s) = e^{s-t}\mathcal{B}(t)$. In particular, if $B(s) \in \mathcal{B}(s)$ and $B(t) \in \mathcal{B}(t)$ are such that $B(t) \subset B(s)$, then $B(s) = e^{s-t}B(t)$ and the conformal factor of the annulus $B(s) \backslash B(t)$ is $\tau = s - t$.

4. For every $t \in \mathbb{R}^+$, $r(\mathcal{B}(t)) = e^t r(\mathcal{B}_0)$.

We now show how to couple lower bounds to the geometric construction. We may think of a function $\mathcal{F} : \mathbb{R}^2 \times \mathbb{R}^+ \to \mathbb{R}^+$ as being defined also for collections of balls, $\mathcal{B}$, via the identifications

$$\mathcal{F}(B(x, r)) = \mathcal{F}(x, r)$$

and

$$\mathcal{F}(\mathcal{B}) = \sum_{B \in \mathcal{B}} \mathcal{F}(B).$$

Here and for the rest of the paper we employ the notation $\bar{B}$ to refer to a specific ball $\bar{B}$ in some collection, and not to refer to the closure of B. We will also abuse notation by writing $\bar{B} \cap \mathcal{B}(t)$ for the collection $\{\bar{B} \cap B \mid B \in \mathcal{B}(t)\}$.

Lemma 2.3. *Let $\mathcal{B}_0$ be a finite collection of disjoint, closed balls, and suppose that $\mathcal{B}(t)$ is the collection of balls obtained from $\mathcal{B}_0$ by growing them according to the ball growth lemma. Fix a time $s > 0$ and suppose that $0 < s_1 < \cdots < s_K \leq s$ denote the times at which mergings occur in the the ball growth lemma, i.e. let the s_i be an increasing enumeration of the set T defined there. Then*

$$\mathcal{F}(\mathcal{B}(s)) - \mathcal{F}(\mathcal{B}_0) = \int_0^s \sum_{B(x,r)\in\mathcal{B}(t)} r\frac{\partial \mathcal{F}}{\partial r}(x,r)\,dt + \sum_{k=1}^{K} \mathcal{F}(\mathcal{B}(s_k)) - \mathcal{F}(\mathcal{B}(s_k))^-, \tag{2.1}$$

where $\mathcal{F}(\mathcal{B}(s_k))^- = \lim_{t\to s_k^-} \mathcal{F}(\mathcal{B}(t))$. Moreover, for any $\bar{B} \in \mathcal{B}(s)$, the following localized version of (2.1) *holds:*

$$\mathcal{F}(\bar{B}) - \mathcal{F}(\bar{B}\cap\mathcal{B}_0) = \int_0^s \sum_{B(x,r)\in\bar{B}\cap\mathcal{B}(t)} r\frac{\partial \mathcal{F}}{\partial r}dt + \sum_{k=1}^{K} \mathcal{F}(\bar{B}\cap\mathcal{B}(s_k)) - \mathcal{F}(\bar{B}\cap\mathcal{B}(s_k))^-. \tag{2.2}$$

Proof. The proof is the same as in Proposition 4.1 of [8], but here we keep the second sum in (2.1) rather than bounding it. □

Note that in the case that

$$\mathcal{F}(x,r) = \int_{B(x,r)} e(u)$$

for some u-dependent energy density $e(u)$, the first term on the right of (2.1) corresponds to integration in polar coordinates on each annulus, and the second corresponds to the energy contained in the non-annular parts of $\mathcal{B}(s)$.

2.2 The radius of a set

In order to effectively use the ball growth lemma to generate lower bounds, it is necessary to first produce a collection of disjoint balls covering the set where u is near 0. We do this by using the concept of the radius of a set, which is useful in two ways. First, it is defined as an infimum over all coverings of the set by collections of balls, so that by exceeding the infimum we may find a covering of the set by balls. Second, it is comparable to the $\mathcal{H}^1$ Hausdorff measure of the boundary, and so it can be used with the co-area formula to produce coverings by balls of the set where $|u|$ is far from unity.

We define the radius of a compact set $\omega \subset \mathbb{R}^2$, written $r(\omega)$, by

$$r(\omega) = \inf\{r(B_1) + \cdots + r(B_k) \mid \omega \subset \cup_{i=1}^k B_i \text{ and } k < \infty\}.$$

We make the following remarks.

1) In the definition we may assume that the balls are disjoint. If they are not, then we

merge balls that meet into a single ball with radius equal to the sum of the radii of the merged balls according to Lemma 2.1.
2) If $A \subseteq B$ then $r(A) \leq r(B)$.
3) The infimum is not necessarily achieved.

It is necessary to also introduce a modification of the radius that measures the radius of the connected components of a compact set ω that lie inside an open set Ω. Indeed, we define

$$r_\Omega(\omega) = \sup\{r(K \cap \omega) \mid K \subset \Omega \text{ s.t. } K \text{ is compact and } \partial K \cap \omega = \varnothing\}.$$

The following lemmas record the crucial properties of these quantities. The omitted proofs may be found in Section 4.4 of [8].

Lemma 2.4. *Let ω be a compact subset of $\mathbb{R}^2$. Then*

$$2r(\omega) \leq \mathcal{H}^1(\partial\omega). \tag{2.3}$$

Lemma 2.5. *Let Ω be open and $\omega \subset \Omega$ be a compact set. Then*

$$2r_\Omega(\omega) \leq \mathcal{H}^1(\partial\omega \cap \Omega). \tag{2.4}$$

Lemma 2.6. *Let ω_1, ω_2 be compact subsets of $\mathbb{R}^2$. Then*

$$r(\omega_1 \cup \omega_2) \leq r(\omega_1) + r(\omega_2). \tag{2.5}$$

Lemma 2.7. *Let ω_1, ω_2 be compact sets, and let $\Omega \subset \mathbb{R}^2$ be an open set. Then*

$$r_\Omega(\omega_1 \cup \omega_2) \leq r_\Omega(\omega_1) + r_\Omega(\omega_2). \tag{2.6}$$

Proof. If $\Omega \subset \omega_1 \cup \omega_2$, then the result is trivial. Suppose otherwise. Let $K \subset \Omega$ be such that K is compact and $\partial K \cap (\omega_1 \cup \omega_2) = \varnothing$. Then $(\partial K \cap \omega_1) \cup (\partial K \cap \omega_2) = \varnothing$, which implies that $\partial K \cap \omega_1 = \varnothing$ and $\partial K \cap \omega_2 = \varnothing$. Hence,

$$\begin{aligned} r(K \cap (\omega_1 \cup \omega_2)) &= r((K \cap \omega_1) \cup (K \cap \omega_2)) \\ &\leq r(K \cap \omega_1) + r(K \cap \omega_2) \\ &\leq r_\Omega(\omega_1) + r_\Omega(\omega_2). \end{aligned} \tag{2.7}$$

Taking the supremum over all such K, we get $r_\Omega(\omega_1 \cup \omega_2) \leq r_\Omega(\omega_1) + r_\Omega(\omega_2)$. $\square$

We will now use these concepts to compare the energy of a real-valued function ρ, defined on an open set Ω, to the radius of the set where ρ is far from unity.

Lemma 2.8. *Let $\rho \in C^1(\Omega, \mathbb{R})$ with $\Omega \subset \mathbb{R}^2$ open and bounded. Let*

$$F_\varepsilon(\rho, \Omega) = \frac{1}{2}\int_\Omega |\nabla\rho|^2 + \frac{1}{2\varepsilon^2}(1-\rho^2)^2. \tag{2.8}$$

Then there is a universal constant C such that

$$r_\Omega\left(\{\rho \leq 1/2\} \cup \{\rho \geq 3/2\}\right) \leq \varepsilon C F_\varepsilon(\rho, \Omega). \tag{2.9}$$

Proof. By the Cauchy-Schwarz inequality and the co-area formula we have that

$$\begin{aligned} F_\varepsilon(\rho,\Omega) &= \frac{1}{2}\int_\Omega |\nabla\rho|^2 + \frac{1}{2\varepsilon^2}(1-\rho^2)^2 \\ &\geq \frac{1}{\varepsilon\sqrt{2}}\int_\Omega |\nabla\rho|\left|1-\rho^2\right| \\ &= \frac{1}{\varepsilon\sqrt{2}}\int_0^\infty\int_{\{\rho=t\}\cap\Omega}\left|1-\rho^2\right| d\mathcal{H}^1 dt \\ &= \frac{1}{\varepsilon\sqrt{2}}\int_0^\infty \left|1-t^2\right|\mathcal{H}^1(\{\rho=t\}\cap\Omega)dt. \end{aligned} \tag{2.10}$$

We break the last integral into two parts and bound

$$\begin{aligned} &\frac{1}{\varepsilon\sqrt{2}}\int_0^\infty \left|1-t^2\right|\mathcal{H}^1(\{\rho=t\}\cap\Omega)dt \\ &\geq \frac{1}{\varepsilon\sqrt{2}}\int_{\frac{1}{2}}^{\frac{3}{4}}(1-t^2)\mathcal{H}^1(\{\rho=t\}\cap\Omega)dt + \frac{1}{\varepsilon\sqrt{2}}\int_{\frac{5}{4}}^{\frac{3}{2}}(t^2-1)\mathcal{H}^1(\{\rho=t\}\cap\Omega)dt \\ &= \frac{1}{\varepsilon 4\sqrt{2}}(1-t_0^2)\mathcal{H}^1(\{\rho=t_0\}\cap\Omega) + \frac{1}{\varepsilon 4\sqrt{2}}(t_1^2-1)\mathcal{H}^1(\{\rho=t_1\}\cap\Omega), \end{aligned} \tag{2.11}$$

where the last equality follows from the mean value theorem, and $t_0 \in (\frac{1}{2},\frac{3}{4})$ and $t_1 \in (\frac{5}{4},\frac{3}{2})$. The bounds on t_0 and t_1 imply that

$$\begin{aligned} (1-t_0^2) &\geq 1-\frac{9}{16} = \frac{7}{16}, \text{ and} \\ (t_1^2-1) &\geq \frac{25}{16}-1 = \frac{9}{16}. \end{aligned} \tag{2.12}$$

Combining (2.10), (2.11), and (2.12), we get

$$\begin{aligned} F_\varepsilon(\rho,\Omega) &\geq \frac{7}{\varepsilon 64\sqrt{2}}\mathcal{H}^1(\{\rho=t_0\}\cap\Omega) + \frac{9}{\varepsilon 64\sqrt{2}}\mathcal{H}^1(\{\rho=t_1\}\cap\Omega) \\ &\geq \frac{7}{\varepsilon 64\sqrt{2}}\left(\mathcal{H}^1(\{\rho=t_0\}\cap\Omega) + \mathcal{H}^1(\{\rho=t_1\}\cap\Omega)\right). \end{aligned} \tag{2.13}$$

Write S_{t_0} and S^{t_1} for the $\mathbb{R}^2$-closures of the sets $\{x\in\Omega \mid \rho(x)\leq t_0\}$ and $\{x\in\Omega \mid \rho(x)\geq t_1\}$ respectively. The bounds $t_0 \geq \frac{1}{2}$, $t_1 \leq \frac{3}{2}$ imply the inclusions $\{\rho \leq 1/2\} \subset S_{t_0}$ and $\{\rho \geq 3/2\} \subset S^{t_1}$. We may then apply lemmas 2.5 and 2.7 to find the bounds

$$\begin{aligned} \mathcal{H}^1(\{\rho=t_0\}\cap\Omega) + \mathcal{H}^1(\{\rho=t_1\}\cap\Omega) &= \mathcal{H}^1(\partial S_{t_0}\cap\Omega) + \mathcal{H}^1(\partial S^{t_1}\cap\Omega) \\ &\geq 2r_\Omega(S_{t_0}) + 2r_\Omega(S^{t_1}) \\ &\geq 2r_\Omega\left(\{\rho\leq 1/2\}\right) + 2r_\Omega\left(\{\rho\geq 3/2\}\right) \\ &\geq 2r_\Omega\left(\{\rho\leq 1/2\}\cup\{\rho\geq 3/2\}\right). \end{aligned} \tag{2.14}$$

Putting (2.14) into (2.13) yields the desired estimate with $C = \frac{32\sqrt{2}}{7}$. □

3 Improved lower bounds on annuli

In this section we will show how to obtain lower bounds for the Ginzburg-Landau energy in terms of the degree. We begin by constructing estimates on circles. The primary difference between our estimates and those constructed previously is that we arrive at our lower bounds by introducing an auxiliary function G and using a completion of the square trick. This allows us to retain terms involving G and thereby create an energy bound with a novel term. Before properly defining G let us prove the lower bounds on circles.

We first record a simple lemma (see for example Lemma 3.4 in [8]).

Lemma 3.1. *Let $u \in H^1(\Omega, \mathbb{C})$ be written (at least locally) $u = \rho v$, where $\rho = |u|$ and $v = e^{i\varphi}$. Then $|\nabla_A u|^2 = |\nabla \rho|^2 + \rho^2 |\nabla\varphi - A|^2 = |\nabla\rho|^2 + \rho^2 |\nabla_A v|^2$.*

Now we prove the lower bounds on circles.

Lemma 3.2. *Let $B := B(a,r) \subset \mathbb{R}^2$, and suppose that $v : \partial B \to \mathbb{S}^1$ and $A : B \to \mathbb{R}^2$ are both C^1. Let $G : \partial B \to \mathbb{R}^2$ be given by $G = \frac{c\tau}{r}$, where τ is the oriented unit tangent vector field to ∂B and c is a constant. Write $d_B := \deg(v, \partial B)$. Then for any $\lambda > 0$,*

$$\frac{1}{2}\int_{\partial B} |\nabla_A v|^2 + \frac{\lambda}{2}\int_B (\operatorname{curl} A)^2 \geq \frac{1}{2}\int_{\partial B} |\nabla_{A+G} v|^2 + \frac{\pi}{r}(2cd_B - c^2) - \frac{\pi c^2}{2\lambda}. \tag{3.1}$$

Proof. Define the quantity

$$X := \int_B \operatorname{curl} A = \int_{\partial B} A \cdot \tau. \tag{3.2}$$

We write $v = e^{i\varphi}$ and recall that $2\pi d_B = \int_{\partial B} \nabla\varphi \cdot \tau$. Using Lemma 3.1, we see

$$\begin{aligned}
\int_{\partial B} |\nabla_{A+G} v|^2 &= \int_{\partial B} |\nabla\varphi - A - G|^2 \\
&= \int_{\partial B} |G|^2 - 2\int_{\partial B} G \cdot (\nabla\varphi - A) + \int_{\partial B} |\nabla\varphi - A|^2 \\
&= \frac{2\pi r c^2}{r^2} - \frac{2c}{r}\int_{\partial B} \nabla\varphi\cdot\tau + \frac{2c}{r}\int_{\partial B} A\cdot\tau + \int_{\partial B} |\nabla_A v|^2 \\
&= \frac{2\pi c^2}{r} - \frac{2c}{r} 2\pi d_B + \frac{2c}{r} X + \int_{\partial B} |\nabla_A v|^2 \\
&= \frac{2\pi(c^2 - 2cd_B)}{r} + \frac{2c}{r} X + \int_{\partial B} |\nabla_A v|^2 .
\end{aligned} \tag{3.3}$$

An application of Hölder's inequality shows that

$$\int_B (\operatorname{curl} A)^2 \geq \frac{1}{\pi r^2}\left(\int_B \operatorname{curl} A\right)^2 = \frac{1}{\pi r^2} X^2. \tag{3.4}$$

Combining (3.3) and (3.4) yields the inequality

$$\frac{1}{2}\int_{\partial B} |\nabla_A v|^2 + \frac{\lambda}{2}\int_B (\operatorname{curl} A)^2 \geq \frac{1}{2}\int_{\partial B} |\nabla_{A+G} v|^2 + \frac{\pi(2cd_B - c^2)}{r} - \frac{c}{r}X + \frac{\lambda}{2\pi r^2}X^2. \tag{3.5}$$

As X varies, the minimum value of the right hand side occurs when $X = \frac{\pi c r}{\lambda}$. Plugging this into (3.5) yields (3.1).

$\square$

For this lemma to be useful we must construct a function $G : \Omega \to \mathbb{R}^2$ compatible with the ball growth lemma. That is, since estimates will ultimately be added up over balls B, G must have the property that on each ∂B, $G = \tau_{\partial B} \frac{c}{r}$ with r the distance to the center of B. We will take advantage of the fact that c was an arbitrary constant; many of the following results are thus valid with any choice of constants, and it is only much later that we choose specific values. Observe already, though, that taking $c = d_B$ yields an improvement by the $\int |\nabla_{A+G} v|^2$ term to the bounds constructed in Lemma 4.4 of [8]. Unfortunately, we must choose a more complicated constant c to make the estimates in Sections 5 and 6 work. We now show how to define such a G so that it will be useful analytically.

Let $\Omega \subset \mathbb{R}^2$ be open and let $\{\mathcal{B}(t)\}_{t \in [0,s]}$ be a family of collections of closed, disjoint balls grown via the ball growth lemma from an initial collection $\mathcal{B}_0$ that covers the set on which u is near 0. Let $\mathcal{G}$ denote the subcollection of balls in $\mathcal{B}(s)$ entirely contained in Ω, and let $\mathcal{G}(t)$ denote the balls in $\mathcal{B}(t)$ that are contained in a ball from $\mathcal{G}$, i.e. that remain inside Ω for all t. For each ball $B \in \mathcal{G}(t)$ we define several quantities. Let $\tau_{\partial B} : \partial B \to \mathbb{R}^2$ denote the oriented unit tangent vector field to ∂B, and let a_B denote the center of B. Let $d_B = \deg(u/|u|, \partial B)$; this is well-defined since the set on which u vanishes is contained in $\mathcal{B}_0$. Let β_B denote a constant, to be specified later, with the property that if $B_1 \in \mathcal{G}(t_1)$, $B_2 \in \mathcal{G}(t_2)$, and $B_2 = e^{t_2 - t_1} B_1$ (i.e. B_2 is grown from B_1 without any mergings) then $\beta_{B_1} = \beta_{B_2}$. In other words, the β_B are constant over each annulus produced by the ball construction. Let $T \subset [0,s]$ denote the finite set of times from the ball growth lemma at which a merging occurs in the growth of $\mathcal{G}(t)$. We then define the function $G : \Omega \to \mathbb{R}^2$ by

$$G(x) = \begin{cases} \tau_{\partial B}(x) \frac{d_B \beta_B}{|x - a_B|} & \text{if } x \in \partial B \text{ for some } B \in \mathcal{G}(t), t \in [0,s] \backslash T \\ 0 & \text{otherwise.} \end{cases} \tag{3.6}$$

The ball growth lemma guarantees that if $x \in \partial B$ for some $B \in \mathcal{G}(t), t \in [0,s] \backslash T$, then that t is unique, and so $G(x)$ is well defined. By construction, $G = 0$ in $\cup_{B \in \mathcal{G}(0)} B$, and so we can use the above definition of G to extend any function previously defined on $\cup_{B \in \mathcal{G}(0)} B$. We will frequently do so.

