MEAN FIELD LIMITS OF THE GROSS-PITAEVSKII AND PARABOLIC GINZBURG-LANDAU EQUATIONS

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Abstract. We prove that in a certain asymptotic regime, solutions of the Gross-Pitaevskii equation converge to solutions of the incompressible Euler equation, and solutions to the parabolic Ginzburg-Landau equations converge to solutions of a limiting equation which we identify.

We work in the setting of the whole plane (and possibly the whole three-dimensional space in the Gross-Pitaevskii case), in the asymptotic limit where ε, the characteristic lengthscale of the vortices, tends to 0, and in a situation where the number of vortices N_ε blows up as $\varepsilon \to 0$. The requirements are that N_ε should blow up faster than $|\log \varepsilon|$ in the Gross-Pitaevskii case, and at most like $|\log \varepsilon|$ in the parabolic case. Both results assume a well-prepared initial condition and regularity of the limiting initial data, and use the regularity of the solution to the limiting equations.

In the case of the parabolic Ginzburg-Landau equation, the limiting mean-field dynamical law that we identify coincides with the one proposed by Chapman-Rubinstein-Schatzman and E in the regime $N_\varepsilon \ll |\log \varepsilon|$, but not if N_ε grows faster.

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1. Introduction

1.1. Problem and background.

We are interested in the Gross-Pitaevskii equation

$$i\partial_t u = \Delta u + \frac{u}{\varepsilon^2}(1-|u|^2) \quad \text{in } \mathbb{R}^2 \tag{1.1}$$

and the parabolic Ginzburg-Landau equation

$$\partial_t u = \Delta u + \frac{u}{\varepsilon^2}(1-|u|^2) \quad \text{in } \mathbb{R}^2 \tag{1.2}$$

in the plane, and also the three-dimensional version of the Gross-Pitaevskii equation

$$i\partial_t u = \Delta u + \frac{u}{\varepsilon^2}(1-|u|^2) \quad \text{in } \mathbb{R}^3, \tag{1.3}$$

all in the asymptotic limit $\varepsilon \to 0$.

These are famous equations of mathematical physics, which arise in the modeling of superfluidity, superconductivity, nonlinear optics, etc. The Gross-Pitaevskii equation is an important instance of a nonlinear Schrödinger equation. These equations also come in a version with gauge, more suitable for the

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modeling of superconductivity, but whose essential mathematical features are similar to these, and which we will discuss briefly below. There is also interest in the "mixed flow" case, sometimes called complex Ginzburg-Landau equation

$$(a+ib)\partial_t u = \Delta u + \frac{u}{\varepsilon^2}(1-|u|^2) \quad \text{in } \mathbb{R}^2. \tag{1.4}$$

For further reference on these models, one can see e.g. [T, TT, AK, SS5].

In these equations, the unknown function u is complex-valued, and it can exhibit vortices, which are zeroes of u with non-zero topological degree, and a core size on the order of ε. In the plane, these vortices are points, whereas in the three-dimensional space they are lines. We are interested in one of the main open problems on Ginzburg-Landau dynamics, which is to understand the dynamics of vortices in the regime in which their number N_ε blows up as $\varepsilon \to 0$. The only available results until now are due to Kurzke and Spirn [KS2] in the case of (1.2) and Jerrard and Spirn [JS2] in the case of (1.1). The latter concern a very dilute limit in which N_ε grows slower than a power of $\log|\log\varepsilon|$ as $\varepsilon \to 0$ (more details are given below).

The dynamics of vortices in (1.1) and (1.2) was first studied in the case where their number N is bounded as $\varepsilon \to 0$ (hence can be assumed to be independent of ε. It was proven, either in the setting of the whole plane or that of a bounded domain, that, for "well-prepared" initial data, after suitable time rescaling, their limiting positions obey the law

$$\frac{da_i}{dt} = (a+\mathbb{J}b)\nabla_i W(a_1,\dots,a_N) \tag{1.5}$$

where $\mathbb{J}$ is the rotation by $\pi/2$ in the plane, and W is in the setting of the plane the so-called Kirchhoff-Onsager energy

$$W(a_1,\dots,a_N) = -\pi\sum_{i\neq j} d_i d_j \log|a_i - a_j| \tag{1.6}$$

where the d_i's are the degrees of the vortices and are assumed to be initially in $\{1,-1\}$. In the setting of a bounded domain and prescribed Dirichlet boundary data, W is (1.6) with some additional terms accounting for boundary effects. It was introduced and derived in that context in the book of Bethuel-Brezis-Hélein [BBH] where it was called the "renormalized energy".

In other words, the vortices move according to the corresponding flow (gradient, Hamiltonian, or mixed) of their limiting interaction energy W. After some formal results based on matched asymptotics by Pismen-Rubinstein and E in [PR, E1], these results were proven in the setting of a bounded domain by Lin [Li1] and Jerrard-Soner [JS1] in the parabolic case, Colliander-Jerrard [CJ1, CJ2] and Lin-Xin [LX2] with later improvements by Jerrard-Spirn [JSp1] in the Schrödinger case, and Kurzke-Melcher-Moser-Spirn [KMMS] in the mixed flow case. In the setting of the whole plane, the analogous results were obtained by Lin-Xin [LX1] in the parabolic case, Bethuel-Jerrard-Smets [BJS] in the Schrödinger case, and Miot [Mi] in the mixed flow case. A proof based on the idea of relating gradient flows and Γ-convergence was also given in [SS4], it was the initial motivation for the abstract scheme of "Γ-convergence of gradient flows" introduced there. Generalizations to the case with gauge, pinning terms and applied electric field terms were also studied [Sp1, Sp2, KS1, Ti, ST2].

All these results hold for well-prepared data and for as long as the points evolving under the dynamical law (1.5) do not collide. In the parabolic case, Bethuel-Orlandi-Smets showed in the series of papers [BOS1, BOS2, BOS3] how to lift the well-prepared condition and handle the difficult issue of collisions and extend the dynamical law (1.5) beyond them. Results of a similar nature were also obtained in [Se1].