Figure 1 shows a simple example of balls grown near the boundary of Ω. Four initial balls, colored light gray, are grown into three final balls, labeled B_1, B_2, B_3. The initial balls are first grown with by a conformal factor of $\tau = \log 2$ until a merging in required in the balls that become B_1. The result of this merging is the white ball contained in B_1. The growth is then continued with a conformal factor of $\tau = \log(6/5)$ to produce the final balls. The annuli on which G is defined are colored in dark gray and black. Since B_3 leaves the domain, G is set to zero on the annuli inside it. G also vanishes on the white region contained in B_1.

With G now properly defined we can show how to couple Lemma 3.2 to the ball growth lemma to produce lower bounds on annuli.

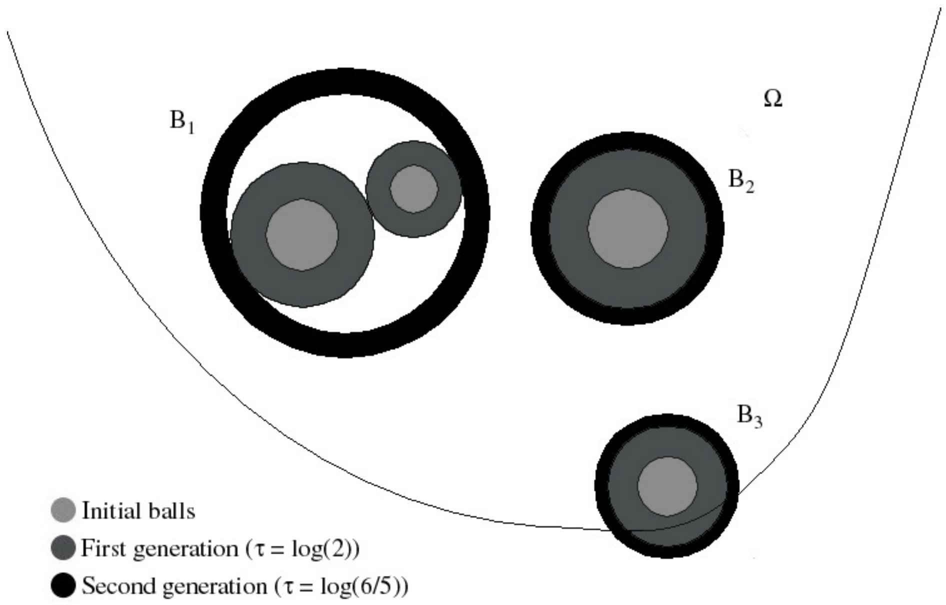

Figure 1: Balls grown near the boundary of Ω

Proposition 3.3. *Let $\mathcal{B}_0$ be a finite, disjoint collection of closed balls and let $\Omega \subseteq \mathbb{R}^2$ be open. Let $\omega = \cup_{B\in\mathcal{B}_0} B$ and denote the collection of balls obtained from $\mathcal{B}_0$ via the ball growth lemma by $\{\mathcal{B}(t)\}$, $t \geq 0$. Suppose that $v : \Omega\backslash\omega \to \mathbb{S}^1$ and $A : \Omega \to \mathbb{R}^2$ are both C^1, and let $G : \Omega \to \mathbb{R}^2$ be the function defined by (3.6). Fix $s > 0$ such that $r(\mathcal{B}(s)) \leq 1$. Then, for any $\bar{B} \in \mathcal{B}(s)$ such that $\bar{B} \subset \Omega$, and any $\lambda > 0$, we have*

$$\begin{aligned} \frac{1}{2}\int_{\bar{B}\backslash\omega} |\nabla_A v|^2 + \frac{r(\bar{B})\lambda}{2}\int_{\bar{B}} (\operatorname{curl} A)^2 - \sum_{B\in\bar{B}\cap\mathcal{B}_0} \frac{r(B)\lambda}{2}\int_B (\operatorname{curl} A)^2 \\ \geq \frac{1}{2}\int_{\bar{B}\backslash\omega} |\nabla_{A+G} v|^2 + \int_0^s \sum_{B\in\bar{B}\cap\mathcal{B}(t)} \pi d_B^2 \left(2\beta_B - \beta_B^2 - \frac{\beta_B^2 r(\mathcal{B}(t))}{2\lambda}\right) dt, \end{aligned} \tag{3.7}$$

where we have written $d_B = \deg(u/|u|, \partial B)$.

Proof. In order to utilize Lemma 2.3 we define the function

$$\mathcal{F}(x,r) = \frac{1}{2}\int_{B(x,r)} |\nabla_A v|^2 + \frac{r\lambda}{2}\int_{B(x,r)} (\operatorname{curl} A)^2. \tag{3.8}$$

Differentiating and using (3.1) with $c = \beta_B d_B$, we arrive at the bound

$$\begin{aligned} \frac{\partial\mathcal{F}}{\partial r} &\geq \frac{1}{2}\int_{\partial B(x,r)} |\nabla_A v|^2 + \frac{\lambda}{2}\int_{B(x,r)} (\operatorname{curl} A)^2 \\ &\geq \frac{1}{2}\int_{\partial B(x,r)} |\nabla_{A+G} v|^2 + \frac{\pi d_B^2}{r}(2\beta_B - \beta_B^2) - \frac{\pi d_B^2\beta_B^2}{2\lambda}. \end{aligned} \tag{3.9}$$

We now recall the notation of Lemma 2.3: $0 < s_1 < \cdots < s_K \le s$ denote the times at which merging occurs in the growth of $\mathcal{B}_0$ to $\mathcal{B}(s)$ via the ball growth lemma, and

$$\mathcal{F}(\bar{B} \cap \mathcal{B}(s_k))^- = \lim_{t \to s_k^-} \mathcal{F}(\bar{B} \cap \mathcal{B}(t)). \tag{3.10}$$

By discarding the terms involving $\operatorname{curl} A$, we see that

$$\begin{aligned} &\sum_{k=1}^{K} \mathcal{F}(\bar{B} \cap \mathcal{B}(s_k)) - \mathcal{F}(\bar{B} \cap \mathcal{B}(s_k))^- \\ &\qquad \ge \sum_{k=1}^{K} \left(\sum_{B \in \bar{B} \cap \mathcal{B}(s_k)} \frac{1}{2} \int_B |\nabla_A v|^2 - \lim_{t \to s_k^-} \sum_{B \in \bar{B} \cap \mathcal{B}(t)} \frac{1}{2} \int_B |\nabla_A v|^2 \right), \end{aligned} \tag{3.11}$$

which corresponds to the integral of $\frac{1}{2} |\nabla_{A+G} v|^2$ over the non-annular parts of $\bar{B} \backslash \omega$ since $G = 0$ there. Since the ball growth lemma makes

$$\frac{d}{dt} r(\mathcal{B}(t)) = r(\mathcal{B}(t)),$$

the expression

$$\int_0^s \sum_{B \in \bar{B} \cap \mathcal{B}(t)} \frac{r(B)}{2} \int_{\partial B} |\nabla_{A+G} v|^2 \, dt$$

corresponds to the integral of $\frac{1}{2} |\nabla_{A+G} v|^2$ over the annular parts of $\bar{B} \backslash \omega$. We now combine this observation, inequalities (3.9) and (3.11), and equality (2.2) to conclude that

$$\begin{aligned} &\mathcal{F}(\bar{B}) - \mathcal{F}(\bar{B} \cap \mathcal{B}_0) \\ &\ge \frac{1}{2} \int_{\bar{B} \backslash \omega} |\nabla_{A+G} v|^2 + \int_0^s \sum_{B \in \bar{B} \cap \mathcal{B}(t)} \pi d_B^2 \left(2\beta_B - \beta_B^2 - \frac{\beta_B^2 r(B)}{2\lambda} \right) dt \\ &\ge \frac{1}{2} \int_{\bar{B} \backslash \omega} |\nabla_{A+G} v|^2 + \int_0^s \sum_{B \in \bar{B} \cap \mathcal{B}(t)} \pi d_B^2 \left(2\beta_B - \beta_B^2 - \frac{\beta_B^2 r(\mathcal{B}(t))}{2\lambda} \right) dt. \end{aligned} \tag{3.12}$$

This is (3.7). $\square$

The following corollary shows that our method, using G, can be used to recover the same estimates found in Proposition 4.3 of [8].

Corollary 3.4. *Under the same assumptions as in Proposition 3.3 we have*

$$\frac{1}{2} \int_{\bar{B} \backslash \omega} |\nabla_A v|^2 + \frac{r(\bar{B})(r_1 - r_0)}{2} \int_{\bar{B}} (\operatorname{curl} A)^2 \ge \int_0^s \sum_{B \in \bar{B} \cap \mathcal{B}(t)} \pi d_B^2 \left(1 - \frac{r(\mathcal{B}(t))}{2(r_1 - r_0)} \right) dt, \tag{3.13}$$

and

$$\frac{1}{2} \int_{\bar{B} \backslash \omega} |\nabla_A v|^2 + \frac{r(\bar{B})(r_1 - r_0)}{2} \int_{\bar{B}} (\operatorname{curl} A)^2 \ge \pi \left| d_{\bar{B}} \right| \left(\log \frac{r_1}{r_0} - \log 2 \right), \tag{3.14}$$

where $r_0 := r(\mathcal{B}_0)$ and $r_1 := r(\mathcal{B}(s)) = e^s r_0$.

Proof. Set $\lambda = r_1 - r_0$, each $\beta_B = 1$, and disregard the $|\nabla_{A+G} v|$ term and the curl terms on $\mathcal{B}_0$ in (3.7) to get (3.13). If $\log \frac{r_1}{r_0} < \log 2$, then (3.14) follows trivially. On the other hand, if $\log \frac{r_1}{r_0} \geq \log 2$, then $r_1 \geq 2r_0$, which implies

$$1 - \frac{r(\mathcal{B}(t))}{2(r_1 - r_0)} \geq 1 - \frac{r_1}{2(r_1 - r_0)} = \frac{r_1 - 2r_0}{2(r_1 - r_0)} \geq 0. \tag{3.15}$$

Then (3.14) follows by noting that $r_1 = e^s r_0$,

$$\frac{d}{dt} r(\mathcal{B}(t)) = r(\mathcal{B}(t)), \tag{3.16}$$

and (see Lemma 4.2 in [8])

$$\sum_{B \in \bar{B} \cap \mathcal{B}(t)} d_B^2 \geq \sum_{B \in \bar{B} \cap \mathcal{B}(t)} |d_B| \geq |d_{\bar{B}}|. \tag{3.17}$$

□

We will need the following modification of the previous corollary later. It is a slight modification of Proposition 4.3 from [8].

Lemma 3.5. *Under the same assumptions as in Proposition 3.3 we have*

$$\frac{1}{2} \int_{\bar{B} \setminus \omega} |\nabla_A v|^2 + \frac{r(\bar{B}) r_1}{2} \int_{\bar{B}} (\operatorname{curl} A)^2 \geq \frac{2\pi}{3} \int_0^s \sum_{B \in \bar{B} \cap \mathcal{B}(t)} d_B^2 \, dt. \tag{3.18}$$

Proof. Lemma 4.4 from [8] provides the lower bound on circles, $\partial B = \partial B(a, r)$:

$$\frac{1}{2} \int_{\partial B} |\nabla_A v|^2 + \frac{\lambda}{2} \int_B (\operatorname{curl} A)^2 \geq \pi \frac{d_B^2}{r} \left(\frac{2\lambda}{2\lambda + r} \right). \tag{3.19}$$

We now set $\lambda = r_1$, bound

$$\frac{2r_1}{2r_1 + r} \geq \frac{2}{3},$$

and proceed as before to conclude.

□

4 Initial and final balls

In this section we record the energy estimates that couple to the ball construction. For technical reasons that will arise in the proof of Theorem 1 we must use the ball growth lemma in two phases, just as in Chapter 4 of [8]. The first phase produces a collection of initial balls that cover the set where $|u|$ is far from unity and on which lower bounds of a type needed in the proof of Theorem 1 are satisfied. This initial collection contains as a subset a collection of balls on which we initially define the function G. The second phase produces a collection of final balls, grown from the initial balls, of a chosen size and on which nice lower bounds hold. In the final section we finally specify the values of the β_B used to define G and show that certain lower bounds hold with this choice of constants.

4.1 The initial balls

Before we can produce the collection of initial balls, we must first produce a collection of balls that covers the set where $|u|$ is far from unity. This is accomplished via the following lemma (Proposition 4.8 from [8]), which shows how the radius of this set is controlled by the energy of $|u|$.

Lemma 4.1. *Let $M, \varepsilon, \delta > 0$ be such that $\varepsilon, \delta < 1$, and let $u \in C^1(\Omega, \mathbb{C})$ satisfy the bound $F_\varepsilon(|u|, \Omega) \leq M$. Then*

$$r(\{x \in \Omega_\varepsilon \mid |u(x) - 1| \geq \delta\}) \leq C\frac{\varepsilon M}{\delta^2} \tag{4.1}$$

where C is a universal constant and $\Omega_\varepsilon = \{x \in \Omega \mid d(x, \partial\Omega) > \varepsilon\}$.

The next technical result shows how to bound from below the modified radius of sub- and super-level sets.

Lemma 4.2. *Let $\Omega \subset \mathbb{R}^2$ be open, $\Omega_\varepsilon = \{x \in \Omega \mid d(x, \partial\Omega) > \varepsilon\}$, and suppose $\mathcal{B}$ is a finite collection of disjoint, closed balls that cover the set*

$$\{x \in \Omega_\varepsilon \mid |u(x) - 1| \geq \delta\}.$$

Let $\mathcal{B}_b$ denote the subcollection of balls in $\mathcal{B}$ that intersect $\partial\Omega_\varepsilon$, and let $\mathcal{B}_i$ denote the subcollection of balls in $\mathcal{B}$ contained in the interior of Ω_ε (i.e. $\mathcal{B} = \mathcal{B}_b \cup \mathcal{B}_i$). Define $\tilde{\Omega} = \Omega_\varepsilon \backslash (\cup_{B\in\mathcal{B}_b} B)$. For $0 < s \leq t$ define the sets $\omega_t = \{x \in \Omega_\varepsilon \mid |u| \leq t\}$, $\omega^t = \{x \in \Omega_\varepsilon \mid |u| \geq t\}$, and $\omega_s^t = \omega_s \cup \omega^t$. Then

$$\begin{aligned} &r_{\Omega_\varepsilon}(\omega_t) \geq r(\omega_t \cap \tilde{\Omega}) \text{ for } t \in (0, 1-\delta), \\ &r_{\Omega_\varepsilon}(\omega^t) \geq r(\omega^t \cap \tilde{\Omega}) \text{ for } t \in (1+\delta, \infty), \text{ and} \\ &r_{\Omega_\varepsilon}(\omega_s^t) \geq r(\omega_s^t \cap \tilde{\Omega}) \text{ for } s \in (0, 1-\delta), t \in (1+\delta, \infty). \end{aligned} \tag{4.2}$$

Proof. Suppose that $t \in (0, 1-\delta)$ and let $\mathrm{Int}(\cdot)$ denote the interior of a set. Write $V = \cup_{B\in\mathcal{B}} B$ and $V_i = \cup_{B\in\mathcal{B}_i} B$. Since the inclusions

$$\mathrm{Int}(V) \supseteq \mathrm{Int}(\{x \in \Omega_\varepsilon \mid |u(x) - 1| \geq \delta\}) \supset \omega_t \tag{4.3}$$

hold, we have that $\omega_t \cap \tilde{\Omega} = \omega_t \cap V_i$, and hence $r(\omega_t \cap \tilde{\Omega}) = r(\omega_t \cap V_i)$. When combined with the fact that V_i is a compact subset of Ω_ε and $\partial V_i \cap \omega_t = \varnothing$, this yields the first estimate in (4.2). Similar arguments prove the second and third assertions. □

We now construct the initial balls. The following proposition is the analogue of Proposition 4.7 of [8], but here we have an extra term of the form

$$\int |\nabla_{A+G} v|^2.$$

Note that items 1, 2, and 3 are the same as those found in [8]; item 4 is new.

Proposition 4.3. *Let $\alpha \in (0,1)$. There exists $\varepsilon_0 > 0$ (depending on α) such that for $\varepsilon \le \varepsilon_0$ and $u \in C^1(\Omega, \mathbb{C})$ with $F_\varepsilon(|u|, \Omega) \le \varepsilon^{\alpha-1}$, the following hold.*

There exists a finite, disjoint collection of closed balls, denoted by $\mathcal{B}_0$, with the following properties.

1. $r(\mathcal{B}_0) = C\varepsilon^{\alpha/2}$, where C is a universal constant.

2. $\{x \in \Omega_\varepsilon \mid |u(x) - 1| \ge \delta\} \subset V_0 := \Omega_\varepsilon \cap (\cup_{B\in\mathcal{B}_0} B)$, where $\delta = \varepsilon^{\alpha/4}$.

3. Write $v = u/|u|$. For $t \in (0, 1-\delta)$ we have the estimate

$$\frac{1}{2}\int_{V_0\backslash\omega_t} |\nabla_A v|^2 + \frac{r(\mathcal{B}_0)^2}{2}\int_{V_0} (\operatorname{curl} A)^2 \ge \pi D_0 \left(\log \frac{r(\mathcal{B}_0)}{r_{\Omega_\varepsilon}(\omega_t)} - C\right), \tag{4.4}$$

where

$$D_0 = \sum_{\substack{B\in\mathcal{B}_0 \\ B\subset\Omega_\varepsilon}} |d_B|. \tag{4.5}$$

4. There exists a family of finite collections of closed, disjoint balls $\{\mathcal{C}(s)\}_{s\in[0,\sigma]}$, all of which are contained in in V_0, and that are grown according to the ball growth lemma from an initial collection, $\mathcal{C}(0)$, that covers the set $\omega_{1/2}^{3/2} \cap V_0$. The number σ is such that $r(\mathcal{C}(\sigma)) = \frac{3}{8}r(\mathcal{B}_0)$. Let $G : V_0 \to \mathbb{R}^2$ be the function defined by using Ω_ε and $\{\mathcal{C}(s)\}_{s\in[0,\sigma]}$ in (3.6) *and then extended by zero to the rest of V_0. For each $\lambda > 0$ we have the estimate*

$$\frac{1}{2}\int_{V_0\backslash\omega_{1/2}^{3/2}} |\nabla_A v|^2 + \sum_{B\in\mathcal{B}_0} \frac{r(B)\lambda}{2}\int_{B\cap\Omega} (\operatorname{curl} A)^2$$

$$\ge \int_0^\sigma \sum_{\substack{\bar{B}\in\mathcal{C}(\sigma) \\ \bar{B}\subset\Omega_\varepsilon}} \sum_{B\in\bar{B}\cap\mathcal{C}(t)} \pi d_B^2 \left(2\beta_B - \beta_B^2 - \frac{\beta_B^2 r(\mathcal{C}(t))}{2\lambda}\right) dt + \frac{1}{2}\int_{V_0\backslash\omega_{1/2}^{3/2}} |\nabla_{A+G} v|^2. \tag{4.6}$$

Proof. We break the proof into six steps. The first four consist of finding four collections of balls that are used to create the initial collection $\mathcal{B}_0$. The last two steps prove the estimates of items 3 and 4.