The expected limiting dynamics of three-dimensional vortex lines under (1.3) is the binormal flow of a curve, but in contrast to the two-dimensional case there are only partial results towards establishing this rigorously [J2].

When the number of points N_ε blows up as $\varepsilon \to 0$, then it is expected that the limiting system of ODEs (1.5) should be replaced by a mean-field evolution for the density of vortices, or vorticity. More precisely, for a family of functions u_ε, one introduces the supercurrent j_ε and the vorticity (or Jacobian) μ_ε of the map u_ε which are defined via

$$j_\varepsilon := \langle iu_\varepsilon, \nabla u_\varepsilon\rangle \qquad \mu_\varepsilon := \operatorname{curl} j_\varepsilon, \tag{1.7}$$

where $\langle x, y\rangle$ stands for the scalar product in $\mathbb{C}$ as identified with $\mathbb{R}^2$ via $\langle x, y\rangle = \frac{1}{2}(\bar{x}y+\bar{y}x)$. The vorticity μ_ε plays the same role as the vorticity in classical fluids, the only difference being that it is essentially quantized at the ε level, as can be seen from the asymptotic estimate $\mu_\varepsilon \simeq 2\pi \sum_i d_i\delta_{a_i}$ as $\varepsilon \to 0$, with $\{a_i\}$ the vortices of u_ε and $d_i \in \mathbb{Z}$ their degrees (these are the so-called Jacobian estimates, which we will recall in the course of the paper).

The mean-field evolution for $\mu = \lim_{\varepsilon\to 0} \mu_\varepsilon/N_\varepsilon$ can be guessed to be the mean-field limit of (1.5) as $N \to \infty$. Proving this essentially amounts to showing that the limits $\varepsilon \to 0$ and $N \to \infty$ can be interchanged, which is a delicate question.

In the case of the Gross-Pitaevskii equation (1.1)-(1.3), it is well-known that the Madelung transform formally yields that the limiting evolution equation should be the incompressible Euler equation (for this and related questions, see for instance the survey [CDS]). In the case of the parabolic Ginzburg-Landau equation, it was proposed, based on heuristic considerations by Chapman-Rubinstein-Schatzman [CRS] and E [E2], that the limiting equation should be

$$\partial_t\mu - \operatorname{div}(\mu\nabla h) = 0 \qquad h = -\Delta^{-1}\mu. \tag{1.8}$$

where μ is the limit of the vortex density, assumed to be nonnegative. In fact, both papers really derived the equation for possibly signed densities, [CRS] did it for the very similar model with gauge in a bounded domain, in which case the coupling $h = -\Delta^{-1}\mu$ is replaced by $h = (-\Delta + I)^{-1}\mu$ (see also Section 1.3.4 below), and [E2] treated both situations with and without gauge, also for signed densities, without discussing the domain boundary.

After this model was proposed, existence, weak notions and properties of solutions to (1.8) were studied in [LZ1, DZ, SV] (see also [MZ] for some related models). They depend greatly on the regularity of the initial data μ. For μ a general probability measure, the product $\mu\nabla h$ does not make sense, and a weak formulation à la Delort [De] must be used; also uniqueness of solutions can fail, although there is always existence of a unique solution which becomes instantaneously L^∞. It also turns out that (1.8) can be interpreted as the gradient flow (as in [O, AGS]) in the space of probability measures equipped with the 2-Wasserstein distance, of the energy functional

$$\Phi(\mu) = \int |\nabla h|^2 \qquad h = -\Delta^{-1}\mu, \tag{1.9}$$

which is also the mean field limit of the usual Ginzburg-Landau energy. The equation (1.8) was also studied with that point of view in [AS] in the bounded domain setting (where the possible entrance and exit of mass creates difficulties). This energy point of view allows one to envision a possible (and so far unsuccessful) energetic proof of the convergence of (1.2) based on the scheme of Gamma-convergence of gradient flows, as described in [Se2].

A proof of the convergence of (1.1), (1.2) or (1.3) to these limiting equations (Euler or (1.8)) has remained elusive for a long time.

The time-independent analogue of this result, i.e. the convergence of solutions to the static Ginzburg-Landau equations to the time-independent version of (1.8) – a suitable weak formulation of $\mu\nabla h = 0$ – in the regime $N_\varepsilon \gg 1$ (the notation means $N_\varepsilon \to +\infty$ as $\varepsilon \to 0$), and without any assumption on the sign of the vorticity, was obtained in [SS2]. As is standard, due to its translation-invariance, the stationary Ginzburg-Landau equation can be rewritten as a conservation law, namely that the so-called stress-energy tensor is divergence free. The method of the proof then consists in passing to the limit in that relation, taking advantage of a good control of the size of the set occupied by vortices. This approach seems to fail to extend to the dynamical setting for lack of extension of this good control.

On the other hand, the proof of convergence of the dynamic equations in the case of bounded number of vortices is usually based on examining the expression for the time-derivative of the energy density and identifying limits for each term (for a quick description, one may refer to the introduction of [ST2]). This proof also seems very hard to extend to the situation of $N_\varepsilon \gg 1$ for the following reasons :

- It relies on estimates on the evolution of the "energy-excess" D, where terms controlled in $\sqrt{D}$ instead of D arise (and these are not amenable to the use of Gronwall's lemma).
- Understanding the evolution of the energy density or excess seems to require controlling the speed of each individual vortex, which is difficult when their number gets large while only averaged information is available.
- The proof works until the first time of possible collision between vortices, which can in principle occur in very short time once the number of vortices blow up, so it seems that one needs good control and understanding of the vortices' mutual distances.