Step 1.

Using $M = \varepsilon^{\alpha-1}$ and $\delta = \varepsilon^{\alpha/4}$ in Lemma 4.1 produces a collection of disjoint, closed balls $\mathcal{E}$ that cover the set $\{x \in \Omega_\varepsilon \mid |u(x) - 1| \ge \delta\}$ such that $R := r(\mathcal{E}) \le C\varepsilon^{\alpha/2}$. We will eventually need to use Lemma 4.2, so we employ its notation by breaking the collection $\mathcal{E}$ into subcollections $\mathcal{E}_i$ and $\mathcal{E}_b$ and defining the set $\tilde{\Omega} = \Omega_\varepsilon\backslash(\cup_{B\in\mathcal{E}_b} B)$.

Step 2.

By the definition of the radius of a set, for any $t \in (0, 1-\delta)$ we can cover $\omega_t \cap \tilde{\Omega}$ by a collection of disjoint balls, denoted by $\mathcal{B}_t^0$, with total radius less than $2r(\omega_t \cap \tilde{\Omega})$. Since $r(\omega_t \cap \tilde{\Omega}) \le R$, we can use Lemma 2.2 to grow the collection $\mathcal{B}_t^0$ into a collection $\mathcal{B}_t$ such that $r(\mathcal{B}_t) = 2R$. We then utilize Corollary 3.4 on each of the balls in $\mathcal{B}_t$ that is contained in $\tilde{\Omega}$ and sum to get the estimate

$$\frac{1}{2}\int_{V_t\setminus\omega_t} |\nabla_A v|^2 + \frac{4R^2}{2}\int_{V_t} (\operatorname{curl} A)^2 \geq \pi D_t\left(\log\frac{2R}{2r(\omega_t\cap\tilde{\Omega})} - \log 2\right), \tag{4.7}$$

where

$$\begin{aligned} V_t &= \tilde{\Omega} \cap (\cup_{B\in\mathcal{B}_t} B), \text{ and} \\ D_t &= \sum_{\substack{B\in\mathcal{B}_t \\ B\subset\tilde{\Omega}}} |d_B|. \end{aligned}$$

Choose $\bar{t} \in (0, 1-\delta)$ such that $D_{\bar{t}}$ is minimal.

Step 3.

Let m denote the supremum of

$$\mathcal{F}(K) := \frac{1}{2}\int_{(K\cap\tilde{\Omega})\setminus\omega} |\nabla_A v|^2 + \frac{4R^2}{2}\int_{K\cap\tilde{\Omega}} (\operatorname{curl} A)^2$$

over compact $K \subset \Omega$ such that $r(K) < 2R$. Choose K so that $r(K) < 2R$ and $\mathcal{F}(K) \geq m - 1$. Cover K by a collection of disjoint, closed balls $\mathcal{K}$ such that $r(\mathcal{K}) = 2R$ (the existence of such a collection is guaranteed by the ball growth lemma).

Step 4.

We can cover $\omega_{1/2}^{3/2} \cap \tilde{\Omega}$ by a collection of disjoint balls, denoted by $\mathcal{C}_0$, with radius less than $\frac{3}{2}r(\omega_{1/2}^{3/2} \cap \tilde{\Omega})$. We use the ball growth lemma, applied to $\mathcal{C}_0$, to produce a family of collections $\{\mathcal{C}(s)\}$ with $s \in (0, \sigma)$,

$$\sigma = \log\left(\frac{3R}{r(\mathcal{C}_0)}\right).$$

Let $\mathcal{C} = \mathcal{C}(\sigma)$ and note that by construction $r(\mathcal{C}) = 3R$.

Step 5.

Define $\mathcal{B}_0$ to be a collection of disjoint balls that cover the balls in $\mathcal{B}_{\bar{t}}$, $\mathcal{K}$, $\mathcal{C}$, and $\mathcal{E}$. We may choose such a collection so that $r(\mathcal{B}_0) = 8R$. Let $V_0 = \Omega_\varepsilon \cap (\cup_{B\in\mathcal{B}_0} B)$. Then

$$I := \frac{1}{2}\int_{V_0\setminus\omega_t} |\nabla_A v|^2 + \frac{r(\mathcal{B}_0)^2}{2}\int_{V_0} (\operatorname{curl} A)^2 \geq \mathcal{F}(K) + \frac{1}{2}\int_{\omega\setminus\omega_t} |\nabla_A v|^2, \tag{4.8}$$

and by the construction of K and V_t for any $t \in (0, 1-\delta)$, this implies

$$
\begin{aligned}
I + 1 &\geq \mathcal{F}(V_t) + \frac{1}{2}\int_{\omega\setminus\omega_t} |\nabla_A v|^2 \\
&\geq \frac{1}{2}\int_{V_t\setminus\omega_t} |\nabla_A v|^2 + \frac{4R^2}{2}\int_{V_t} (\operatorname{curl} A)^2 \\
&\geq \pi D_t \left(\log \frac{2R}{2r(\omega_t \cap \tilde{\Omega})} - \log 2\right) \\
&\geq \pi D_t \left(\log \frac{r(\mathcal{B}_0)}{r_{\Omega_\varepsilon}(\omega_t)} - C\right),
\end{aligned}
\tag{4.9}
$$

where the last line follows from (4.2) and the fact that $r(\mathcal{B}_0) = 8R$. By the choice of $\bar{t}$,

$$
D_t \geq D_{\bar{t}} = \sum_{\substack{B \in \mathcal{B}_{\bar{t}} \\ B \subset \tilde{\Omega}}} |d_B| \,. \tag{4.10}
$$

We break the collection of balls in the last sum in (4.10) into two subcollections:

$$
\begin{aligned}
I_1 &:= \{B \in \mathcal{B}_{\bar{t}} \mid B \subseteq \tilde{\Omega}, B \subseteq B' \in \mathcal{B}_0 \text{ so that } B' \cap \partial\Omega_\varepsilon \neq \varnothing\} \\
I_2 &:= \{B \in \mathcal{B}_{\bar{t}} \mid B \subseteq \tilde{\Omega}, B \subseteq B' \in \mathcal{B}_0 \text{ so that } B' \subseteq \Omega_\varepsilon\}.
\end{aligned}
$$

Then

$$
\sum_{\substack{B \in \mathcal{B}_{\bar{t}} \\ B \subset \tilde{\Omega}}} |d_B| = \sum_{B \in I_1} |d_B| + \sum_{B \in I_2} |d_B| \geq 0 + \sum_{\substack{B \in \mathcal{B}_0 \\ B \subset \Omega_\varepsilon}} |d_B| = D_0, \tag{4.11}
$$

where the inequality follows from Lemma 4.2 in [8]. Combining (4.9), (4.10), and (4.11) yields (4.4).

Step 6.

Let U be the union of the balls in $\mathcal{C}_0$ that are contained in Ω_ε and W be the union of the balls in $\mathcal{C}$ that are contained in Ω_ε. Then applying Proposition 3.3 to each $\bar{B} \in \mathcal{C}$ such that $\bar{B} \subset \Omega_\varepsilon$ and summing, we get the estimate

$$
\begin{aligned}
&\frac{1}{2}\int_{W\setminus U} |\nabla_A v|^2 + \sum_{\substack{\bar{B} \in \mathcal{C} \\ \bar{B} \subset \Omega_\varepsilon}} \frac{r(\bar{B})\lambda}{2} \int_{\bar{B}} (\operatorname{curl} A)^2 \\
&\qquad \geq \frac{1}{2}\int_{W\setminus U} |\nabla_{A+G} v|^2 + \int_0^\sigma \sum_{\substack{\bar{B} \in \mathcal{C} \\ \bar{B} \subset \Omega_\varepsilon}} \sum_{B \in \bar{B} \cap \mathcal{C}(t)} \pi d_B^2 \left(2\beta_B - \beta_B^2 - \frac{\beta_B^2 r(\mathcal{C}(t))}{2\lambda}\right) dt.
\end{aligned}
\tag{4.12}
$$

G vanishes in the regions $V_0 \backslash W$ and $U \backslash \omega_{1/2}^{3/2}$, so

$$\frac{1}{2} \int_{(V_0 \backslash W) \cup (U \backslash \omega_{1/2}^{3/2})} |\nabla_A v|^2 = \frac{1}{2} \int_{(V_0 \backslash W) \cup (U \backslash \omega_{1/2}^{3/2})} |\nabla_{A+G} v|^2 . \tag{4.13}$$

Adding (4.13) to both sides of (4.12) and noting that

$$\sum_{\substack{\bar{B} \in \mathcal{C} \\ \bar{B} \subset \Omega_\varepsilon}} \frac{r(\bar{B})\lambda}{2} \int_{\bar{B}} (\operatorname{curl} A)^2 \le \sum_{B \in \mathcal{B}_0} \frac{r(B)\lambda}{2} \int_{B \cap \Omega} (\operatorname{curl} A)^2 \tag{4.14}$$

yields (4.6). $\square$

4.2 The final balls

The next proposition constructs the final balls from the initial ones constructed in Proposition 4.3. Items 1, 2, and 3 are the same as those of Theorem 4.1 of [8]; item 4 contains the novel estimate with the G-term.

Proposition 4.4. *Let $\alpha \in (0,1)$. There exists $\varepsilon_0 > 0$ (depending on α) such that for $\varepsilon \le \varepsilon_0$ and $u \in C^1(\Omega, \mathbb{C})$ with $F_\varepsilon(|u|, \Omega) \le \varepsilon^{\alpha-1}$, the following hold.*

For any $1 > r > C\varepsilon^{\alpha/2}$, where C is a universal constant, there exists a finite, disjoint collection of closed balls, denoted by $\mathcal{B}$, with the following properties.

1. $r(\mathcal{B}) = r$.

2. $\{x \in \Omega_\varepsilon \mid |u(x) - 1| \ge \delta\} \subset V := \Omega_\varepsilon \cap (\cup_{B \in \mathcal{B}} B)$, where $\delta = \varepsilon^{\alpha/4}$.

3. Write $v = u/|u|$. For $t \in (0, 1-\delta)$ we have the estimate

$$\frac{1}{2} \int_{V \backslash \omega_t} |\nabla_A v|^2 + \frac{r^2}{2} \int_V (\operatorname{curl} A)^2 \ge \pi D \left(\log \frac{r}{r_{\Omega_\varepsilon}(\omega_t)} - C \right), \tag{4.15}$$

where

$$D = \sum_{\substack{B \in \mathcal{B} \\ B \subset \Omega_\varepsilon}} |d_B| . \tag{4.16}$$

4. Let $G : \Omega \to \mathbb{R}^2$ be the extension, according to (3.6), of the G from item 4 in Proposition 4.3. Write $s = \log \frac{r}{r(\mathcal{B}_0)}$. Then

$$\begin{aligned} \frac{1}{2} \int_{V \backslash \omega_{1/2}^{3/2}} |\nabla_A v|^2 + \sum_{\bar{B} \in \mathcal{B}} \frac{r(\bar{B})(r - r(\mathcal{B}_0))}{2} \int_{\bar{B} \cap \Omega} (\operatorname{curl} A)^2 &\ge \frac{1}{2} \int_{V \backslash \omega_{1/2}^{3/2}} |\nabla_{A+G} v|^2 \\ + \int_0^s \sum_{\substack{\bar{B} \in \mathcal{B} \\ \bar{B} \subset \Omega_\varepsilon}} \sum_{B \in \bar{B} \cap \mathcal{B}(t)} \pi d_B^2 \left(2\beta_B - \beta_B^2 - \frac{\beta_B^2 r(\mathcal{B}(t))}{2(r - r(\mathcal{B}_0))} \right) dt& \\ + \int_0^\sigma \sum_{\substack{\bar{B} \in \mathcal{C}(\sigma) \\ \bar{B} \subset \Omega_\varepsilon}} \sum_{B \in \bar{B} \cap \mathcal{C}(t)} \pi d_B^2 \left(2\beta_B - \beta_B^2 - \frac{\beta_B^2 r(\mathcal{C}(t))}{2(r(\mathcal{C}(\sigma)) - r(\mathcal{C}_0))} \right) dt.& \end{aligned} \tag{4.17}$$

Proof. Lemma 4.3 provides an initial set of disjoint, closed balls $\mathcal{B}_0$. We grow these according to the ball growth lemma to produce $\{\mathcal{B}(t)\}_{t\in[0,s]}$ with s chosen so that $r(\mathcal{B}(s)) = r$, i.e. $s = \log \frac{r}{r(\mathcal{B}_0)}$. By construction, items 1 and 2 are proved. Let $\mathcal{B} = \mathcal{B}(s)$, and write $V = \Omega_\varepsilon \cap \cup_{B\in\mathcal{B}} B$, $V_0 = \Omega_\varepsilon \cap \cup_{B\in\mathcal{B}_0} B$. Let $G : V_0 \to \mathbb{R}^2$ be the function defined in item 4 of Proposition 4.3. We then use $\mathcal{B}_0$ and $\mathcal{B}$ to extend $G : \Omega \to \mathbb{R}^2$ according to (3.6).

We analyze the balls in $\mathcal{B}$ according to whether or not they are contained entirely in Ω_ε. For balls $\bar{B} \in \mathcal{B}$ such that $\bar{B} \subset \Omega_\varepsilon$, we use (3.14), and for the other balls we use the trivial non-negative bound. Summing over all balls in $\mathcal{B}$, we get

$$\frac{1}{2}\int_{V\setminus V_0} |\nabla_A v|^2 + \sum_{\bar{B}\in\mathcal{B}} \frac{r(\bar{B})(r - r(\mathcal{B}_0))}{2} \int_{\bar{B}\cap\Omega} (\operatorname{curl} A)^2 \geq \pi D \left(\log \frac{r}{r(\mathcal{B}_0)} - \log 2\right). \quad (4.18)$$

Adding (4.4) to (4.18) and noting that $D_0 \geq D$ then yields (4.15).

To prove (4.17) we proceed similarly, using different estimates for the balls in $\mathcal{B}$ according to whether or not they are contained in Ω_ε. For balls $\bar{B} \in \mathcal{B}$ such that $\bar{B} \subset \Omega_\varepsilon$ we use Proposition 3.3 to get the estimate

$$\begin{aligned}\frac{1}{2}\int_{\bar{B}\setminus V_0} |\nabla_A v|^2 + \frac{r(\bar{B})\lambda}{2}\int_{\bar{B}} (\operatorname{curl} A)^2 - \sum_{B\in\bar{B}\cap\mathcal{B}_0} \frac{r(B)\lambda}{2}\int_B (\operatorname{curl} A)^2 \\ \geq \frac{1}{2}\int_{\bar{B}\setminus V_0} |\nabla_{A+G} v|^2 + \int_0^s \sum_{B\in\bar{B}\cap\mathcal{B}(t)} \pi d_B^2 \left(2\beta_B - \beta_B^2 - \frac{\beta_B^2 r(\mathcal{B}(t))}{2\lambda}\right).\end{aligned} \quad (4.19)$$

On the other hand, the construction of G guarantees that it vanishes on all balls $\bar{B} \in \mathcal{B}$ such that $\bar{B} \cap \partial\Omega_\varepsilon \neq \varnothing$, and so for such $\bar{B}$ we trivially have the estimate

$$\begin{aligned}\frac{1}{2}\int_{(\bar{B}\cap\Omega)\setminus V_0} |\nabla_A v|^2 + \frac{r(\bar{B})\lambda}{2}\int_{\bar{B}\cap\Omega} (\operatorname{curl} A)^2 - \sum_{B\in\bar{B}\cap\mathcal{B}_0} \frac{r(B)\lambda}{2}\int_{B\cap\Omega} (\operatorname{curl} A)^2 \\ \geq \frac{1}{2}\int_{(\bar{B}\cap\Omega)\setminus V_0} |\nabla_{A+G} v|^2.\end{aligned} \quad (4.20)$$

Summing (4.19) and (4.20) over all balls in $\mathcal{B}$ then yields the estimate

$$\begin{aligned}\frac{1}{2}\int_{V\setminus V_0} |\nabla_A v|^2 + \sum_{\bar{B}\in\mathcal{B}} \frac{r(\bar{B})\lambda}{2}\int_{\bar{B}\cap\Omega} (\operatorname{curl} A)^2 - \sum_{B\in\mathcal{B}_0} \frac{r(B)\lambda}{2}\int_{B\cap\Omega} (\operatorname{curl} A)^2 \\ \geq \frac{1}{2}\int_{V\setminus V_0} |\nabla_{A+G} v|^2 + \int_0^s \sum_{\substack{\bar{B}\in\mathcal{B} \\ \bar{B}\subset\Omega_\varepsilon}} \sum_{B\in\bar{B}\cap\mathcal{B}(t)} \pi d_B^2 \left(2\beta_B - \beta_B^2 - \frac{\beta_B^2 r(\mathcal{B}(t))}{2\lambda}\right) dt.\end{aligned} \quad (4.21)$$

We insert $\lambda = r - r(\mathcal{B}_0)$ into (4.21) and $\lambda = r(\mathcal{C}(\sigma)) - r(\mathcal{C}_0) = \frac{3r(\mathcal{B}_0)}{8} - r(\mathcal{C}_0)$ into (4.6) and add the estimates together. Noting that

$$\frac{3r(\mathcal{B}_0)}{8} - r(\mathcal{C}_0) - r + r(\mathcal{B}_0) \leq C\varepsilon^{\alpha/2} - r \leq 0, \quad (4.22)$$

we arrive at the estimate (4.17).

□

4.3 Degree analysis and selection of the β_B values

We will now select the values of the β_B used to define G. Ultimately, later in Theorem 2, we will get rid of G altogether by bounding its $L^{2,\infty}$ norm by a term of the order D^2. This bound, the proof of which is Proposition 6.4, requires the values of the β_B to be small. However, since they play a role in the lower bounds of Proposition 4.4, we can not choose the β_B to be too small. We balance these two demands by introducing a parameter η to measure when β_B must be small and when it can assume the natural choice for its value, 1.

The next two results establish that for a ball $\bar{B} \in \mathcal{B}(s)$ there is a transition time (depending on η) in the family $\bar{B} \cap \mathcal{B}(t)$ before which we can take $\beta_B = 1$, and after which we must use something more complicated.

Lemma 4.5. *Let $\mathcal{B}_0$ be a finite collection of disjoint, closed balls. Suppose further that the collection $\mathcal{B}_0$ has the degree covering property that for all balls $B \subset \Omega \backslash (\cup_{S \in \mathcal{B}_0} S)$, it is the case that $d_B = 0$. In other words, the collection $\mathcal{B}_0$ covers all of the vortices. Let $\mathcal{B}(t)$, $t \in [0, s]$, be a t-parameterized family of finite collections of disjoint, closed balls. Suppose that $\mathcal{B}_0 = \mathcal{B}(0)$ and that*

$$\bigcup_{B \in \mathcal{B}(t_1)} B \subseteq \bigcup_{B \in \mathcal{B}(t_2)} B \text{ for } t_1 \leq t_2. \tag{4.23}$$

Fix $\bar{B} \in \mathcal{B}(s)$. Define the negative and positive vorticity masses by

$$\begin{aligned} N(t) &:= \sum_{\substack{B \in \bar{B} \cap \mathcal{B}(t) \\ d_B < 0}} |d_B| \\ P(t) &:= \sum_{\substack{B \in \bar{B} \cap \mathcal{B}(t) \\ d_B > 0}} d_B. \end{aligned} \tag{4.24}$$

Then for any $\eta \in (0, 1)$, the following hold.

1. If $d_{\bar{B}} \geq 0$ and the inequality

$$N(s_0) \leq \eta P(s_0) \tag{4.25}$$

holds for some $s_0 \in [0, s]$, then $N(t) \leq \eta P(t)$ for all $t \in [s_0, s]$.

2. If $d_{\bar{B}} < 0$ and the inequality

$$P(s_0) \leq \eta N(s_0) \tag{4.26}$$

holds for some $s_0 \in [0, s]$, then $P(t) \leq \eta N(t)$ for all $t \in [s_0, s]$.

Proof. Take $d_{\bar{B}} \geq 0$; the following proves (4.25), and a similar argument with $d_{\bar{B}} < 0$ proves (4.26). Let $n(t) = \#\mathcal{B}(t)$. Then by the inclusion property (4.23), $n(t)$ is a decreasing $\mathbb{N}$-valued function. Hence there exist finitely many times $0 = t_0 < \cdots < t_K = s$ such that $n(t)$ is constant on (t_i, t_{i+1}). This implies that for $t_i < s < t < t_{i+1}$ and $B \in \mathcal{B}(t)$, there exists exactly one ball $B' \in \mathcal{B}(s)$ such that $B' \subseteq B$, and by the degree covering property, $d_B = d_{B'}$. It follows that $N(t)$ and $P(t)$ are also constant on each (t_i, t_{i+1}). Then it suffices to show that if $N(t_k) \leq \eta P(t_k)$, then $N(t_{k+1}) \leq \eta P(t_{k+1})$.

Given a ball $C \in \mathcal{B}(t_{k+1})$, the inclusion property guarantees that there is a finite collection $\{B_1, \dots, B_j\} \subseteq \mathcal{B}(t_k)$ such that $B_i \subseteq C$ for $i = 1, \dots, j$. We then get

$$
\begin{aligned}
|d_C| &= -\sum_{\substack{i \in \{1,\dots,j\} \\ d_{B_i} \geq 0}} d_{B_i} + \sum_{\substack{i \in \{1,\dots,j\} \\ d_{B_i} < 0}} |d_{B_i}| \ \text{ if } d_C < 0, \text{ and} \\
|d_C| &= \sum_{\substack{i \in \{1,\dots,j\} \\ d_{B_i} \geq 0}} d_{B_i} - \sum_{\substack{i \in \{1,\dots,j\} \\ d_{B_i} < 0}} |d_{B_i}| \ \text{ if } d_C \geq 0.
\end{aligned}
\tag{4.27}
$$

We must now subdivide the collection $\bar{B} \cap \mathcal{B}(t_k)$ according to the degrees of balls in $\bar{B} \cap \mathcal{B}(t_{k+1})$. Define the collections

$$
\begin{aligned}
I_{-,-} &= \{B \in \bar{B} \cap \mathcal{B}(t_k) \mid d_B < 0, \exists B' \in \bar{B} \cap \mathcal{B}(t_{k+1}) \text{ s.t. } B \subset B', d_{B'} < 0\} \\
I_{-,+} &= \{B \in \bar{B} \cap \mathcal{B}(t_k) \mid d_B < 0, \exists B' \in \bar{B} \cap \mathcal{B}(t_{k+1}) \text{ s.t. } B \subset B', d_{B'} \geq 0\} \\
I_{+,-} &= \{B \in \bar{B} \cap \mathcal{B}(t_k) \mid d_B \geq 0, \exists B' \in \bar{B} \cap \mathcal{B}(t_{k+1}) \text{ s.t. } B \subset B', d_{B'} < 0\} \\
I_{+,+} &= \{B \in \bar{B} \cap \mathcal{B}(t_k) \mid d_B \geq 0, \exists B' \in \bar{B} \cap \mathcal{B}(t_{k+1}) \text{ s.t. } B \subset B', d_{B'} \geq 0\}.
\end{aligned}
$$

Now we can estimate

$$
\begin{aligned}
\eta \sum_{B \in I_{-,+}} |d_B| + \sum_{B \in I_{-,-}} |d_B| &\leq \sum_{B \in I_{-,+}} |d_B| + \sum_{B \in I_{-,-}} |d_B| = N(t_k) \\
&\leq \eta P(t_k) = \eta \sum_{B \in I_{+,-}} d_B + \eta \sum_{B \in I_{+,+}} d_B \leq \sum_{B \in I_{+,-}} d_B + \eta \sum_{B \in I_{+,+}} d_B.
\end{aligned}
\tag{4.28}
$$

After regrouping terms according to containment and using (4.27) and (4.28) we conclude

$$
N(t_{k+1}) = \sum_{B \in I_{-,-}} |d_B| - \sum_{B \in I_{+,-}} d_B \leq \eta \sum_{B \in I_{+,+}} d_B - \eta \sum_{B \in I_{-,+}} |d_B| = \eta P(t_{k+1}). \tag{4.29}
$$

$\square$

We use this lemma to define the transition times.

Corollary 4.6. *Assume the hypotheses and notation of Lemma 4.5. If $d_{\bar{B}} \geq 0$ then there exists $t_0 \in [0, s]$ such that $\eta P(t) < N(t)$ for $t \in [0, t_0)$ and $N(t) \leq \eta P(t)$ for $t \in [t_0, s]$. Similarly, if $d_{\bar{B}} < 0$ then there exists $t_0 \in [0, s]$ such that $\eta N(t) < P(t)$ for $t \in [0, t_0)$ and $P(t) \leq \eta N(t)$ for $t \in [t_0, s]$. We call these times, t_0, the transition times.*

Proof. Assume $d_{\bar{B}} \geq 0$. Since there is only one ball in $\bar{B} \cap \mathcal{B}(s)$, and the degree in $\bar{B}$ is nonnegative, the inequality $N(s) \leq \eta P(s)$ is satisfied trivially. An application of Lemma 4.5 proves the existence of t_0. A similar argument works for the case when $d_{\bar{B}} < 0$. $\square$

With the transition times defined we can finally set the values of the β_B. Define the collection $\{\mathcal{D}(t)\}_{t\in[0,s+\sigma]}$ by

$$\mathcal{D}(t) = \begin{cases} \mathcal{C}(t), & t\in[0,\sigma) \\ \mathcal{B}(t-\sigma), & t\in[\sigma,s+\sigma]. \end{cases} \tag{4.30}$$

Let $\eta\in(0,1)$. For each $\bar{B}\in\mathcal{B}$ let $t_{\bar{B}}\in[0,s+\sigma]$ denote the transition time for the collection $\bar{B}\cap\mathcal{D}(t)$ obtained from Corollary 4.6 (the times depend on η). We now specify the values of β_B in the definition of G. Note that the construction of G only requires specifying the values of β_B for those balls B such that $B\subset\bar{B}\in\mathcal{B}$ with $\bar{B}\subset\Omega_\varepsilon$. Then for $B\in\bar{B}\cap\mathcal{D}(t)$ for some $\bar{B}\in\mathcal{B}$, we define

$$\beta_B = \begin{cases} 1, & \text{if } t\in[0,t_{\bar{B}}) \\ |d_{\bar{B}}|^{\frac{1}{2}}\left(\sum_{B'\in\bar{B}\cap\mathcal{D}(t)} d_{B'}^2\right)^{-\frac{1}{2}}, & \text{if } t\in[t_{\bar{B}},s+\sigma]. \end{cases} \tag{4.31}$$

Note that if

$$\sum_{B'\in\bar{B}\cap\mathcal{D}(t)} d_{B'}^2 = 0,$$

then $d_{\bar{B}}=0$ as well, and we take the second case in (4.31) to equal 0. Further, note that in the second case, the β_B are chosen so that for $t\in[t_{\bar{B}},s+\sigma]$

$$\sum_{B\in\bar{B}\cap\mathcal{D}(t)} d_B^2\beta_B^2 = |d_{\bar{B}}|. \tag{4.32}$$

The following proposition shows that G is still useful for the lower bounds with these values of β_B.

Proposition 4.7. *With G defined as above, and under the assumptions of Proposition 4.4, we have the estimate*

$$\frac{1}{2}\int_{V\setminus\omega_{1/2}^{3/2}}|\nabla_A v|^2+\frac{r^2}{2}\int_V(\operatorname{curl}A)^2 \geq \frac{1}{2}\int_{V\setminus\omega_{1/2}^{3/2}}|\nabla_{A+G}v|^2+\pi D\left(\log\frac{r}{r_{\Omega_\varepsilon}(\omega_{1/2}^{3/2})}-C\right). \tag{4.33}$$

Proof. To prove (4.33) we must deal with the sums in the integrands in (4.17). We begin by showing that the terms in parentheses are nonnegative. Since $r(\mathcal{B}_0)=C\varepsilon^{\alpha/2}$ and $\beta_B\leq 1$, we can estimate

$$\begin{aligned} 2\beta_B-\beta_B^2-\frac{\beta_B^2 r(\mathcal{B}(t))}{2(r-r(\mathcal{B}_0))} &= \beta_B^2\left(\frac{2}{\beta_B}-1-\frac{r(\mathcal{B}(t))}{2(r-r(\mathcal{B}_0))}\right) \\ \geq \beta_B^2\left(1-\frac{r(\mathcal{B}(t))}{2(r-r(\mathcal{B}_0))}\right) &\geq \beta_B^2\left(\frac{r-2r(\mathcal{B}_0)}{2(r-r(\mathcal{B}_0))}\right)\geq 0. \end{aligned} \tag{4.34}$$

By construction,

$$r(\mathcal{C}_0) < \frac{3}{2} r(\omega_{1/2}^{3/2} \cap \tilde{\Omega}) \leq \frac{3}{2} R = \frac{1}{2} r(\mathcal{C}(\sigma)), \tag{4.35}$$

and so we can similarly conclude that

$$2\beta_B - \beta_B^2 - \frac{\beta_B^2 r(\mathcal{B}(t))}{2(r(\mathcal{C}(\sigma)) - r(\mathcal{C}_0))} \geq 0. \tag{4.36}$$

A simple change of variables $t \mapsto t + \sigma$ allows us to rewrite

$$\begin{aligned}
&\int_0^s \sum_{\substack{\bar{B} \in \mathcal{B} \\ \bar{B} \subset \Omega_\varepsilon}} \sum_{B \in \bar{B} \cap \mathcal{B}(t)} \pi d_B^2 \left(2\beta_B - \beta_B^2 - \frac{\beta_B^2 r(\mathcal{B}(t))}{2(r - r(\mathcal{B}_0))} \right) dt \\
&+ \int_0^\sigma \sum_{\substack{\bar{B} \in \mathcal{C}(\sigma) \\ \bar{B} \subset \Omega_\varepsilon}} \sum_{B \in \bar{B} \cap \mathcal{C}(t)} \pi d_B^2 \left(2\beta_B - \beta_B^2 - \frac{\beta_B^2 r(\mathcal{C}(t))}{2(r(\mathcal{C}(\sigma)) - r(\mathcal{C}_0))} \right) dt \\
&= \sum_{\substack{\bar{B} \in \mathcal{B} \\ \bar{B} \subset \Omega_\varepsilon}} \int_0^{s+\sigma} \sum_{B \in \bar{B} \cap \mathcal{D}(t)} \pi d_B^2 \left(2\beta_B - \beta_B^2 - \frac{\beta_B^2 r(\mathcal{D}(t))}{2\lambda(t)} \right) dt,
\end{aligned} \tag{4.37}$$

where

$$\lambda(t) = \begin{cases} r(\mathcal{C}(\sigma)) - r(\mathcal{C}_0), & t \in [0, \sigma) \\ r - r(\mathcal{B}_0), & t \in [\sigma, s + \sigma]. \end{cases}$$

Fix $\bar{B} \in \mathcal{B}$ such that $\bar{B} \subset \Omega_\varepsilon$. For $t \in [0, t_{\bar{B}})$ we have that $\beta_B = 1$, and hence

$$\sum_{B \in \bar{B} \cap \mathcal{D}(t)} \pi d_B^2 \left(2\beta_B - \beta_B^2 - \frac{\beta_B^2 r(\mathcal{D}(t))}{2\lambda(t)} \right) \geq \pi d_{\bar{B}} \left(1 - \frac{r(\mathcal{D}(t))}{2\lambda(t)} \right). \tag{4.38}$$

For $t \in [t_{\bar{B}}, s + \sigma]$ we similarly estimate

$$\begin{aligned}
\sum_{B \in \bar{B} \cap \mathcal{D}(t)} d_B^2 \left(2\beta_B - \beta_B^2 - \frac{\beta_B^2 r(\mathcal{D}(t))}{2\lambda(t)} \right) &= 2\, |d_{\bar{B}}|^{\frac{1}{2}} \left(\sum_{B \in \bar{B} \cap \mathcal{D}(t)} d_B^2 \right)^{\frac{1}{2}} - |d_{\bar{B}}| \left(1 + \frac{r(\mathcal{D}(t))}{2\lambda(t)} \right) \\
&\geq 2\, |d_{\bar{B}}|^{\frac{1}{2}}\, |d_{\bar{B}}|^{\frac{1}{2}} - |d_{\bar{B}}| \left(1 + \frac{r(\mathcal{D}(t))}{2\lambda(t)} \right) \\
&= |d_{\bar{B}}| \left(1 - \frac{r(\mathcal{D}(t))}{2\lambda(t)} \right).
\end{aligned} \tag{4.39}$$

This proves that

$$\begin{aligned}
&\sum_{\substack{\bar{B}\in\mathcal{B}\\ \bar{B}\subset\Omega_\varepsilon}} \int_0^{s+\sigma} \sum_{B\in\bar{B}\cap\mathcal{D}(t)} \pi d_B^2 \left(2\beta_B - \beta_B^2 - \frac{\beta_B^2 r(\mathcal{D}(t))}{2\lambda(t)}\right) dt \\
&\geq \pi \sum_{\substack{\bar{B}\in\mathcal{B}\\ \bar{B}\subset\Omega_\varepsilon}} |d_{\bar{B}}| \int_0^{s+\sigma} \left(1 - \frac{r(\mathcal{D}(t))}{2\lambda(t)}\right) dt \\
&= \pi D(s+\sigma-1),
\end{aligned} \tag{4.40}$$

where the last equality follows since $r(\mathcal{D}(t))' = r(\mathcal{D}(t))$ for $t \in [0, s+\sigma]\backslash\{\sigma\}$ and $\lambda(t)$ is piecewise constant.

An application of Lemma 4.2 and the bound (4.35) show that

$$r(\mathcal{C}_0) \leq \frac{3}{2} r_{\Omega_\varepsilon}(\omega_{1/2}^{3/2}). \tag{4.41}$$

Recall that $r(\mathcal{C}(\sigma)) = 3r(\mathcal{B}_0)/8$. This and (4.41) provide the bound

$$\begin{aligned}
s+\sigma-1 &= \left(\log\frac{r}{r(\mathcal{B}_0)} + \log\frac{r(\mathcal{C}(\sigma))}{r(\mathcal{C}_0)} - 1\right) \\
&\geq \left(\log\frac{r}{r(\mathcal{B}_0)} + \log\frac{r(\mathcal{B}_0)}{4r_{\Omega_\varepsilon}(\omega_{1/2}^{3/2})} - 1\right) \\
&= \left(\log\frac{r}{r_{\Omega_\varepsilon}(\omega_{1/2}^{3/2})} - C\right).
\end{aligned} \tag{4.42}$$

Plugging (4.40) and (4.42) into (4.17) yields (4.33).

□

5 Proof of the main results

With our technical tools sufficiently developed, we may now assemble them for use in proving the main theorems.

We begin with a lemma on the use of the co-area formula in conjunction with sub- and super-level sets.