Recently Spirn-Kurzke [KS2] and Jerrard-Spirn [JS1] were however able to make the first breakthrough by accomplishing this program for (1.2) and (1.1) respectively, in the case where N_ε grows very slowly: $N_\varepsilon \ll (\log\log|\log\varepsilon|)^{1/4}$ in the parabolic Ginzburg-Landau case, and $N_\varepsilon \ll (\log|\log\varepsilon|)^{1/2}$ in the two-dimensional Gross-Pitaevskii case, assuming some specific well-preparedness conditions on the initial data. Relying on their previous work [KS1, JSp1], they showed that the method of proof for finite number of vortices can be made more quantitative and pushed beyond bounded N_ε, controlling the vortex distances and proving that their positions remain close to those of the N_ε points solving the ODE system (1.5), and then finally passing to the limit for that system by applying classical "point vortex methods", in the manner of, say, Schochet [Sch]. There is however very little hope for extending such an approach to larger values of N_ε.

By a different proof method based on the evolution of a "modulated energy", we will establish

- the mean-field limit evolution of (1.1)–(1.3) in the regime $|\log\varepsilon| \ll N_\varepsilon \ll 1/\varepsilon$
- the mean-field limit evolution of (1.2) in the regime $1 \ll N_\varepsilon \le O(|\log\varepsilon|)$

both at the level of convergence of the supercurrents, and not just the vorticity. We note that the condition $N_\varepsilon \le O(|\log\varepsilon|)$ allows to reach a physically very relevant regime: in the case of the equation with gauge, $|\log\varepsilon|$ is the order of the number of vortices that are present just after one reached the so-called first critical field H_{c_1}, itself of the order of $|\log\varepsilon|$ (cf. [SS5]).

Our method relies on the assumed regularity of the solution to the limiting equation, thus is restricted to limiting vorticities which are sufficiently regular. In contrast, although they concern very dilute limits, the results of [KS2, JS1] allow for general (possibly irregular) limiting vorticities.

1.2. **Our setting and results.**

1.2.1. *Scaling of the equation.* In order to obtain a nontrivial limiting evolution, the appropriate scaling of the equation to consider is

$$\begin{cases} N_\varepsilon\left(\frac{\alpha}{|\log\varepsilon|}+i\beta\right)\partial_t u = \Delta u + \frac{1}{\varepsilon^2}u(1-|u|^2) & \text{in } \mathbb{R}^2 \\ u(\cdot,0)=u^0. \end{cases} \tag{1.10}$$

Here we have put both the 2D Gross-Pitaevskii and parabolic equations in the same framework. To obtain Gross-Pitaevskii, one should set $\alpha = 0$ and $\beta = 1$ and to obtain the parabolic Ginzburg-Landau equation, one should set $\alpha = 1$ and $\beta = 0$. We will also comment later on the mixed case where one would have α and β nonzero, and say, $\alpha^2 + \beta^2 = 1$, and we will describe the adaptation to the 3D Gross-Pitaevskii case as we go. In all the paper, when we write $\mathbb{R}^n$, we mean the whole Euclidean space with n being either 2 or 3.

1.2.2. *Limiting equation.* Throughout the paper, for vector fields X in the plane, we use the notation

$$X^\perp = (-X_2, X_1) \qquad \nabla^\perp = (-\partial_2, \partial_1)$$

and this way

$$\operatorname{curl} X = \partial_1 X_2 - \partial_2 X_1 = -\operatorname{div} X^\perp.$$

In the 2D Gross-Pitaevskii case, or in the parabolic case with $N_\varepsilon \ll |\log \varepsilon|$, then the limiting equation will be the incompressible equation

$$(1.11)\qquad \boxed{\begin{cases} \partial_t \mathrm{v} = 2\beta \mathrm{v}^\perp \operatorname{curl} \mathrm{v} - 2\alpha \mathrm{v} \operatorname{curl} \mathrm{v} + \nabla \mathrm{p} & \text{in } \mathbb{R}^2 \\ \operatorname{div} \mathrm{v} = 0 & \text{in } \mathbb{R}^2, \end{cases}}$$

with p the pressure associated to the divergence-free condition. In the parabolic case with N_ε comparable to $|\log \varepsilon|$, then without loss of generality, we may assume that

$$(1.12)\qquad \lambda := \lim_{\varepsilon \to 0} \frac{|\log \varepsilon|}{N_\varepsilon}$$

exists and is positive and finite. In that case, the limiting equation will be

$$(1.13)\qquad \boxed{\partial_t \mathrm{v} = \lambda \nabla \operatorname{div} \mathrm{v} - 2\mathrm{v} \operatorname{curl} \mathrm{v} \quad \text{in } \mathbb{R}^2.}$$

This is a particular case of the mixed flow equation

$$(1.14)\qquad \partial_t \mathrm{v} = \frac{\lambda}{\alpha} \nabla \operatorname{div} \mathrm{v} + 2\beta \mathrm{v}^\perp \operatorname{curl} \mathrm{v} - 2\alpha \mathrm{v} \operatorname{curl} \mathrm{v} \quad \text{in } \mathbb{R}^2.$$

The incompressible Euler equation is typically written as

$$(1.15)\qquad \begin{cases} \partial_t \mathrm{v} + \mathrm{v} \cdot \nabla \mathrm{v} + \nabla \mathrm{p} = 0 & \text{in } \mathbb{R}^n \\ \operatorname{div} \mathrm{v} = 0 & \text{in } \mathbb{R}^n, \end{cases}$$

where p is the pressure. But when $\operatorname{div} \mathrm{v} = 0$ and $n = 2$, one has the identity

$$(1.16)\qquad \mathrm{v} \cdot \nabla \mathrm{v} = \operatorname{div} (\mathrm{v} \otimes \mathrm{v}) = \mathrm{v}^\perp \operatorname{curl} \mathrm{v} + \frac{1}{2} \nabla |\mathrm{v}|^2,$$

so when $\alpha = 0$, (1.11) is exactly the 2D incompressible Euler equation (up to a time rescaling by a factor of 2), with the new pressure equal to the old one plus $|\mathrm{v}|^2$. Existence, uniqueness, and regularity of its solutions are well-known since Volibner and Yudovich (one can refer to textbooks such as [MB, Ch1] and references therein).