Lemma 5.1. *Let $u : \Omega \to \mathbb{C}$ and $A : \Omega \to \mathbb{R}^2$ both be C^1 and write (at least locally) $u = \rho v$ with $\rho = |u|$. Fix $t_0 > 0$ and $V \subset \Omega$ to be compact. Then*

$$\begin{aligned}
\frac{1}{2}\int_{V\cap\{\rho\geq t_0\}} \rho^2 |\nabla_A v|^2 &= \int_{t_0}^{\infty} -t^2 \frac{d}{dt}\left(\frac{1}{2}\int_{V\cap\{\rho\geq t\}} |\nabla_A v|^2\right) dt \\
&= \frac{t_0^2}{2}\int_{V\cap\{\rho\geq t_0\}} |\nabla_A v|^2 + \int_{t_0}^{\infty} 2t\left(\frac{1}{2}\int_{V\cap\{\rho\geq t\}} |\nabla_A v|^2\right) dt
\end{aligned} \tag{5.1}$$

and

$$\begin{aligned}\frac{1}{2}\int_{V\cap\{\rho\le t_0\}}\rho^2|\nabla_A v|^2 &= \int_0^{t_0} -t^2\frac{d}{dt}\left(\frac{1}{2}\int_{V\cap\{\rho\ge t\}\cap\{\rho\le t_0\}}|\nabla_A v|^2\right)dt \\ &= \int_0^{t_0} 2t\left(\frac{1}{2}\int_{V\cap\{\rho\ge t\}\cap\{\rho\le t_0\}}|\nabla_A v|^2\right)dt.\end{aligned} \tag{5.2}$$

Proof. The first equality in (5.1) follows from the co-area formula, and the second follows by integrating by parts. The same argument proves (5.2). □

5.1 Proof of Theorem 1

Theorem 1 is an improvement on Theorem 4.1 of [8] that incorporates the G term into the lower bounds on the vortex balls. The crucial difference between this result and those in the previous section is that this one bounds the energy of the function $u:\Omega\to\mathbb{C}$, whereas the previous results were for the $\mathbb{S}^1$-valued map $v=u/|u|:\Omega\to\mathbb{S}^1\hookrightarrow\mathbb{C}$. The statement made in the introduction of Theorem 1 should be understood with $G:\Omega\to\mathbb{R}^2$ the function defined in item 4 of Proposition 4.4 with β_B values given by (4.31).

Proposition 4.4 produces the collection $\mathcal{B}$ and guarantees items 1 and 2. The rest of the proof is devoted to showing that (1.9) holds. By Lemma 3.1 we have, writing $u=\rho v$,

$$\begin{aligned}&\frac{1}{2}\int_V |\nabla_A u|^2+\frac{1}{2\varepsilon^2}(1-|u|^2)^2+r^2(\operatorname{curl}A)^2 \\ &\qquad=\frac{1}{2}\int_V |\nabla\rho|^2+\frac{1}{2\varepsilon^2}(1-\rho^2)^2+\rho^2|\nabla_A v|^2+r^2(\operatorname{curl}A)^2.\end{aligned} \tag{5.3}$$

An application of the co-area formula and integration by parts, the same as that used in Lemma 5.1, shows that

$$\frac{1}{2}\int_V \rho^2|\nabla_A v|^2=\int_0^\infty 2t\left(\frac{1}{2}\int_{V\setminus\omega_t}|\nabla_A v|^2\right)dt. \tag{5.4}$$

Then

$$\begin{aligned}&\frac{1}{2}\int_V \rho^2|\nabla_A v|^2+r^2(\operatorname{curl}A)^2 \\ &\ge \int_0^\infty 2t\left(\frac{1}{2}\int_{V\setminus\omega_t}|\nabla_A v|^2\right)dt+\int_0^{1-\delta}2t\left(\frac{r^2}{2}\int_V(\operatorname{curl}A)^2\right)dt \\ &=\int_0^{\frac{1}{2}}2t\left(\frac{1}{2}\int_{V\setminus\omega_t}|\nabla_A v|^2+\frac{r^2}{2}\int_V(\operatorname{curl}A)^2\right)dt+\int_{1-\delta}^\infty 2t\left(\frac{1}{2}\int_{V\setminus\omega_t}|\nabla_A v|^2\right)dt \\ &+\int_{\frac{1}{2}}^{1-\delta}2t\left(\frac{1}{2}\int_{V\setminus\omega_t}|\nabla_A v|^2+\frac{r^2}{2}\int_V(\operatorname{curl}A)^2\right)dt \\ &:=A_1+A_2+A_3.\end{aligned} \tag{5.5}$$

We further break up the first term on the right side of (5.5):

$$
\begin{aligned}
A_1 &= \int_0^{\frac{1}{2}} 2t \left(\frac{1}{2} \int_{V \setminus \omega_t} |\nabla_A v|^2 + \frac{r^2}{2} \int_V (\operatorname{curl} A)^2 \right) dt \\
&= \int_0^{\frac{1}{2}} 2t \left(\frac{1}{2} \int_{\omega_{1/2}^{3/2} \setminus \omega_t} |\nabla_A v|^2 \right) dt + \int_0^{\frac{1}{2}} 2t \left(\frac{1}{2} \int_{V \setminus \omega_{1/2}^{3/2}} |\nabla_A v|^2 + \frac{r^2}{2} \int_V (\operatorname{curl} A)^2 \right) dt.
\end{aligned} \tag{5.6}
$$

Then, by writing $\omega_{1/2}^{3/2} \setminus \omega_t = \omega^{3/2} \cup \omega_{1/2} \setminus \omega_t$, noting that $\omega_{1/2} \subset V$, and applying (5.2) with $t_0 = 1/2$, we may conclude that

$$
\begin{aligned}
\int_0^{\frac{1}{2}} 2t \left(\frac{1}{2} \int_{\omega_{1/2}^{3/2} \setminus \omega_t} |\nabla_A v|^2 \right) dt &= \int_0^{\frac{1}{2}} 2t \left(\frac{1}{2} \int_{\omega^{3/2}} |\nabla_A v|^2 + \frac{1}{2} \int_{\omega_{1/2} \setminus \omega_t} |\nabla_A v|^2 \right) dt \\
&= \frac{1}{8} \int_{\omega^{3/2}} |\nabla_A v|^2 + \frac{1}{2} \int_{\omega_{1/2}} \rho^2 |\nabla_A v|^2 .
\end{aligned} \tag{5.7}
$$

Since the integrand does not depend on t, we have

$$
\int_0^{\frac{1}{2}} 2t \left(\frac{1}{2} \int_{V \setminus \omega_{1/2}^{3/2}} |\nabla_A v|^2 + \frac{r^2}{2} \int_V (\operatorname{curl} A)^2 \right) dt = \frac{1}{4} \left(\frac{1}{2} \int_{V \setminus \omega_{1/2}^{3/2}} |\nabla_A v|^2 + \frac{r^2}{2} \int_V (\operatorname{curl} A)^2 \right). \tag{5.8}
$$

From (5.1), applied with $t_0 = 1 - \delta$, we bound the second term in (5.5)

$$
\begin{aligned}
A_2 = \int_{1-\delta}^{\infty} 2t \left(\frac{1}{2} \int_{V \setminus \omega_t} |\nabla_A v|^2 \right) dt &= \frac{1}{2} \int_{V \setminus \omega_{1-\delta}} (\rho^2 - (1-\delta)^2) |\nabla_A v|^2 \\
\geq \frac{1}{2} \int_{\omega^{3/2}} (\rho^2 - 1) |\nabla_A v|^2 .
\end{aligned} \tag{5.9}
$$

When $\rho \geq \frac{3}{2}$, the inequality $\rho^2 - \frac{3}{4} \geq \frac{2}{3}\rho^2$ holds; hence,

$$
\frac{1}{2} \int_{\omega^{3/2}} (\rho^2 - 1) |\nabla_A v|^2 + \frac{1}{8} \int_{\omega^{3/2}} |\nabla_A v|^2 \geq \frac{1}{3} \int_{\omega^{3/2}} \rho^2 |\nabla_A v|^2 . \tag{5.10}
$$

We now combine (5.5) – (5.10), leaving A_3 as it was, and arrive at the bound

$$
\begin{aligned}
\frac{1}{2} \int_V \rho^2 |\nabla_A v|^2 + r^2 (\operatorname{curl} A)^2 \geq \frac{1}{4} \left(\frac{1}{2} \int_{V \setminus \omega_{1/2}^{3/2}} |\nabla_A v|^2 + \frac{r^2}{2} \int_V (\operatorname{curl} A)^2 \right) \\
+ \int_{\frac{1}{2}}^{1-\delta} 2t \left(\frac{1}{2} \int_{V \setminus \omega_t} |\nabla_A v|^2 + \frac{r^2}{2} \int_V (\operatorname{curl} A)^2 \right) dt + \frac{1}{3} \int_{\omega_{1/2}^{3/2}} \rho^2 |\nabla_A v|^2 .
\end{aligned} \tag{5.11}
$$

Recalling the notation

$$
F_\varepsilon(\rho, V) = \frac{1}{2} \int_V |\nabla \rho|^2 + \frac{1}{2\varepsilon^2} (1 - \rho^2)^2
$$

and the decomposition (5.3), we can use (5.11) to see that

$$\frac{1}{2}\int_V |\nabla_A u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2 + r^2(\operatorname{curl} A)^2 = F_\varepsilon(\rho, V) + \frac{1}{2}\int_V \rho^2 |\nabla_A v|^2 + r^2(\operatorname{curl} A)^2$$
$$\geq B_1 + B_2 + B_3, \tag{5.12}$$

where

$$B_1 := \frac{1}{4}\left(F_\varepsilon(\rho, V) + \frac{1}{2}\int_{V\backslash\omega_{1/2}^{3/2}} |\nabla_A v|^2 + \frac{r^2}{2}\int_V (\operatorname{curl} A)^2\right),$$
$$B_2 := \frac{3\beta}{4}F_\varepsilon(\rho, V) + \int_{\frac{1}{2}}^{1-\delta} 2t\left(\frac{1}{2}\int_{V\backslash\omega_t} |\nabla_A v|^2 + \frac{r^2}{2}\int_V (\operatorname{curl} A)^2\right) dt,$$
$$B_3 := \frac{3(1-\beta)}{4}F_\varepsilon(\rho, V) + \frac{1}{3}\int_{\omega_{1/2}^{3/2}} \rho^2 |\nabla_A v|^2,$$

and $\beta \in (0,1)$ is to be chosen later in the proof.

To bound B_1, we employ Proposition 4.7 to see that

$$\begin{aligned} &\frac{1}{2}\int_{V\backslash\omega_{1/2}^{3/2}} |\nabla_A v|^2 + \frac{r^2}{2}\int_V (\operatorname{curl} A)^2 + \frac{1}{2}\int_V |\nabla\rho|^2 + \frac{1}{2\varepsilon^2}(1-\rho^2)^2 \\ &\geq \frac{1}{2}\int_{V\backslash\omega_{1/2}^{3/2}} |\nabla_{A+G} v|^2 + \pi D\left(\log\frac{r}{r_{\Omega_\varepsilon}(\omega_{1/2}^{3/2})} - C\right) + F_\varepsilon(\rho, V). \end{aligned} \tag{5.13}$$

Then, an application of Lemma 2.8 shows that

$$\begin{aligned} \pi D\left(\log\frac{r}{r_{\Omega_\varepsilon}(\omega_{1/2}^{3/2})} - C\right) + F_\varepsilon(\rho, V) &\geq \pi D\left(\log\frac{r}{C\varepsilon F_\varepsilon(\rho, V)} - C\right) + F_\varepsilon(\rho, V) \\ &\geq \pi D\left(\log\frac{r}{\varepsilon D} - C\right) + F_\varepsilon(\rho, V) - \pi D\log\frac{F_\varepsilon(\rho, V)}{\pi D} \\ &\geq \pi D\left(\log\frac{r}{\varepsilon D} - C\right), \end{aligned} \tag{5.14}$$

where the last line follows from the inequality $x - a\log\frac{x}{a} \geq 0$. On the set $V\backslash\omega_{1/2}^{3/2}$ it is the case that $1/2 \leq \rho \leq 3/2$, and so $1 \geq 4\rho^2/9$. Hence, from this bound, (5.13), and (5.14), we may conclude that

$$\begin{aligned} B_1 &\geq \frac{1}{4}\left(\frac{1}{2}\int_{V\backslash\omega_{1/2}^{3/2}} |\nabla_{A+G} v|^2 + \pi D\left(\log\frac{r}{\varepsilon D} - C\right)\right) \\ &\geq \frac{1}{18}\int_{V\backslash\omega_{1/2}^{3/2}} \rho^2 |\nabla_{A+G} v|^2 + \frac{\pi D}{4}\left(\log\frac{r}{\varepsilon D} - C\right). \end{aligned} \tag{5.15}$$

To control B_2, we begin by using (2.10) and Lemma 2.5 to find the bound

$$\begin{aligned}\frac{3\beta}{4}F_\varepsilon(\rho,V) &\geq \frac{3\sqrt{2}\beta}{8\varepsilon}\int_0^\infty \left|1-t^2\right| \mathcal{H}^1(\{\rho=t\})dt \\ &\geq \frac{3\sqrt{2}\beta}{4\varepsilon}\int_{\frac{1}{2}}^{1-\delta}(1-t^2)r_{\Omega_\varepsilon}(\omega_t)dt.\end{aligned} \tag{5.16}$$

Then (5.16) and (4.15) prove that

$$B_2 \geq \int_{\frac{1}{2}}^{1-\delta}\left(2t\pi D\left(\log\frac{r}{r_{\Omega_\varepsilon}(\omega_t)}-C\right)+\frac{3\sqrt{2}\beta}{4\varepsilon}(1-t^2)r_{\Omega_\varepsilon}(\omega_t)\right)\,dt. \tag{5.17}$$

As $r_{\Omega_\varepsilon}(\omega_t)$ varies, the integrand on the right hand side of (5.17) achieves its minimum at

$$r_{\Omega_\varepsilon}(\omega_t)=\frac{8\pi Dt\varepsilon}{3\sqrt{2}\beta(1-t^2)}.$$

Plugging this in, we get the estimate

$$\begin{aligned}B_2 &\geq \int_{\frac{1}{2}}^{1-\delta}2\pi Dt\left(\log\frac{3\sqrt{2}r\beta(1-t^2)}{8\pi Dt\varepsilon}-C+1\right)dt \\ &= \int_{\frac{1}{2}}^{1-\delta}2\pi Dt\left(\log\frac{r}{\varepsilon D}+\log\frac{3\sqrt{2}\beta(1-t^2)}{8\pi t}-C\right)dt \\ &= \pi D\left(\left((1-\delta)^2-\frac{1}{4}\right)\log\frac{r}{\varepsilon D}-C\right).\end{aligned} \tag{5.18}$$

We now choose $\beta=\frac{23}{27}$ so that $\frac{3(1-\beta)}{8}=\frac{1}{18}$. Then

$$B_1+B_3 \geq \frac{1}{18}\int_V\left|\nabla_{A+G}u\right|^2+\frac{1}{2\varepsilon^2}(1-|u|^2)^2+\frac{\pi D}{4}\left(\log\frac{r}{\varepsilon D}-C\right). \tag{5.19}$$

Using (5.18) and (5.19) in (5.12) then shows that

$$\begin{aligned}&\frac{1}{2}\int_V\left|\nabla_A u\right|^2+\frac{1}{2\varepsilon^2}(1-|u|^2)^2+r^2(\operatorname{curl}A)^2 \\ &\geq \pi D\left((1-\delta)^2\log\frac{r}{\varepsilon D}-C\right)+\frac{1}{18}\int_V\left|\nabla_{A+G}u\right|^2+\frac{1}{2\varepsilon^2}(1-|u|^2)^2.\end{aligned} \tag{5.20}$$

Now, by assumption $r\leq 1\leq D$, so $\log\frac{r}{D}\leq 0$. Since $\delta=\varepsilon^{\alpha/4}$, we have that for $\varepsilon\leq\varepsilon_0=\varepsilon_0(\alpha)$, the inequalities

$$\begin{aligned}&\delta^2-\delta\leq 0 \\ &(2\delta-\delta^2)\log\varepsilon\geq -1\end{aligned} \tag{5.21}$$

both hold. Hence, for $\varepsilon \le \varepsilon_0$,

$$\begin{aligned}(1-\delta)^2 \log \frac{r}{\varepsilon D} - C &= \log \frac{r}{\varepsilon D} - C + (\delta^2 - 2\delta) \log \frac{r}{D} + (2\delta - \delta^2) \log \varepsilon \\ &\ge \log \frac{r}{\varepsilon D} - C - 1.\end{aligned} \tag{5.22}$$

Combining (5.20) with (5.22) gives (1.9).

□

5.2 Proof of Theorem 2 and corollaries

Proof of Theorem 2. Theorem 2 justifies the selection of the function G. It has been chosen so that $\|G\|_{L^{2,\infty}}$ only depends on the final data of Theorem 1, that is on natural quantities. This estimate of $\|G\|_{L^{2,\infty}}$, Proposition 6.4, is quite technical and is thus reserved for the next section. A more thorough discussion of the space $L^{2,\infty}$, also known as weak-L^2, is also reserved for the next section.

We begin by noting that $\nabla_A u = \nabla_{A+G} u + iGu$. This and the fact that $\|f\|_{L^{2,\infty}(V)} \le \|g\|_{L^{2,\infty}(V)}$ if $|f| \le |g|$ allow us to estimate

$$\begin{aligned}\frac{1}{2} \|\nabla_A u\|^2_{L^{2,\infty}(V)} &\le \|\nabla_{A+G} u\|^2_{L^{2,\infty}(V)} + \|iGu\|^2_{L^{2,\infty}(V)} \\ &\le \|\nabla_{A+G} u\|^2_{L^2(V)} + \frac{9}{4} \|G\|^2_{L^{2,\infty}(V)}.\end{aligned} \tag{5.23}$$

The second inequality follows since $|u| \le \frac{3}{2}$ on the support of G. Write

$$F^r_\varepsilon(u, A, V) = \frac{1}{2} \int_V |\nabla_A u|^2 + \frac{1}{2\varepsilon^2}(1 - |u|^2)^2 + r^2 (\operatorname{curl} A)^2.$$

We now employ Theorem 1 to bound

$$\|\nabla_{A+G} u\|^2_{L^2(V)} \le 18 \left(F^r_\varepsilon(u, A, V) - \pi D \left(\log \frac{r}{\varepsilon D} - C \right) \right). \tag{5.24}$$

We will show in Proposition 6.4 that

$$\begin{aligned}\frac{9}{4} \|G\|^2_{L^{2,\infty}(V)} \le{}& \frac{216(1+\eta)}{2\eta - 1} \left(F^r_\varepsilon(u, A, V) - \pi D \left(\log \frac{r}{\varepsilon D} - C \right) \right) \\ &+ \pi \frac{9(1+\eta)}{1-\eta} \sum_{\substack{\bar{B} \in \mathcal{B} \\ \bar{B} \subset \Omega_\varepsilon}} d^2_{\bar{B}}.\end{aligned} \tag{5.25}$$

Now choose $\eta = \frac{5+\sqrt{2785}}{60} \approx .962$ so that

$$18 + \frac{216(1+\eta)}{2\eta - 1} = \frac{9(1+\eta)}{1-\eta}.$$

Combining (5.23) – (5.25) yields (1.11) with constant $C = (1-\eta)/(18(1+\eta)) \approx 1/951$.

□

The previous theorem dealt with the energy content of the set $V \subset \Omega$. We can deduce a slightly stronger version of Corollary 1.2.