In the three-dimensional Gross-Pitaevskii case, our limiting equation will be the time-rescaled incompressible Euler equation rewritten again as

$$(1.17)\qquad \boxed{\begin{cases} \partial_t \mathrm{v} = 2 \operatorname{div} \left(\mathrm{v} \otimes \mathrm{v} - \frac{1}{2} |\mathrm{v}|^2 I \right) + \nabla \mathrm{p} & \text{in } \mathbb{R}^3 \\ \operatorname{div} \mathrm{v} = 0 & \text{in } \mathbb{R}^3. \end{cases}}$$

Taking the curl of (1.11), one obtains

$$\partial_t \omega = 2 \operatorname{div} ((\alpha \mathrm{v}^\perp + \beta \mathrm{v}) \omega) \qquad \omega = \operatorname{curl} \mathrm{v}, \ \operatorname{div} \mathrm{v} = 0,$$

so in the case $\alpha = 0$, $\beta = 1$, we of course retrieve the vorticity form of the Euler equation, and in the case $\alpha = 1$, $\beta = 0$ and $N_\varepsilon \ll |\log \varepsilon|$, we see that we retrieve the Chapman-Rubinstein-Schatzman-E equation (1.8). But in the case $0 < \lambda < +\infty$ the curl of (1.13) is *not* (1.8). In fact the divergence of v is not zero, even if we assume it vanishes at initial time, and this affects the dynamics of the vortices. This phenomenon can be seen as a drift on the vortices created by the phase of u, a "phase-vortex interaction", as was also observed in [BOS1].

The existence and uniqueness of global regular solutions to (1.11) can be obtained exactly as in the case of the Euler equation, see for example [Ch1]. It suffices to write the equation in vorticity form as $\partial_t \omega = A(\omega) \cdot \nabla \omega$ where $A(\omega)$ is a Fourier operator of order -1 (we do not give details here). In contrast, the equation (1.13) is new in the literature, and for the sake of completeness we present in an appendix (cf. Theorem 3) a result of local existence and uniqueness of $C^{1,\gamma}$ solutions to the general equation (1.14), which suffices for our purposes. More results, including global existence of such regular solutions are established in [Du].

We note that the transition from one limiting equation to the other happens in the parabolic case, in the regime where N_ε is proportional to $|\log \varepsilon|$. This is a natural regime, which corresponds to the situation where the number of points is of the same order as the "self-interaction" energy of each vortex.

1.2.3. *Method of proof.* Our method of proof bypasses issues such as taking limits in the energy evolution or vorticity evolution relations and controlling vortex distances. Instead, it takes advantage of the regularity and stability of the solution to the limiting equation. More precisely, we introduce what can be called a "modulated Ginzburg-Landau energy"

$$\mathcal{E}_\varepsilon(u,t) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u - iuN_\varepsilon \mathrm{v}(t)|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2} + N_\varepsilon^2(1-|u|^2)\psi(t), \tag{1.18}$$

where $\mathrm{v}(t)$ is a regular solution to the desired limiting equation, and

$$\psi(t) = \begin{cases} \mathrm{p}(t) - |\mathrm{v}(t)|^2 & \text{in the Gross-Pitaevskii case} \\ -|\mathrm{v}(t)|^2 & \text{in the parabolic case.} \end{cases} \tag{1.19}$$

This quantity is modelled on the Ginzburg-Landau energy, and controls the L^2 distance between the supercurrent $j_\varepsilon = \langle iu_\varepsilon, \nabla u_\varepsilon\rangle$ normalized by N_ε and the limiting velocity field v, because in the regimes we consider, the term $\int_{\mathbb{R}^n} N_\varepsilon^2(1-|u|^2)\psi(t)$ is a small perturbation (that term will however play a role in algebraic cancellations).

One of the difficulties in the proof is that the convergence of $j_\varepsilon/N_\varepsilon$ to v is not strong in L^2, in general, but rather weak in L^2, due to a concentration of an amount $\pi|\log\varepsilon|$ of energy at each of the vortex points (this energy concentration can be seen as a defect measure in the convergence of $j_\varepsilon/N_\varepsilon$ to v). In order to take that concentration into account, we need to subtract off of $\mathcal{E}_\varepsilon$ the constant quantity $\pi N_\varepsilon|\log\varepsilon|$. In the regime where $N_\varepsilon \gg |\log\varepsilon|$, then $\pi N_\varepsilon|\log\varepsilon| = o(N_\varepsilon^2)$ and this quantity (or the concentration) happens to become negligible, which is what will make the proof in the Gross-Pitaevskii case much simpler and applicable to the three-dimensional setting as well, but of course restricted to that regime.

The main point of the proof consists in differentiating $\mathcal{E}_\varepsilon$ in time, and showing that $\frac{d}{dt}\mathcal{E}_\varepsilon(u_\varepsilon(t),t) \le C(\mathcal{E}_\varepsilon(u_\varepsilon(t),t) - \pi N_\varepsilon|\log\varepsilon|)$, which allows us to apply Gronwall's lemma, and conclude that if $\mathcal{E}_\varepsilon - \pi N_\varepsilon|\log\varepsilon|$ is initially small, it remains so. The difficulty for this is to show that a control in $C(\mathcal{E}_\varepsilon - \pi N_\varepsilon|\log\varepsilon|)$ instead of $C\sqrt{\mathcal{E}_\varepsilon}$, is possible, even though the terms involved initially appear to be of order $\sqrt{\mathcal{E}_\varepsilon}$. This is made possible by a series of algebraic simplifications in the computations that reveal only quadratic terms. An important insight leading to these simplifications is that one should work, by analogy with the gauged Ginzburg-Landau model, with gauged quantities, where $N_\varepsilon \mathrm{v}$ plays the role of a space gauge vector-field, and $N_\varepsilon \phi$ plays the role of a temporal gauge, ϕ being defined by

$$\phi(t) = \begin{cases} \mathrm{p}(t) & \text{in cases leading to } (1.11), (1.17) \\ \lambda \operatorname{div} \mathrm{v}(t) & \text{in cases leading to } (1.13). \end{cases} \tag{1.20}$$

The idea of proving convergence via a Gronwall argument on the modulated energy, while assuming and using the regularity of the limiting solution is similar to the relative entropy method for establishing (the stability of) hydrodynamic limits, first introduced in [Yau] and used for quantum many body problems, mean-field theory and semiclassical limits, one example of the latter arising precisely for the limit of the Gross-Pitaevskii equation in [LZ2]; or Brenier's modulated entropy method to establish kinetic to fluid limits such as the derivation of the Euler equation from the Boltzmann or Vlasov equations (see for instance [SR] and references therein).