Corollary 5.2. *Assume the hypotheses of Theorem 1. Then*

$$C \|\nabla_A u\|^2_{L^{2,\infty}(\Omega)} \le F^r_\varepsilon(u, A, \Omega) - \pi D \left(\log \frac{r}{\varepsilon D} - C\right) + \pi \sum_{\substack{B \in \mathcal{B} \\ B \subset \Omega_\varepsilon}} d_B^2. \tag{5.26}$$

Proof. Add $F^r_\varepsilon(u, A, \Omega \backslash V)$ to both sides of (1.11). We then bound

$$\begin{aligned} &C \|\nabla_A u\|^2_{L^{2,\infty}(V)} + F^r_\varepsilon(u, A, \Omega \backslash V) \\ &\ge C \|\nabla_A u\|^2_{L^{2,\infty}(V)} + \|\nabla_A u\|^2_{L^2(\Omega \backslash V)} \\ &\ge C \|\nabla_A u\|^2_{L^{2,\infty}(V)} + \|\nabla_A u\|^2_{L^{2,\infty}(\Omega \backslash V)} \\ &\ge C \|\nabla_A u\|^2_{L^{2,\infty}(\Omega)}, \end{aligned} \tag{5.27}$$

where the last inequality follows by using the convexity of norms, and C is a different constant. The result follows. □

Proof of Proposition 1.3. It is proved in Theorem 0.5 of [1] that minimizers of E_ε with this constraint have exactly d zeroes of degree 1 which converge to d distinct points $a_1, \ldots, a_d$, minimizing W_g. They also prove that their energy is

$$\min E_\varepsilon = \pi d |\log \varepsilon| + \min W_g + d\gamma + o(1), \tag{5.28}$$

where γ is a universal constant. Let us apply the vortex-ball construction to these solutions, choosing for final radius $r = \frac{1}{4} \min_{i,j} (\operatorname{dist}(a_i, \partial\Omega), |a_i - a_j|)$. Since the final balls $B \in \mathcal{B}$ cover all the zeroes of u, and there is exactly one zero b_i^ε with nonzero degree, converging to each a_i, there is one ball B_i in the collection containing b_i^ε. Since $d_i = \deg(u_\varepsilon, \partial B_i) = 1$, and there are no other zeroes of u_ε, we have $D = d$ (with our previous notation) and Corollary 1.2 (taken with $A \equiv 0$) gives us

$$E_\varepsilon(u_\varepsilon) + \pi d \ge C \|\nabla u_\varepsilon\|^2_{L^{2,\infty}(\Omega)} + \pi d(|\log \varepsilon| - C - \log d),$$

where C is a universal constant. In view of (5.28), this implies that

$$C \|\nabla u_\varepsilon\|^2_{L^{2,\infty}(\Omega)} \le \min W_g + d\gamma + Cd + \pi d \log d + o(1),$$

and the first result follows.

Since $L^{2,\infty}$ is a dual Banach space, we deduce from this bound that, as $\varepsilon \to 0$, up to extraction, ∇u_ε converges weakly-$*$ in $L^{2,\infty}$, to its distributional limit. But it is proved in [1] that $\nabla u_\varepsilon \to \nabla u_\star$ uniformly away from $a_1, \cdots, a_d$ (in fact in C^k_{loc}), where $u_\star$ is given by

$$u_\star(x) = e^{iH(x)} \prod_{k=1}^d \frac{x - a_k}{|x - a_k|}$$

with H a harmonic function. Note in particular that $u_\star \in W^{1,p}(\Omega)$ for $p < 2$.

We claim that $\nabla u_\varepsilon \to \nabla u_\star$ in the sense of distributions on Ω. Indeed, let X be a smooth compactly supported test vector field. Fix $\rho > 0$ and let us write

$$\int_\Omega (\nabla u_\varepsilon - \nabla u_\star) \cdot X = \int_{\Omega \setminus \cup_i B(a_i,\rho)} (\nabla u_\varepsilon - \nabla u_\star) \cdot X + \sum_i \int_{B(a_i,\rho)} (\nabla u_\varepsilon - \nabla u_\star) \cdot X.$$

The first term in the right-hand side tends to 0 by uniform convergence of ∇u_ε to $\nabla u_\star$ away from the a_i's. The second term is bounded by Hölder's inequality by $C\|X\|_{L^\infty}\|\nabla u_\varepsilon - \nabla u_\star\|_{L^p(\Omega)}\rho^{2/q}$, where $p < 2$ and $1/p + 1/q = 1$. This is bounded by $C\rho^{2/q}\|X\|_{L^\infty}$ since $\nabla u_\star \in L^p(\Omega)$ for all $p < 2$ and ∇u_ε is bounded in $L^p(\Omega)$ for all $p < 2$ ($L^{2,\infty}(\Omega)$ embeds in $L^p(\Omega)$ for all $p < 2$). Letting ρ tend to 0 we conclude that $\int_\Omega (\nabla u_\varepsilon - \nabla u_\star) \cdot X \to 0$ and finally that $\nabla u_\varepsilon \rightharpoonup \nabla u_\star$ weakly-$*$ in $L^{2,\infty}(\Omega)$.

$\square$

6 The $L^{2,\infty}$ norm of G

6.1 Definitions and preliminary results

We begin with a discussion of the various quantities needed to define and norm the space $L^{2,\infty}$. For a function $f : \Omega \to \mathbb{R}^k$, $k \geq 1$, we define the distribution function of f by

$$\lambda_f(t) = |\{x \in \Omega \mid |f(x)| > t\}|. \tag{6.1}$$

This allows us to define the decreasing rearrangement of f as $f^* : \mathbb{R}^+ \to \mathbb{R}^+$, where

$$f^*(t) = \inf\{s > 0 \mid \lambda_f(s) \leq t\}. \tag{6.2}$$

We then define the quantity

$$|||f|||_{L^{2,\infty}} = \sqrt{\sup_{t>0} t^2 \lambda_f(t)} = \sup_{t>0} t \lambda_f(t)^{\frac{1}{2}} = \sup_{t>0} t^{\frac{1}{2}} f^*(t), \tag{6.3}$$

and $L^{2,\infty}(\Omega) = \{f \mid |||f|||_{L^{2,\infty}} < \infty\}$. Unfortunately, this does not define a norm, but rather a quasi-norm. That is, $|||\cdot|||_{L^{2,\infty}}$ satisfies

$$\begin{cases} |||\alpha f|||_{L^{2,\infty}} = |\alpha| \, |||f|||_{L^{2,\infty}} \\ |||f|||_{L^{2,\infty}} = 0 \text{ if and only if } f = 0 \text{ a.e.} \\ |||f + g|||_{L^{2,\infty}} \leq C(|||f|||_{L^{2,\infty}} + |||g|||_{L^{2,\infty}}) \text{ for some } C \geq 1. \end{cases}$$

It can be shown that with $|||\cdot|||_{L^{2,\infty}}$, $L^{2,\infty}$ is a quasi-Banach space, i.e. a linear space in which every quasi-norm Cauchy sequence converges in the quasi-norm. However, as the next lemma shows, the space can, in fact, be normed. We define

$$\begin{aligned}\|f\|_{L^{2,\infty}} &= \sup_{|E|<\infty} |E|^{-1/2} \int_E |f(x)|\,dx \\ &= \sup_{t>0} \frac{1}{t^{\frac{1}{2}}} \sup_{|E|=t} \int_E |f(x)|\,dx \\ &= \sup_{t>0} \frac{1}{t^{\frac{1}{2}}} \int_0^t f^*(s)ds,\end{aligned} \tag{6.4}$$

which is obviously a norm.

Lemma 6.1. *$L^{2,\infty}$ is a Banach space with norm $\|\cdot\|_{L^{2,\infty}}$, and*

$$|||f|||_{L^{2,\infty}} \leq \|f\|_{L^{2,\infty}} \leq 2|||f|||_{L^{2,\infty}}. \tag{6.5}$$

Proof. Since f^* is decreasing, we see that

$$\|f\|_{L^{2,\infty}} = \sup_{t>0} \frac{1}{t^{\frac{1}{2}}} \int_0^t f^*(s)ds \geq \sup_{t>0} \frac{1}{t^{\frac{1}{2}}} t f^*(t) = \sup_{t>0} t^{\frac{1}{2}} f^*(t) = |||f|||_{L^{2,\infty}}. \tag{6.6}$$

For the second inequality we note that

$$\frac{1}{t^{\frac{1}{2}}} \int_0^t f^*(s)ds = \frac{1}{t^{\frac{1}{2}}} \int_0^t (s^{\frac{1}{2}} f^*(s)) \frac{ds}{s^{\frac{1}{2}}} \leq |||f|||_{L^{2,\infty}} \frac{2t^{\frac{1}{2}}}{t^{\frac{1}{2}}} = 2|||f|||_{L^{2,\infty}}. \tag{6.7}$$

This also shows how to construct a function that makes the inequalities sharp: any f so that $f^*(s) = \frac{c}{\sqrt{s}}$ will do. This is the case for $f(x) = 1/|x|$ in $\mathbb{R}^2$. □

We now present the

Proof of Proposition 1.4. First rewrite the L^2 integral using the distribution function:

$$\int_\Omega |f|^2 = \int_0^\infty 2t\lambda_f(t)dt. \tag{6.8}$$

We break this integral into two parts and utilize the boundedness of f and the trivial inequality $\lambda_f(t) \leq |\Omega|$ for all $t > 0$. Indeed,

$$\begin{aligned}\int_0^\infty 2t\lambda_f(t)dt &= \int_0^C 2t\lambda_f(t)dt + \int_C^{\frac{C}{\varepsilon}} 2t\lambda_f(t)dt \\ &\leq |\Omega| \int_0^C 2tdt + 2\sup_{t>0}(t^2\lambda_f(t)) \int_C^{\frac{C}{\varepsilon}} \frac{dt}{t} \\ &\leq |\Omega|\, C^2 + 2\, \|f\|^2_{L^{2,\infty}(\Omega)} \log \frac{C}{C\varepsilon},\end{aligned} \tag{6.9}$$

where we have used Lemma 6.1 in the last inequality. The result follows by dividing both sides by $2\,|\log \varepsilon|$. □

6.2 The calculation

Before proving the main result we prove some quasi-norm estimates for simplified versions of G. The main result breaks G into various simplified components in order to utilize these estimates.

Lemma 6.2. *Suppose we are given a collection of disjoint annuli $\{A_i\}$, $i = 1, \dots, n$, where*

$$A_i = \{r_i < |x - c_i| \le s_i\} \subset \mathbb{R}^2,$$

c_i denotes the center of A_i, and r_i and s_i are the inner and outer radii respectively. Let

$$f(x) = \sum_{i=1}^{n} \chi_{A_i}(x) v_i(x) \frac{a_i}{|x - c_i|}, \tag{6.10}$$

where $v_i : A_i \longrightarrow \mathbb{R}^k$ is a vector field so that $|v_i| = 1$ and a_i is a constant for $i = 1, \dots, n$. Write $\tau_i = \log \frac{s_i}{r_i}$ for the conformal factor of A_i. Then for $t > 0$,

$$t^2 \lambda_f(t) \le \pi \sum_{i=1}^{n} a_i^2 \left(1 - e^{-2\tau_i}\right). \tag{6.11}$$

Proof. We begin by noting that on the annulus A_i it is the case that

$$\frac{|a_i|}{s_i} \le |f| < \frac{|a_i|}{r_i}. \tag{6.12}$$

Then for any $t > 0$ and any annulus A_i, the measure of the set in A_i where $f > t$ is simple to calculate. Indeed, if $t \le |a_i| / s_i$, then $f > t$ on the whole annulus, which has measure $\pi(s_i^2 - r_i^2)$. If $t \ge |a_i| / r_i$, then $f < t$ everywhere on the annulus, and so the measure is zero. Finally, if $|a_i| / s_i < t < |a_i| / r_i$, then $f > t$ exactly on the subannulus $\{r_i < |x - c_i| \le \rho_i\}$, where

$$\rho_i = \frac{|a_i|}{t}, \tag{6.13}$$

which has measure $\pi(a_i^2/t^2 - r_i^2)$.

Combining these, for any $t > 0$ we may then write

$$\lambda_f(t) = \sum_{\{i \,|\, \frac{|a_i|}{s_i} < t < \frac{|a_i|}{r_i}\}} \pi \left(\frac{a_i^2}{t^2} - r_i^2 \right) + \sum_{\{i \,|\, t \le \frac{|a_i|}{s_i}\}} \pi(s_i^2 - r_i^2). \tag{6.14}$$

Then

$$\begin{aligned} t^2 \lambda_f(t) &= \sum_{\{i \,|\, \frac{|a_i|}{s_i} < t < \frac{|a_i|}{r_i}\}} \pi(a_i^2 - t^2 r_i^2) + \sum_{\{i \,|\, t \le \frac{|a_i|}{s_i}\}} \pi(s_i^2 - r_i^2) t^2 \\ &\le \sum_{\{i \,|\, \frac{|a_i|}{s_i} < t < \frac{|a_i|}{r_i}\}} \pi a_i^2 \left(1 - \frac{r_i^2}{s_i^2}\right) + \sum_{\{i \,|\, t \le \frac{|a_i|}{s_i}\}} \pi a_i^2 \left(1 - \frac{r_i^2}{s_i^2}\right) \qquad (6.15) \\ &\le \sum_{i=1}^{n} \pi a_i^2 \left(1 - \frac{r_i^2}{s_i^2}\right). \end{aligned}$$

Plugging in $\tau_i = \log \frac{s_i}{r_i}$ proves the result.

□

The next lemma tells us that a collection of annuli with uniformly bounded degrees and the property that they can be rearranged to fit concentrically inside each other can, for the purposes of estimating the $L^{2,\infty}$ quasi-norm, be regarded as a single annulus.

Lemma 6.3. *Suppose $\{A_i\}$, $i = 1, \ldots, n$, is a collection of disjoint annuli, where*

$$A_i = \{r_i < |x - c_i| \leq s_i\} \subset \mathbb{R}^2,$$

c_i denotes the center of A_i, and r_i and s_i are the inner and outer radii respectively. Suppose further that the annuli can be arranged concentrically without overlap. That is, suppose that

$$r_1 < s_1 \leq r_2 < s_2 \leq r_3 \leq \cdots < s_{n-1} \leq r_n < s_n.$$

Let

$$f(x) = \sum_{i=1}^{n} \chi_{A_i}(x) v_i(x) \frac{a_i}{|x - c_i|}, \tag{6.16}$$

where the a_i are constants such that $|a_i| \leq |a|$ and $v_i : A_i \to \mathbb{R}^k$ is a vector field so that $|v_i| = 1$ for $i = 1, \ldots, n$. Then

$$t^2 \lambda_f(t) = t^2 \sum_{i=1}^{n} |A_i \cap \{|f| > t\}| \leq \pi a^2. \tag{6.17}$$

Proof. Since the distribution function is invariant under translations, without loss of generality we may assume that the annuli are concentric with common center c. This reduces f to the form

$$f(x) = \sum_{i=1}^{n} \chi_{A_i}(x) v_i(x) \frac{a_i}{|x - c|}. \tag{6.18}$$

Consider the function

$$g(x) = \frac{a e_1}{|x - c|}, \tag{6.19}$$

where $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^k$. The pointwise bound $|f(x)| \leq |g(x)|$ yields the bound $\lambda_f(t) \leq \lambda_g(t)$ for all $t > 0$. It is a simple matter to see that

$$\lambda_g(t) = \pi \frac{a^2}{t^2}, \tag{6.20}$$

and hence,

$$t^2 \lambda_f(t) \leq t^2 \lambda_g(t) = \pi a^2. \tag{6.21}$$

□

We are now ready to prove the main result of this section.

Proposition 6.4. *Let $G : \Omega \to \mathbb{R}^2$ be the function defined in Proposition 4.4 with $\eta \in (0,1)$ fixed and the β_B values given by* (4.31)*. Write*

$$F_\varepsilon^r(u, A, V) = \frac{1}{2}\int_V |\nabla_A u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2 + r^2(\operatorname{curl} A)^2.$$

Then

$$\begin{aligned}\|G\|^2_{L^{2,\infty}(V)} \le \frac{96(1+\eta)}{2\eta - 1}&\left(F_\varepsilon^r(u, A, V) - \pi D\left(\log\frac{r}{\varepsilon D} - C\right)\right)\\ &+ \pi\frac{4(1+\eta)}{1-\eta}\sum_{\substack{\bar{B}\in\mathcal{B}\\ \bar{B}\subset\Omega_\varepsilon}} d_{\bar{B}}^2.\end{aligned} \tag{6.22}$$

Proof. Step 1

To begin we must translate the notation used to define G into different notation that is more cumbersome but that will allow a more exact enumeration of the objects generated by the ball construction. Recall that to define G, the collection $\{\mathcal{D}(t)\}_{t\in[0,s+\sigma]}$ defined by (4.30) is refined to the subcollection $\{\mathcal{G}(t)\}_{t\in[0,s+\sigma]}$ that consists of all balls that stay entirely inside Ω_ε. Let N be the number of balls in $\mathcal{G}(s+\sigma) = \{\bar{B}_1, \dots, \bar{B}_N\}$, i.e. the number of final balls. Let T be the finite set of merging times in the growth of $\mathcal{G}(t)$, where here we count $t = \sigma$, the time when the collection shifts from $\mathcal{C}(\sigma)$ to $\mathcal{B}(0)$, as a merging time. Let $0 = t_0 < t_1 < \dots < t_{K-1} < t_K = s+\sigma$ be an enumeration of $T \cup \{0, s+\sigma\}$. For $k = 1, \dots, K$ and $t \in [t_{k-1}, t_k)$ we call all balls in $\mathcal{G}(t)$ members of the k^{th} generation. We write $\mathcal{G}(t_k^-)$ for the collection of balls obtained as $t \to t_k^-$, i.e. the collection of pre-merged balls at time $t = t_k$. Similarly, when we write $\mathcal{G}(t_k)$ we refer to the post-merged balls. For $k = 1, \dots, K$ and $n = 1, \dots, N$ we enumerate

$$\begin{aligned}\{B_{i,k,n}\}_{i=1}^{M_{k,n}} &= \{B \in \mathcal{G}(t_k^-) \mid B \subset \bar{B}_n\}, \text{ and}\\ \{\tilde{B}_{i,k,n}\}_{i=1}^{M_{k,n}} &= \{B \in \mathcal{G}(t_{k-1}) \mid B \subset \bar{B}_n\},\end{aligned}$$

in such a way that $\tilde{B}_{i,k,n} \subset B_{i,k,n}$. We define the annuli $A_{i,k,n} = B_{i,k,n}\backslash\tilde{B}_{i,k,n}$ and write $d_{i,k,n} = \deg(u, \partial B_{i,k,n})$ for the degree of u in the annulus $A_{i,k,n}$. For fixed $k = 1, \dots, K$ we say the annuli $\{A_{i,k,n}\}$ are k^{th} generation annuli. Without loss of generality we may assume that the indices are ordered so that $|d_{i,k,n}|$ is a decreasing sequence with respect to i for k and n fixed. Write $D_n = d_{\bar{B}_n}$. We define the conformal growth factor in the k^{th} generation, denoted τ_k, by

$$\tau_k = \log\frac{r(\mathcal{G}(t_k^-))}{r(\mathcal{G}(t_{k-1}))}.$$

Recall that for each $\bar{B}_n$, $n = 1, \dots, N$, Corollary 4.6 provides a transition time $t_{\bar{B}_n}$ (depending on η). In the current setting, the more natural notion is that of transition generation, and in fact, the proof of Lemma 4.5 shows that the transition time actually occurs at one of the t_k for $k = 0, \dots, K-1$. We then define the transition generation k_n as the unique k such that $t_{\bar{B}_n} \in [t_{k-1}, t_k)$. If we define generational versions of the negative and positive vorticity masses $N(t)$ and $P(t)$ from (4.24) by

$$N(k,n) := \sum_{\substack{1 \le i \le M_{k,n} \\ d_{i,k,n} < 0}} |d_{i,k,n}|$$
$$P(k,n) := \sum_{\substack{1 \le i \le M_{k,n} \\ d_{i,k,n} \ge 0}} d_{i,k,n},$$

then the definition of k_n and Corollary 4.6 allow us to conclude

$$D_n \ge 0 \Rightarrow \begin{cases} \eta P(k,n) < N(k,n) & \text{for } 1 \le k \le k_n - 1 \\ N(k,n) \le \eta P(k,n) & \text{for } k_n \le k \le K \end{cases} \tag{6.23}$$

$$D_n < 0 \Rightarrow \begin{cases} \eta N(k,n) < P(k,n) & \text{for } 1 \le k \le k_n - 1 \\ P(k,n) \le \eta N(k,n) & \text{for } k_n \le k \le K. \end{cases} \tag{6.24}$$

Translating the definition of the β_B from (4.31) into the new notation, we see that

$$\beta_{i,k,n} = \begin{cases} 1 & \text{for } 1 \le k < k_n,\ 1 \le i \le M_{k,n} \\ |D_n|^{1/2} \left(\sum_{i=1}^{M_{k,n}} d_{i,k,n}^2 \right)^{-1/2} & \text{for } k_n \le k \le K,\ 1 \le i \le M_{k,n}. \end{cases} \tag{6.25}$$

This means that G can be written

$$G(x) = \sum_{n=1}^{N} \sum_{k=1}^{K} \sum_{i=1}^{M_{k,n}} \chi_{A_{i,k,n}}(x) \frac{d_{i,k,n}\beta_{i,k,n}}{|x - c_{i,k,n}|} \tau_{i,k,n}(x), \tag{6.26}$$

where $\tau_{i,k,n}$ is the unit tangent vector field in $A_{i,k,n}$. In order to somewhat ease the notational burden, we define the following sets of indices. The early and later generations are given respectively by

$$S_e = \{(n,k) \mid 1 \le n \le N, 1 \le k \le k_n - 1\}$$
$$S_l = \{(n,k) \mid 1 \le n \le N, k_n \le k \le K\},$$

and we similarly define the sets of early and later annuli by

$$T_e = \{(n,k,i) \mid (n,k) \in S_e, 1 \le i \le M_{k,n}\}$$
$$T_l = \{(n,k,i) \mid (n,k) \in S_l, 1 \le i \le M_{k,n}\}.$$

Step 2.

In this step we will prove an intermediate bound on $t^2\lambda_G(t)$. We begin by breaking the distribution function for G up into two components determined by the value of k_n. Indeed,

$$\begin{aligned}\lambda_G(t) &= \sum_{n,k,i} |A_{i,k,n} \cap \{|G| > t\}| \\ &= \sum_{T_e} |A_{i,k,n} \cap \{|G| > t\}| + \sum_{T_l} |A_{i,k,n} \cap \{|G| > t\}| \\ &:= A_1 + A_2.\end{aligned} \tag{6.27}$$

Applying Lemma 6.2 to A_1, we see that

$$t^2 A_1 \leq \pi \sum_{T_e} d_{i,k,n}^2 (1 - e^{-2\tau_k}). \tag{6.28}$$

To analyze the A_2 term we must take advantage of all of the notation created in the first step. Particular attention must be paid to the generations after k_n that come about as the result of mergings in which balls of nonzero degree are merged only with balls of zero degree. These generations, which we call zero-merging generations, throw off a counting argument that we will use to bound the number of later generations (after k_n) in terms of the degrees of the balls in the k_n^{th} generation. Generations that are not zero-merging generations we call effective-merging generations. The degrees of the annuli are not changed in a zero-merging generation, and the annuli of such a generation can be rearranged to fit concentrically outside the annuli of the previous generation. Our strategy for dealing with zero-merging generations, then, is to collect successive zero-merging generations, group them with the preceding effective-merging generation, and utilize Lemma 6.3 to regard the group as a single collection of annuli.

To this end, for each n we define the sets

$$Z_n = \{k \in \{k_n, \ldots, K\} \mid \text{each ball in } \mathcal{G}(t_k) \text{ contains at most one ball in } \mathcal{G}(t_k^-) \text{ of nonzero degree}\},$$

and

$$I_n = \{k_n, \ldots, K\} \backslash Z_n.$$

The generations in Z_n are the zero-merging generations, and those in I_n are the effective-merging generations.

Since $|d_{i,k,n}|$ is a decreasing sequence with respect to i for k, n fixed, there must exist an integer $P_{k,n} \in \{1, \ldots, M_{k,n}\}$ so that $d_{i,k,n} \neq 0$ for $i = 1, \ldots, P_{k,n}$ and $d_{i,k,n} = 0$ for $i = P_{k,n} + 1, \ldots, M_{k,n}$. Since the annuli of a zero-merging generation have the same degrees as the previous generation, we have that $P_{k,n} = P_{k-1,n}$. We may assume, without loss of generality, that the ball ordering is such that $B_{i,k-1,n} \subset B_{i,k,n}$ and $d_{i,k,n} = d_{i,k-1,n}$ for $k \in Z_n$ and $i = 1, \ldots, P_{k,n}$. To identify sequences of zero-merging generations that happen one after the other we write $Z_n = Z_n^1 \cup \cdots \cup Z_n^{m_n}$, where the Z_n^j are maximal subsets of sequential integers, i.e. the integer connected components of Z_n. All of the generations in

Z_n^j will be grouped with the generation preceding Z_n^j and analyzed as a single entity with Lemma 6.3. This preceding effective generation occurs at generation $l_n^j := \min(Z_N^j) - 1$. We group it together with the generations in Z_n^j by forming the collections $\tilde{Z}_n^j = Z_n^j \cup \{l_n^j\}$. Write the modified collection $\tilde{Z}_n = \tilde{Z}_n^1 \cup \cdots \cup \tilde{Z}_n^{m_n}$, and $\tilde{I}_n = I_n \backslash Z_n$. Note that $P_{k,n}$ is constant for $k \in \tilde{Z}_n^j$; we call this common value P_n^j.

We now split A_2 again:

$$\begin{aligned} A_2 &= \sum_{T_l} |A_{i,k,n} \cap \{|G| > t\}| \\ &= \sum_{n=1}^{N} \sum_{k \in \tilde{I}_n} \sum_{i=1}^{P_{k,n}} |A_{i,k,n} \cap \{|G| > t\}| + \sum_{n=1}^{N} \sum_{k \in \tilde{Z}_n} \sum_{i=1}^{P_{k,n}} |A_{i,k,n} \cap \{|G| > t\}| \\ &:= B_1 + B_2. \end{aligned} \tag{6.29}$$

Applying Lemma 6.2 to B_1, we get

$$t^2 B_1 \leq \pi \sum_{n=1}^{N} \sum_{k \in \tilde{I}_n} \sum_{i=1}^{P_{k,n}} (d_{i,k,n} \beta_{i,k,n})^2 (1 - e^{-2\tau_k}) \leq \pi \sum_{n=1}^{N} \sum_{k \in \tilde{I}_n} \sum_{i=1}^{P_{k,n}} (d_{i,k,n} \beta_{i,k,n})^2. \tag{6.30}$$

Upon inserting the values of $\beta_{i,k,n}$ from (6.25), we find that

$$t^2 B_1 \leq \pi \sum_{n=1}^{N} \sum_{k \in \tilde{I}_n} |D_n| = \pi \sum_{n=1}^{N} \#(\tilde{I}_n) \, |D_n| \, , \tag{6.31}$$

where $\#(\tilde{I}_n)$ denotes the cardinality of $\tilde{I}_n$.

To handle the B_2 term we note that

$$\begin{aligned} &\{(n,k,i) \mid 1 \leq n \leq N, k \in \tilde{Z}_n, 1 \leq i \leq P_{k,n}\} \\ &= \bigcup_{\substack{1 \leq n \leq N \\ 1 \leq j \leq m_n}} \{(n,k,i) \mid 1 \leq i \leq P_n^j, k \in \tilde{Z}_n^j\}, \end{aligned} \tag{6.32}$$

and hence

$$B_2 = \sum_{n=1}^{N} \sum_{j=1}^{m_n} \sum_{i=1}^{P_n^j} \sum_{k \in \tilde{Z}_n^j} |A_{i,k,n} \cap \{|G| > t\}| \, . \tag{6.33}$$

When a zero-merging happens to a ball B of nonzero degree, it is merged with a number of balls of zero degree. The resulting ball has the same degree as B, and its radius is strictly larger than the radius of B. Thus, we see that the radii hypothesis of Lemma 6.3 is satisfied by $\{A_{i,k,n}\}$ for $k \in \tilde{Z}_n^j$, $i = 1, \ldots, P_{k,n}$. Moreover, for $k \in \tilde{Z}_n^j$, we have that $d_{i,k,n} = d_{i,l_n^j,n}$ and $\beta_{i,k,n} = \beta_{i,l_n^j,n}$. All hypotheses of Lemma 6.3 are thus satisfied; applying it, for each j, n we may bound

$$t^2 \sum_{k \in \tilde{Z}_n^j} |A_{i,k,n} \cap \{|G| > t\}| \leq \pi (d_{i,l_n^j,n} \beta_{i,l_n^j,n})^2. \tag{6.34}$$

Plugging in the values of $\beta_{i,k,n}$ from (6.25) then shows that

$$t^2 B_2 \leq \sum_{n=1}^{N} \sum_{j=1}^{m_n} \pi \left| D_n \right| = \sum_{n=1}^{N} \pi m_n \left| D_n \right|. \tag{6.35}$$

Recall that $I_n = \tilde{I}_n \cup \{l_n^1, \ldots, l_n^{m_n}\}$. Hence $\#(I_n) = \#(\tilde{I}_n) + m_n$. We then combine (6.29), (6.31), and (6.35) to get the estimate

$$t^2 A_2 \leq \pi \sum_{n=1}^{N} \#(I_n) \left| D_n \right|. \tag{6.36}$$

Together, (6.27), (6.28), and (6.36) prove that

$$t^2 \lambda_G(t) \leq \pi \sum_{T_e} d_{i,k,n}^2 \left(1 - e^{-2\tau_k}\right) + \pi \sum_{n=1}^{N} \#(I_n) \left| D_n \right|, \tag{6.37}$$

where $\#(I_n)$ is the cardinality of I_n.

Step 3.

In this step we will utilize the η inequalities (6.23) and (6.24) to show that the energy excess, $F_\varepsilon(u, A) - \pi D \left(\log \frac{r}{\varepsilon D} - C\right)$, controls the first term on the right side of (6.37). To begin we modify an argument from the beginning of the proof of Theorem 1. Define V to be the union of the balls in $\mathcal{G}(s + \sigma)$. Then, copying (5.5), we can bound

$$\begin{aligned} F_\varepsilon^r(u, A, V) &= \frac{1}{2} \int_V \rho^2 \left| \nabla_A v \right|^2 + \frac{1}{2\varepsilon^2}(1 - \rho^2)^2 + \left| \nabla \rho \right|^2 + r^2 (\operatorname{curl} A)^2 \\ &\geq F_\varepsilon(\rho, V) + \int_0^{\frac{1}{2}} 2t \left(\frac{1}{2} \int_{V \setminus \omega_t} \left| \nabla_A v \right|^2 \right) dt + \frac{r^2}{8} \int_V (\operatorname{curl} A)^2 \\ &+ \int_{\frac{1}{2}}^{1-\delta} 2t \left(\frac{1}{2} \int_{V \setminus \omega_t} \left| \nabla_A v \right|^2 + \frac{r^2}{2} \int_V (\operatorname{curl} A)^2 \right) dt. \end{aligned} \tag{6.38}$$

For $t \in [0, 1/2]$ the inclusions

$$V \setminus \omega_t \supseteq V \setminus \omega_{1/2} \supseteq V \setminus \omega_{1/2}^{3/2} \tag{6.39}$$

hold, and hence

$$\begin{aligned} \int_0^{\frac{1}{2}} 2t \left(\frac{1}{2} \int_{V \setminus \omega_t} \left| \nabla_A v \right|^2 \right) dt &\geq \int_0^{\frac{1}{2}} 2t \left(\frac{1}{2} \int_{V \setminus \omega_{1/2}^{3/2}} \left| \nabla_A v \right|^2 \right) dt \\ &= \frac{1}{8} \int_{V \setminus \omega_{1/2}^{3/2}} \left| \nabla_A v \right|^2. \end{aligned} \tag{6.40}$$

We now use (5.18) and (5.22) from Theorem 1 to bound

$$\begin{aligned}&\int_{1/2}^{1-\delta} 2t\left(\frac{1}{2}\int_{V\setminus\omega_t}|\nabla_A v|^2+\frac{r^2}{2}\int_V(\operatorname{curl}A)^2\right)dt+\frac{3}{4}F_\varepsilon(\rho,V)\\&\geq \pi D\left(\frac{3}{4}\log\frac{r}{\varepsilon D}-C\right).\end{aligned}\tag{6.41}$$

Here we have used $D=\sum_{n=1}^N D_n$. Assembling the bounds (6.38), (6.40), and (6.41) produces the bound

$$\begin{aligned}&F_\varepsilon^r(u,A,V)-\pi D\left(\log\frac{r}{\varepsilon D}-C\right)\\&\geq\frac{1}{4}\left(F_\varepsilon(\rho,V)+\frac{1}{2}\int_{V\setminus\omega_{1/2}^{3/2}}|\nabla_A v|^2+\frac{r^2}{2}\int_V(\operatorname{curl}A)^2-\pi D\log\frac{r}{\varepsilon D}\right).\end{aligned}\tag{6.42}$$

The argument in (5.14) shows that

$$F_\varepsilon(\rho,V)-\pi D\left(\log\frac{r}{\varepsilon D}-C\right)\geq\pi D\left(\log\frac{r}{r_{\Omega_\varepsilon}(\omega_{1/2}^{3/2})}-C\right).\tag{6.43}$$

In order to use the logarithm terms they must be translated into the new notation. Recalling (4.42) and changing the constant C (larger but still universal), we see that

$$\begin{aligned}&\log\frac{r}{r_{\Omega_\varepsilon}(\omega_{1/2}^{3/2})}-C=\log\frac{r}{r(\mathcal{B}_0)}+\log\frac{3r(\mathcal{B}_0)}{16r_{\Omega_\varepsilon}(\omega_{1/2}^{3/2})}\\&\leq\log\frac{r}{r(\mathcal{B}_0)}+\log\frac{r(\mathcal{C}(\sigma))}{r(\mathcal{C}_0)}=\sum_{k=1}^K\tau_k.\end{aligned}\tag{6.44}$$

Combining (6.42) - (6.44) and again changing the constant, we arrive at

$$\begin{aligned}&F_\varepsilon^r(u,A,V)-\pi D\left(\log\frac{r}{\varepsilon D}-C\right)\\&\geq\frac{1}{4}\left(\frac{1}{2}\int_{V\setminus\omega_{1/2}^{3/2}}|\nabla_A v|^2+\frac{r^2}{2}\int_V(\operatorname{curl}A)^2-\pi D\sum_{k=1}^K\tau_k\right).\end{aligned}\tag{6.45}$$

We now translate the term on the right side of inequality (6.45) into the new notation and break it into two parts according to whether the generation is before or after generation k_n. Indeed,

$$\begin{aligned}&\frac{1}{2}\int_{V\setminus\omega_{1/2}^{3/2}}|\nabla_A v|^2+\frac{r^2}{2}\int_V(\operatorname{curl}A)^2-\pi D\sum_{k=1}^K\tau_k\geq\frac{1}{2}\sum_{T_e}\int_{A_{i,k,n}}|\nabla_A v|^2\;-\pi\sum_{S_e}|D_n|\,\tau_k\\&+\frac{1}{2}\sum_{T_l}\int_{A_{i,k,n}}|\nabla_A v|^2+r^2(\operatorname{curl}A)^2-\pi\sum_{S_l}|D_n|\,\tau_k+\sum_{n=1}^N\sum_{B\in\bar{B}_n\cap\mathcal{G}(t_{\bar{B}_n})}\frac{r^2}{2}\int_B(\operatorname{curl}A)^2.\end{aligned}\tag{6.46}$$

For each $\bar{B}_n \in \mathcal{G}(s+\sigma)$ we consider $\bar{B}_n$ to have been grown from $\bar{B}_n \cap \mathcal{G}(t_{\bar{B}_n})$ and apply Corollary 3.4; summing over n gives

$$\frac{1}{2}\sum_{T_l}\int_{A_{i,k,n}} |\nabla_A v|^2 + r^2(\operatorname{curl} A)^2 \geq \pi D\left(\log\frac{r}{r(\mathcal{G}(t_{\bar{B}_n}))} - \log 2\right). \tag{6.47}$$

Note that if $t_{\bar{B}_n} \geq \sigma$, then

$$\sum_{k=k_n}^{K} \tau_k = \log\frac{r}{r(\mathcal{G}(t_{\bar{B}_n}))},$$

whereas if $t_{\bar{B}_n} < \sigma$, then

$$\sum_{k=k_n}^{K} \tau_k = \log\frac{r}{r(\mathcal{B}_0)} + \log\frac{r(\mathcal{G}(\sigma))}{r(\mathcal{G}(t_{\bar{B}_n}))} = \log\frac{r}{r(\mathcal{G}(t_{\bar{B}_n}))} + \log\frac{3}{8}$$

since $r(\mathcal{G}(\sigma)) = r(\mathcal{C}(\sigma)) = 3r(\mathcal{B}_0)/8$ (see item 4 of Proposition 4.3). Then

$$\frac{1}{2}\sum_{T_l}\int_{A_{i,k,n}} |\nabla_A v|^2 + r^2(\operatorname{curl} A)^2 - \pi\sum_{S_l}|D_n|\,\tau_k \geq -\pi C D, \tag{6.48}$$

where C is universal.