In the point of view of that method, $\int_{\mathbb{R}^n} |\nabla u - iuN_\varepsilon \mathrm{v}|^2$ is the modulated energy, while $\int_{\mathbb{R}^n} |\partial_t u - iuN_\varepsilon \phi|^2$ is the modulated energy-dissipation.

1.2.4. *Well-posedness of the Cauchy problem.* The equations (1.1) and (1.3) are shown in [Ge, Ga] to be globally well-posed in the natural energy space

$$E = \{u \in H^1_{loc}(\mathbb{R}^n), \nabla u \in L^2(\mathbb{R}^n), |u|^2 - 1 \in L^2(\mathbb{R}^n)\}. \tag{1.21}$$

This is the setting we will consider in dimension 3 and corresponds to solutions which have zero total degree at infinity. But in general this is too restrictive for our purposes: the problem is that, when working in the whole space, the natural energy is infinite as soon as the total degree of u at infinity is not zero. It thus needs to be renormalized by substracting off the energy of some fixed map U_{D_ε} which behaves at infinity like u_ε, i.e. typically like $e^{iN_\varepsilon \theta}$ for example. To be more specific, in the two-dimensional case we consider as in [BS, Mi], for each integer D, a reference map U_D, which is smooth in $\mathbb{R}^2$ and such that

$$U_D = \left(\frac{z}{|z|}\right)^D \quad \text{outside of } B(0,1). \tag{1.22}$$

The well-posedness of the Cauchy problem in that context was established in [BS] in the Gross-Pitaevskii case and [Mi] in the mixed flow (hence possibly parabolic) case : they show that given $u_\varepsilon^0 \in U_{D_\varepsilon} + H^1(\mathbb{R}^2)$ (in fact they even consider a slightly wider class than this), there exists a unique global solution to (1.10) such that $u_\varepsilon(t) - U_{D_\varepsilon} \in C^0(\mathbb{R}, H^1(\mathbb{R}^2))$, and satisfying other properties that will be listed in Section 2.1. This is the set-up that we will use, as is also done in [JSp2]. Without loss of generality, we may assume that $D_\varepsilon \geq 0$. In the Gross-Pitaevskii case, we will allow for D_ε (the total degree) to be possibly different from N_ε (the total number of vortices), which corresponds to a vorticity which does not have a distinguished sign. For simplicity, we will then assume that $D_\varepsilon / N_\varepsilon = d \leq 1$, a number independent of ε. In the parabolic case, we need to have a distinguished sign and we will take $D_\varepsilon = N_\varepsilon$.

Let us point out that the use of the modulated energy will simplify the proofs in that respect, in the sense that it naturally provides a finite energy and thus replaces having to "renormalize" the infinite energy as in [BJS, Mi, JSp2].

1.2.5. *Main results.* We may now state our main results, starting with the Gross-Pitaevskii case. In all the paper, we denote by C^γ the space of functions which are bounded and Hölder continuous with exponent γ, and by $C^{1,\gamma}$ the space of functions which are bounded and whose derivative is bounded and C^γ. We use the standard notation $a \ll b$ to mean $\lim a/b = 0$, and the standard o notation, all asymptotics being taken as $\varepsilon \to 0$.

Theorem 1 (Gross-Pitaevskii case)**.** *Assume N_ε satisfies*

$$|\log \varepsilon| \ll N_\varepsilon \ll \frac{1}{\varepsilon} \quad \textit{as } \varepsilon \to 0. \tag{1.23}$$

Let $\{u_\varepsilon\}_{\varepsilon>0}$ be solutions to

$$\begin{cases} iN_\varepsilon \partial_t u_\varepsilon = \Delta u_\varepsilon + \frac{u_\varepsilon}{\varepsilon^2}(1-|u_\varepsilon|^2) & \textit{in } \mathbb{R}^n \\ u_\varepsilon(\cdot,0) = u_\varepsilon^0. \end{cases} \tag{1.24}$$

If $n = 2$ we assume $u_\varepsilon^0 \in U_{D_\varepsilon} + H^2(\mathbb{R}^2)$ where we take $0 \le D_\varepsilon \le N_\varepsilon$ with $D_\varepsilon/N_\varepsilon = d$, U_{D_ε} is as in (1.22)*, and* v *is a solution to* (1.11)*, such that* $\mathrm{v}(0) - d\langle \nabla U_1, iU_1\rangle \in L^2(\mathbb{R}^2)$*,* $\mathrm{v}(t) \in L^\infty(\mathbb{R}_+, C^{0,1}(\mathbb{R}^2))$*, and* $\mathrm{curl}\, \mathrm{v}(t) \in L^\infty(\mathbb{R}_+, L^1(\mathbb{R}^2))$*.*
If $n = 3$ we assume $u_\varepsilon^0 \in E$ as in (1.21) *with $\Delta u_\varepsilon^0 \in L^2(\mathbb{R}^3)$, and* v *is a solution to* (1.17) *such that* $\mathrm{v}(t) \in L^\infty([0,T], C^{0,1}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3))$*,* $\partial_t \mathrm{v} \in L^\infty([0,T], L^\infty(\mathbb{R}^3) \cap L^2(\mathbb{R}^3))$ *and* $\mathrm{curl}\, \mathrm{v}(t) \in L^\infty([0,T], L^1(\mathbb{R}^3))$*.*