It remains to control the term corresponding to the early generations:

$$Q := \frac{1}{2}\sum_{T_e}\int_{A_{i,k,n}} |\nabla_A v|^2 \;- \pi\sum_{S_e}|D_n|\,\tau_k + \sum_{n=1}^{N}\sum_{B\in\bar{B}_n\cap\mathcal{G}(t_{\bar{B}_n})}\frac{r^2}{2}\int_B(\operatorname{curl} A)^2.$$

We apply Lemma 3.5 to each $B \in \bar{B}_n \cap \mathcal{G}(t_{\bar{B}_n})$ and sum to get

$$Q \geq \pi\sum_{S_e}\tau_k\left(\frac{2}{3}\sum_{i=1}^{M_{k,n}} d_{i,k,n}^2 - |D_n|\right). \tag{6.49}$$

In order to control the difference in (6.49) we must now turn to the η inequalities for generations before k_n. If $D_n \geq 0$, $1 \leq k < k_n$, the inequality (6.23) allows us to estimate

$$\begin{aligned}\sum_{i=1}^{M_{k,n}} d_{i,k,n}^2 &\geq \sum_{i=1}^{M_{k,n}} |d_{i,k,n}| = \sum_{\substack{1\leq i\leq M_{k,n}\\ d_{i,k,n}\geq 0}} d_{i,k,n} + \sum_{\substack{1\leq i\leq M_{k,n}\\ d_{i,k,n}<0}} |d_{i,k,n}| \\ &> (1+\eta)\sum_{\substack{1\leq i\leq M_{k,n}\\ d_{i,k,n}\geq 0}} d_{i,k,n} \\ &\geq (1+\eta)D_n = (1+\eta)\,|D_n|\,.\end{aligned} \tag{6.50}$$

If $D_n < 0$, we similarly get

$$\sum_{i=1}^{M_{k,n}} d_{i,k,n}^2 > (1+\eta)\,|D_n|\,,$$

and so in either case we arrive at the estimate

$$-|D_n| \geq -\frac{1}{1+\eta}\sum_{i=1}^{M_{k,n}} d_{i,k,n}^2. \tag{6.51}$$

Putting (6.51) into (6.49) then shows that

$$\begin{aligned} Q &\geq \pi\frac{2\eta-1}{3(1+\eta)}\sum_{T_e}\tau_k d_{i,k,n}^2 \\ &\geq \pi\frac{2\eta-1}{6(1+\eta)}\sum_{T_e} d_{i,k,n}^2(1-e^{-2\tau_k}), \end{aligned} \tag{6.52}$$

where in the last inequality we have used the fact that

$$x \geq \frac{1}{2}(1-e^{-2x}) \text{ for } x \geq 0.$$

Finally, we use (6.45) – (6.48) and (6.52) to conclude

$$F_\varepsilon(u,A,V) - \pi D\left(\log\frac{r}{\varepsilon D} - C\right) \geq \pi\frac{2\eta-1}{24(1+\eta)}\sum_{T_e} d_{i,k,n}^2(1-e^{-2\tau_k}). \tag{6.53}$$

Step 4.

In this step we use the η inequalities to provide an upper bound for the second term on the right side of (6.37) by bounding $\#(I_n)$ in terms of $|D_n|$ and η. Fix n and suppose that $k_n \leq k \leq K$. For now take $D_n \geq 0$. The inequality (6.23) allows us to bound

$$\sum_{\substack{1\leq i\leq M_{k,n}\\ d_{i,k,n}\geq 0}} d_{i,k,n} = D_n + \sum_{\substack{1\leq i\leq M_{k,n}\\ d_{i,k,n}<0}} |d_{i,k,n}| \leq D_n + \eta\sum_{\substack{1\leq i\leq M_{k,n}\\ d_{i,k,n}\geq 0}} d_{i,k,n},$$

and so we can conclude that

$$\sum_{\substack{1\leq i\leq M_{k,n}\\ d_{i,k,n}\geq 0}} d_{i,k,n} \leq \frac{|D_n|}{1-\eta}. \tag{6.54}$$

We can use this estimate to bound $\#(I_n)$. Each generation in I_n is an effective-merging generation. As such, the mergings of that generation include at least one ball of nonzero degree merging with another ball of nonzero degree, resulting in a decrease in the number of balls of nonzero degree. So, the number of effective generations, $\#(I_n)$, is bounded by the number of nonzero degree balls in the k_n generation. This quantity can then be bounded in terms of D_n and η. Indeed,

$$
\begin{aligned}
\#(I_n) &\leq \#\text{ of nonzero degree balls in generation } k_n \\
&\leq \sum_{i=1}^{M_{k_n,n}} |d_{i,k_n,n}| = \sum_{\substack{1\leq i\leq M_{k_n,n} \\ d_{i,k_n,n}\geq 0}} |d_{i,k_n,n}| + \sum_{\substack{1\leq i\leq M_{k_n,n} \\ d_{i,k_n,n}<0}} |d_{i,k_n,n}| \\
&\leq (1+\eta) \sum_{\substack{1\leq i\leq M_{k_n,n} \\ d_{i,k_n,n}\geq 0}} d_{i,k_n,n} \\
&\leq \frac{1+\eta}{1-\eta} |D_n| .
\end{aligned}
\tag{6.55}
$$

If $D_n < 0$ then (6.24) and a similar argument show that (6.55) still holds. Hence

$$
\pi \sum_{n=1}^{N} \#(I_n) |D_n| \leq \pi \frac{1+\eta}{1-\eta} \sum_{n=1}^{N} |D_n|^2 . \tag{6.56}
$$

Step 5.

We now conclude the proof by combining (6.37), (6.53), and (6.56) to get the inequality

$$
t^2 \lambda_G(t) \leq \pi \frac{1+\eta}{1-\eta} \sum_{n=1}^{N} |D_n|^2 + \frac{24(1+\eta)}{2\eta - 1} \left(F_\varepsilon^r(u, A, V) - \pi D \left(\log \frac{r}{\varepsilon D} - C \right) \right) . \tag{6.57}
$$

Using Lemma 6.1 and switching back to our original notation then proves (6.22). □

7 Jerrard's construction

In the above results we have modified and improved the vortex ball construction of Sandier, introduced in [6], and presented in an updated form in [8]. The purpose of this section is to show that the methods of this paper can be applied equally well to the other version of the vortex ball construction, developed by Jerrard in [3]. The two constructions are not at all dissimilar, so it is no surprise that the above methods still work. For completeness, though, we highlight the differences in the two constructions and outline the modifications necessary to make the above ideas work with Jerrard's construction. In the interest of brevity we discuss only the case without magnetic field.

There are three main differences between the ball construction employed above and that of [3]. The Jerrard construction grows finite collections of disjoint balls from an initial small collection to a final large collection, employing mergings when grown balls become tangent. However, a collection of disjoint balls $\{B_i\}$ is not grown uniformly, as we grow them above, but instead according to the parameter

$$s = \min_i \frac{r_i}{|d_i|},$$

where $d_i = \deg(u, \partial B_i)$ and r_i is the radius of B_i. There is no guarantee that this parameter is uniform throughout the collection (hence the minimum in the definition of s), and as a result, only balls for which the minimum s is achieved are grown. Note that as a ball is grown without merging, its degree does not vary, so increasing s amounts to increasing the radius of the ball. Moreover, for the subcollection of balls in $\{B_i\}$ that achieve s, if we write s^{new} for the increased parameter and r_i^{new} for the increased radii, we see that

$$\frac{s^{new}}{s} = \frac{r_i^{new}}{d_i}\frac{d_i}{r_i} = \frac{r_i^{new}}{r_i},$$

and so all of the annuli formed by deleting the old balls from the new ones have the same conformal type. The use of this parameter causes trouble above since $r(\mathcal{B}(t)) \neq e^t r(\mathcal{B}_0)$.

The second major difference in the two methods is in how they pass from lower bounds on circles, which in both methods are most conveniently calculated by estimating $\frac{1}{2}\int_{\partial B(a,r)} |\nabla v|^2$ from below, to lower bounds of $\frac{1}{2}\int |\nabla u|^2$ on annuli and balls. Above we employ the co-area formula in Lemma 5.1 and in (5.5) of Theorem 1 to accomplish this. The Jerrard method writes $u = \rho v$, with $\rho = |u|$, and expands the energy as

$$\frac{1}{2}\int_{\partial B(a,r)} |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2 = \frac{1}{2}\int_{\partial B(a,r)} |\nabla \rho|^2 + \frac{1}{2\varepsilon^2}(1-\rho^2)^2 + \frac{1}{2}\int_{\partial B(a,r)} \rho^2 |\nabla v|^2.$$

Lemmas 2.4 and 2.5 of [3] then show that

$$\frac{1}{2}\int_{\partial B(a,r)} \rho^2 |\nabla v|^2 \geq \pi \frac{m^2 d^2}{r},$$

and

$$\frac{1}{2}\int_{\partial B(a,r)} |\nabla \rho|^2 + \frac{1}{2\varepsilon^2}(1-\rho^2)^2 \geq \frac{1}{c\varepsilon}(1-m)^2,$$

where c is a universal constant and $m = \min\{1, \inf\limits_{\partial B(a,r)} \rho\}$. These two bounds are combined with the energy expansion to find

$$\frac{1}{2}\int_{\partial B(a,r)} |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2 \geq \inf_{m\in[0,1]} \left(\pi\frac{m^2 d^2}{r} + \frac{1}{c\varepsilon}(1-m)^2\right) =: \lambda_\varepsilon(r,d).$$

One readily verifies that $\lambda_\varepsilon(r,d) \geq \lambda_\varepsilon(r/|d|, 1)$ and that

$$\lambda_\varepsilon(r,1) = \frac{\pi}{r + c\varepsilon\pi}. \tag{7.1}$$

The function $\Lambda_\varepsilon(s) = \int_0^s \lambda_\varepsilon(r,1)dr = \pi \log(1 + \frac{s}{c\varepsilon\pi})$ is then introduced, and lower bounds on annuli are calculated by integrating on circles:

$$\begin{aligned}\frac{1}{2}\int_{B(a,r_1)\setminus B(a,r_0)} |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2 \geq \int_{r_0}^{r_1} \lambda_\varepsilon(r,d)dr \geq |d| \int_{r_0/|d|}^{r_1/|d|} \lambda_\varepsilon(r,1)dr \\ = |d|\left(\Lambda_\varepsilon(r_1/|d|) - \Lambda_\varepsilon(r_0/|d|)\right).\end{aligned}$$

Note that this bound justifies the use of $s = r/d$ as the growth parameter.

The third major difference is in the nature of the lower bounds. The method above produces lower bounds on the total collection of balls but can not say much about the energy content of any given ball in the collection. Because of its use of the Λ_ε function, which only depends on the parameter s, the Jerrard construction can localize the lower bounds to each ball in the collection. In particular, Proposition 4.1 of [3], the analogue of our Theorem 1, shows that there exists a σ_0 such that for any $0 \leq \sigma \leq \sigma_0$ there exists a collection of disjoint balls $\{B_i\}$ with radii r_i and degrees d_i such that

$$\frac{1}{2}\int_{B_i\cap\Omega} |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2 \geq \frac{r_i}{s}\Lambda_\varepsilon(s),$$

where $s = \min_i(r_i/|d_i|) \in [\sigma/2, \sigma]$. In particular this implies that

$$\frac{1}{2}\int_{B_i\cap\Omega} |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2 \geq \pi |d_i| \log\left(1 + \frac{\sigma}{2c\pi\varepsilon}\right).$$

The proof of this result follows from a line of reasoning similar to what led to Theorem 1. An initial collection of balls $\{B_i\}$ with radii $r_i \geq \varepsilon$ is found (Proposition 3.3 of [3]) that covers $\{|u| \leq 1/2\}$ and on which

$$\frac{1}{2}\int_{B_i\cap\Omega} |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2 \geq c_0\frac{r_i}{\varepsilon} \geq \frac{r_i}{s}\Lambda_\varepsilon(s), \tag{7.2}$$

where c_0 is a universal constant. These balls are then grown into the final balls according to the ball growth lemma, but used with the parameter s as the growth parameter. It is then shown that growth and merging preserves the form of the lower bound (7.2), i.e. that if the bound holds with one value of s, it also holds with the value of s obtained after growing the balls.

In order to utilize our completion of the square trick to extract the new term we must only present a modification of Lemma 3.2 designed to work with the minimization of m trick. The rest of the argument follows from simple modifications of the arguments in [3] that we will only sketch.

Lemma 7.1. *Let $B = B(a,r)$ and suppose that $u : \partial B \to \mathbb{C}$ is C^1 and that $|u| > c \geq 0$ on ∂B. Write $u = \rho v$ with $\rho = |u|$, and define the function*

$$G = \frac{dm^2\beta}{\rho^2 r}\tau, \tag{7.3}$$

where $d = \deg(u, \partial B)$, $m = \min\{1, \inf_{\partial B(a,r)} \rho\}$, τ *is the oriented unit tangent vector field to* ∂B, *and* $\beta \in [0,1]$ *is a constant. Then*

$$\frac{1}{2}\int_{\partial B} \rho^2 |\nabla v|^2 \geq \frac{1}{2}\int_{\partial B} \rho^2 |\nabla v - G|^2 + \pi \frac{d^2 m^2 \beta}{r}. \tag{7.4}$$

Proof. Arguing as in Lemma 3.2, we find that

$$\frac{1}{2}\int_{\partial B} \rho^2 |\nabla v|^2 = \frac{1}{2}\int_{\partial B} \rho^2 |\nabla v - G|^2 + 2\pi d \frac{dm^2\beta}{r} - \frac{d^2 m^4 \beta^2}{2r^2}\int_{\partial B} \frac{1}{\rho^2}. \tag{7.5}$$

Then the definition of m implies that

$$2\pi d \frac{dm^2\beta}{r} - \frac{d^2 m^4 \beta^2}{2r^2}\int_{\partial B}\frac{1}{\rho^2} \geq \pi \frac{d^2 m^2}{r}(2\beta - \beta^2) \geq \pi \frac{d^2 m^2 \beta}{r}, \tag{7.6}$$

where the last inequality follows from the fact that $0 \leq \beta \leq 1$. This proves the result. □

This result may be used in conjunction with Lemma 2.5 of [3], borrowing half of that energy to absorb into the novel term, to arrive at the lower bound

$$\frac{1}{2}\int_{\partial B} |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2 \geq \frac{1}{4}\int_{\partial B} |\nabla u - iuG|^2 + \inf_{m\in[0,1]}\left(\pi \frac{m^2 d^2 \beta}{r} + \frac{1}{c\varepsilon}(1-m)^2\right). \tag{7.7}$$

In order to gain the ability to localize the estimates in each ball, we must have that $\lambda_\varepsilon(r,d)$ is independent of β and that the homogeneity inequality $\lambda_\varepsilon(r,d) \geq \lambda_\varepsilon(r/|d|, 1)$ holds. The first of these requires us to set $\beta = 1$ in the above, which precludes the special choice of β needed to make Proposition 6.4 work. The second requires us to throw away the d^2 terms in favor of $|d|$. So, there is a tradeoff: the price we pay for localizing the estimates is a loss of control of the $L^{2,\infty}$ norm of the auxiliary function G. This choice leads to the lower bound on circles

$$\frac{1}{2}\int_{\partial B} |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2 \geq \frac{1}{4}\int_{\partial B} |\nabla u - iuG|^2 + \lambda_\varepsilon(r/|d|, 1), \tag{7.8}$$

where λ_ε is as defined in (7.1), but with the universal constant doubled, and $G = \frac{dm^2}{\rho^2 r}\tau$. The bound on circles leads to bounds on annuli by integrating; indeed,

$$\begin{aligned}\frac{1}{2}\int_{B(a,r_1)\setminus B(a,r_0)} |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2 \geq \frac{1}{4}&\int_{B(a,r_1)\setminus B(a,r_0)} |\nabla u - iuG|^2 \\ &+ |d|\left(\Lambda_\varepsilon(r_1/|d|) - \Lambda_\varepsilon(r_0/|d|)\right),\end{aligned} \tag{7.9}$$

where now we take $G(x) = \frac{dm^2}{\rho(x)^2|x-a|}\tau(x)$.

Now, to achieve a bound of the form (7.2) but with the L^2 difference with iuG included, we use Lemma 7.1 in the Jerrard construction. As above, we define the function G to vanish in the initial collection of balls obtained in Proposition 3.3 of [3]. Then we trivially modify (7.2) to read (since $G = 0$ there)

$$\begin{aligned}\frac{1}{2}\int_{B_i\cap\Omega} |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2 &\geq \frac{c_0 r_i}{2\varepsilon} + \frac{1}{4}\int_{B_i\cap\Omega} |\nabla u|^2 \\ &\geq \frac{r_i}{s}\Lambda_\varepsilon(s) + \frac{1}{4}\int_{B_i\cap\Omega} |\nabla u - iuG|^2 .\end{aligned} \tag{7.10}$$

We then take G to vanish in all of the non-annular regions of the balls constructed in Proposition 4.1 of [3]. The estimates in these balls, like the original Sandier estimates, discard the energy of the non-annular regions. We retain it and rewrite it as a $\int |\nabla u - iuG|^2$ term, which is possible since $G = 0$ there. Then, adding in the extra G term in the annular regions, we arrive at the modification.

Proposition 7.2. *There exists a σ_0 such that for any $0 \leq \sigma \leq \sigma_0$ there exists a collection of disjoint balls $\{B_i\}$ with radii r_i and degrees d_i such that*

$$\frac{1}{2}\int_{B_i\cap\Omega} |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2 \geq \frac{1}{4}\int_{B_i\cap\Omega} |\nabla u - iuG|^2 + \frac{r_i}{s}\Lambda_\varepsilon(s),$$

where $s = \min_i(r_i/|d_i|) \in [\sigma/2, \sigma]$. In particular this implies that

$$\frac{1}{2}\int_{B_i\cap\Omega} |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2 \geq \frac{1}{4}\int_{B_i\cap\Omega} |\nabla u - iuG|^2 + \pi|d_i|\log\left(1+\frac{\sigma}{2c\pi\varepsilon}\right).$$

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