Letting $\mathcal{E}_\varepsilon$ be defined from $\mathrm{v}(t)$ *via* (1.18)*, assume that*

$$\mathcal{E}_\varepsilon(u_\varepsilon^0, 0) \le o(N_\varepsilon^2). \tag{1.25}$$

Then, for every $t \ge 0$ (resp. $t \le T$), we have $\mathcal{E}_\varepsilon(u_\varepsilon(t), t) \le o(N_\varepsilon^2)$, and in particular we have

$$\frac{j_\varepsilon}{N_\varepsilon} := \frac{\langle \nabla u_\varepsilon, iu_\varepsilon\rangle}{N_\varepsilon} \to \mathrm{v} \quad \textit{strongly in } L^1_{loc}(\mathbb{R}^n). \tag{1.26}$$

If we know in addition that u_ε is bounded in $L^\infty_{loc}(\mathbb{R}_+, L^\infty(\mathbb{R}^n))$ uniformly in ε, then the convergence is strong in $L^2(\mathbb{R}^n)$.

The restriction $N_\varepsilon \gg |\log \varepsilon|$ is a technical obstruction caused by the difficulty in controlling the velocity of the individual vortices because of the lack of control of $\int_{\mathbb{R}^n} |\partial_t u_\varepsilon|^2$. On the other hand, the restriction $N_\varepsilon \ll \frac{1}{\varepsilon}$ seems quite natural, since when N_ε is larger, the modulus of u should enter the limiting equation, giving rise to compressible Euler equations. On that aspect we refer to the survey [BDGSS] and results quoted therein.

In the parabolic case, we have the following result

Theorem 2 (Parabolic case)**.** *Assume N_ε satisfies*

$$1 \ll N_\varepsilon \le O(|\log \varepsilon|) \quad \textit{as } \varepsilon \to 0, \tag{1.27}$$

and let λ be as in (1.12)*.*
Let $\{u_\varepsilon\}_{\varepsilon>0}$ be solutions to (1.10) *with $\beta = 0$ and $\alpha = 1$, associated to initial*

conditions $u_\varepsilon^0 \in U_{N_\varepsilon} + H^1(\mathbb{R}^2)$ *where* U_{N_ε} *is as in* (1.22)*. Assume* v *is a solution to* (1.11) *if* $N_\varepsilon \ll |\log \varepsilon|$*, and to* (1.13) *otherwise, such that* $\mathrm{v}(0) - \langle \nabla U_1, iU_1\rangle \in L^2(\mathbb{R}^2)$*,* $\mathrm{curl}\, \mathrm{v}(0) \ge 0$*, and belonging to* $L^\infty([0,\mathrm{T}], C^{1,\gamma}(\mathbb{R}^2))$ *for some* $\gamma > 0$ *and some* $\mathrm{T} > 0$ *(possibly infinite). Letting* $\mathcal{E}_\varepsilon$ *be defined from* $\mathrm{v}(t)$ *via* (1.18)*, assume that*

$$\mathcal{E}_\varepsilon(u_\varepsilon^0, 0) \le \pi N_\varepsilon |\log \varepsilon| + o(N_\varepsilon^2). \tag{1.28}$$

Then, for every $t \in [0,\mathrm{T}]$*, we have* $\mathcal{E}_\varepsilon(u_\varepsilon(t), t) \le \pi N_\varepsilon |\log\varepsilon| + o(N_\varepsilon^2)$ *and*

$$\frac{\nabla u_\varepsilon - i u_\varepsilon N_\varepsilon \mathrm{v}}{N_\varepsilon} \to 0 \text{ strongly in } L^p_{loc}(\mathbb{R}^2) \text{ for } p<2, \tag{1.29}$$
$$\text{and weakly in } L^2(\mathbb{R}^2) \text{ if in addition } \lambda < \infty,$$

$$\frac{j_\varepsilon}{N_\varepsilon} := \frac{\langle \nabla u_\varepsilon, i u_\varepsilon\rangle}{N_\varepsilon} \to \mathrm{v} \text{ strongly in } L^1_{loc}(\mathbb{R}^2). \tag{1.30}$$

If we know in addition that u_ε *is bounded in* $L^\infty_{loc}(\mathbb{R}_+, L^\infty(\mathbb{R}^2))$ *uniformly in* ε*, then the convergence of* $j_\varepsilon/N_\varepsilon$ *is in the same sense as in* (1.29)*.*

We note that in both theorems, we obtain the convergence of the solutions of (1.10) at the level of their supercurrents j_ε, which is obviously stronger than the convergence of the vorticity $\mu_\varepsilon/N_\varepsilon = \mathrm{curl}\, j_\varepsilon/N_\varepsilon$ to $\mathrm{curl}\,\mathrm{v}$, which it implies.

The additional condition on the uniform boundedness of u_ε is easy to verify in the parabolic case: for example if the initial data satisfies $|u_\varepsilon^0| \le 1$, then this is preserved along the flow of (1.2) by the maximum principle.

The conditions (1.25) and (1.28) are well-preparedness conditions. It is fairly standard to check that one can build configurations u_ε^0 that satisfy them, for example proceeding as in [SS5, Section 7.3].

In the parabolic case, for $u_\varepsilon^0 \in U_{N_\varepsilon} + H^1(\mathbb{R}^2)$ and (1.28) to hold, the initial configuration should have most of its vortices of degree 1, and thus $\mathrm{curl}\,\mathrm{v}(0)$ must be nonnegative (it is automatically of mass 2π by the condition $\mathrm{v}(0) - \langle \nabla U_1, iU_1\rangle \in L^2(\mathbb{R}^2)$ so the assumption $\mathrm{curl}\,\mathrm{v}(0) \ge 0$ is redundant). We take advantage of these well-preparedness conditions as well as of the regularity of the solutions to the limiting equations in crucial ways. Since regularity propagates in time in these limiting equations, then the regularity assumptions we have placed really amount to just another assumption on the initial data. It is of course significantly more challenging and an open problem to prove convergence without such assumptions, in particular without knowing in the parabolic case that the initial limiting vorticity has a sign.

The reason for the restriction $N_\varepsilon \le O(|\log\varepsilon|)$ will become clear in the course of the proof: the factor of growth of the modulated energy in Gronwall's lemma is bounded by $CN_\varepsilon/|\log\varepsilon|$ and thus becomes too large otherwise. We are not even sure whether the formal analogue of (1.13), i.e. the equation with $\lambda = 0$ (shown to be locally well-posed in [Du]), is the correct limiting equation.

1.3. **Other settings.**

1.3.1. *The mixed flow case.* With our method of proof, we can prove that if $\alpha > 0$, $\beta > 0$, and $\alpha^2 + \beta^2 = 1$, the same results as Theorem 2 hold, with convergence to the limiting equation (1.11), respectively (1.14), under the additional condition $N_\varepsilon \gg \log|\log\varepsilon|$, cf. Remark 4.10. For a sketch of the proof and quantities to use, one can refer to Appendix C, setting the gauge fields to 0.

1.3.2. *The torus.* We have chosen to work in the whole plane or space, but one can easily check that the proof works with no change in the case of a torus. The proof is of course easier since there is no need for controlling infinity, and there are no boundary terms in integrations by parts. However, this gives rise to a nontrivial situation only in the Gross-Pitaevskii case, since in the parabolic case we have to assume that the vorticity has a distinguished sign, while at the same time its integral over the torus vanishes. In the parabolic case, to have a nontrivial situation one should instead consider the case with gauge as described just below in Section 1.3.4, where the total vorticity over a period does not have to vanish.

1.3.3. *Bounded domains.* On the contrary, working in a bounded domain entails significant difficulty in the parabolic case : one basically needs to control the change in energy due to the entrance and exit of vortices (one can see the occurence of this difficulty at the limiting equation level in [AS]), and the necessary tools are not yet available.

1.3.4. *The case with gauge.* Our proof adapts well to the cause with gauge, which is the true physical model for superconductors, again in the setting of the infinite plane or the torus. The corresponding evolution equations are then the so-called Gorkov-Eliashberg equations

$$\begin{cases} N_\varepsilon \left(\frac{\alpha}{|\log\varepsilon|} + i\beta\right)(\partial_t u - iu\Phi) = (\nabla_A)^2 u + \frac{u}{\varepsilon^2}(1-|u|^2) & \text{in } \mathbb{R}^2 \\ \sigma(\partial_t A - \nabla\Phi) = \nabla^\perp \operatorname{curl} A + \langle iu, \nabla_A u\rangle & \text{in } \mathbb{R}^2. \end{cases} \tag{1.31}$$

These were first derived by Schmid [Sch] and Gorkov-Eliashberg [GE], and the mixed flow equation was also suggested as a good model for the classical Hall effect by Dorsey [Do] and Kopnin et al. [KIK]. In this system the unknown functions are u, the complex-valued order parameter, A the gauge of the magnetic field (a real-valued vector field), and Φ the gauge of the electric field (a real-valued function). The notation ∇_A denotes the covariant derivative $\nabla - iA$. The magnetic field in the sample is deduced from A by $h = \operatorname{curl} A$, and the electric field by $E = -\partial_t A + \nabla\Phi$. Finally, $\sigma \geq 0$ is a real parameter, the characteristic relaxation time of the magnetic field, which may be taken to be 0. The dynamics of a finite number of vortices under such flows was established rigorously in [Sp1, Sp2, SS4, KS1, Ti, ST2], in a manner analogous to that described in the case without gauge. A dynamical law for the limit of the vorticity was formally proposed in [CRS, E2], the analogue of (1.8) mentioned above.

Natural physical quantities associated to this model are the gauge-invariant supercurrent

$$j_\varepsilon = \langle iu_\varepsilon, \nabla_{A_\varepsilon} u_\varepsilon\rangle,$$

the gauge-invariant vorticity

$$\mu_\varepsilon = \operatorname{curl}(j_\varepsilon + A_\varepsilon)$$

and the electric field

$$E_\varepsilon = -\partial_t A_\varepsilon + \nabla \Phi_\varepsilon.$$

In Appendix C we explain how to adapt the computations made in the planar case without gauge to the case with gauge, in order to derive the following limiting equations: if $\alpha = 0$ or $N_\varepsilon \ll |\log \varepsilon|$,

$$\begin{cases} \partial_t \mathrm{v} - \mathrm{E} = (-2\alpha \mathrm{v} + 2\beta \mathrm{v}^\perp)(\operatorname{curl} \mathrm{v} + \mathrm{h}) + \nabla \mathrm{p} \\ \operatorname{div} \mathrm{v} = 0 \\ -\sigma \mathrm{E} = \mathrm{v} + \nabla^\perp \mathrm{h} \\ \partial_t \mathrm{h} = -\operatorname{curl} \mathrm{E}, \end{cases} \tag{1.32}$$

and if $\alpha \neq 0$ and $\lim_{\varepsilon \to 0} \frac{|\log \varepsilon|}{N_\varepsilon} = \lambda$ is positive and finite,

$$\begin{cases} \partial_t \mathrm{v} - \mathrm{E} = (-2\alpha \mathrm{v} + 2\beta \mathrm{v}^\perp)(\operatorname{curl} \mathrm{v} + \mathrm{h}) + \frac{\lambda}{\alpha} \nabla \operatorname{div} \mathrm{v} \\ -\sigma \mathrm{E} = \mathrm{v} + \nabla^\perp \mathrm{h} \\ \partial_t \mathrm{h} = -\operatorname{curl} \mathrm{E}. \end{cases} \tag{1.33}$$

The corresponding results to Theorems 1 and 2 are then the convergences

$$\frac{j_\varepsilon}{N_\varepsilon} \to \mathrm{v}, \quad \frac{\mu_\varepsilon}{N_\varepsilon} \to \mu := \operatorname{curl} \mathrm{v} + \mathrm{h}, \quad \frac{\operatorname{curl} A_\varepsilon}{N_\varepsilon} \to \mathrm{h}, \quad \frac{E_\varepsilon}{N_\varepsilon} \to \mathrm{E} \tag{1.34}$$

in the case $\alpha = 0$ and $|\log \varepsilon| \ll N_\varepsilon \ll \frac{1}{\varepsilon}$ to (1.32), and in the case $\beta = 0$ and $1 \ll N_\varepsilon \le O(|\log \varepsilon|)$ to (1.32) or (1.33) according to the situation.

We note that if $\sigma \neq 0$, (1.32), resp. (1.33), can be rewritten as a system of equations on only two unknowns v and h :

$$\begin{cases} \partial_t \mathrm{v} + \frac{1}{\sigma}(\mathrm{v} + \nabla^\perp \mathrm{h}) = (-2\alpha \mathrm{v} + 2\beta \mathrm{v}^\perp)(\operatorname{curl} \mathrm{v} + \mathrm{h}) + \nabla p \\ \operatorname{div} \mathrm{v} = 0 \\ \sigma \partial_t \mathrm{h} = \operatorname{curl} \mathrm{v} + \Delta \mathrm{h}, \end{cases} \tag{1.35}$$

respectively

$$\begin{cases} \partial_t \mathrm{v} + \frac{1}{\sigma}(\mathrm{v} + \nabla^\perp \mathrm{h}) = (-2\alpha \mathrm{v} + 2\beta \mathrm{v}^\perp)(\operatorname{curl} \mathrm{v} + \mathrm{h}) + \frac{\lambda}{\alpha} \nabla \operatorname{div} \mathrm{v} \\ \sigma \partial_t \mathrm{h} = \operatorname{curl} \mathrm{v} + \Delta \mathrm{h}. \end{cases} \tag{1.36}$$

We can also note that taking the curl of (1.32) or (1.33) gives (with μ the limiting vorticity as in (1.34))

$$\begin{cases} \partial_t \mu = \operatorname{div}((2\alpha \mathrm{v}^\perp + 2\beta \mathrm{v})\mu) \\ \sigma \partial_t \mathrm{h} - \Delta \mathrm{h} + \mathrm{h} = \mu \\ \operatorname{div} \mathrm{v} = 0 \text{ or } \partial_t \operatorname{div} \mathrm{v} = -\frac{1}{\sigma} \operatorname{div} \mathrm{v} + \operatorname{div}((-2\alpha \mathrm{v} + 2\beta \mathrm{v}^\perp)\mu) - \frac{\lambda}{\alpha} \Delta \operatorname{div} \mathrm{v}. \end{cases}$$

This is a transport equation for μ, coupled with a linear heat equation for h. As before, when choosing $\beta = 0$, this equation coincides with the model of

[CRS, E2] when $N_\varepsilon \ll |\log \varepsilon|$, but not if N_ε is larger and λ is finite.

The rest of the paper is organized as follows: We start with some preliminaries in which we recall some properties of the solutions and a priori bounds, introduce the basic quantities like the stress-energy tensor, the velocity and the modulated energy, and present explicit computations on them.

Then we present the proofs in increasing order of complexity: we start with the proof of Theorem 1 which is remarkably short. We then present the proof of Theorem 2, which requires using all the by now standard tools of vortex analysis techniques : vortex-balls constructions, Jacobian estimates, and product estimate. An appendix is devoted to the proof of the short-time existence and uniqueness of solutions to (1.13), and another one to the computations in the gauge case.

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2. Preliminaries

In these preliminaries, we will work in the general setting of (1.10) (or its three-dimensional version) with α, β, nonnegative satisfying $\alpha^2+\beta^2=1$, which allows us to treat the Gross-Pitaevskii and parabolic (and mixed) cases at once. We note that with the choice (1.20), the limiting equations (1.11), (1.13) can always be rewritten

$$\partial_t \mathrm{v} = \nabla\phi + 2\beta \mathrm{v}^\perp \operatorname{curl} \mathrm{v} - 2\alpha \mathrm{v} \operatorname{curl} \mathrm{v}. \tag{2.1}$$

Here and in all the paper, C will denote some positive constant independent of ε, but which may depend on the various bounds on v.

Also $C^{-1+\gamma}(\mathbb{R}^n)$ denotes functions that are a sum of derivatives of $C^\gamma(\mathbb{R}^n)$ functions and $C^\infty(\mathbb{R}^n)$ bounded functions, and $C^{1+\gamma}(\mathbb{R}^n)$ is the same as $C^{1,\gamma}(\mathbb{R}^n)$, i.e. functions which are bounded and whose derivative is bounded and C^γ.

2.1. A priori bounds.

2.1.1. *A priori estimates on* v. We first gather a few facts about the solutions v to (1.11), (1.13) and (1.17) that we consider.

Lemma 2.1. *Let* v *be a solution to* (1.11) *in* $L^\infty([0,\infty],C^{0,1}(\mathbb{R}^2))$ *such that* $\mathrm{v}(0)-d\langle\nabla U_1, iU_1\rangle \in L^2(\mathbb{R}^2)$ *and* $\operatorname{curl}\mathrm{v}\in L^\infty(\mathbb{R}_+,L^1(\mathbb{R}^2))$. *There exists a* p *such that* (1.11) *holds and such that for any* $0<\mathrm{T}<\infty$, $\mathrm{v}-d\langle\nabla U_1, iU_1\rangle \in L^\infty([0,\mathrm{T}],L^2(\mathbb{R}^2))$, $\mathrm{v}\in L^\infty([0,\mathrm{T}],L^4(\mathbb{R}^2))$, $\partial_t\mathrm{v}\in L^\infty([0,\mathrm{T}],L^2(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2))$, $\mathrm{p}\in L^\infty([0,\mathrm{T}],C^{0,1}(\mathbb{R}^2)\cap L^2(\mathbb{R}^2))$, *and* $\partial_t\mathrm{p},\nabla\mathrm{p}\in L^\infty([0,\mathrm{T}],L^2(\mathbb{R}^2))$.

Let v *be a solution to* (1.17) *such that* $\mathrm{v}\in L^\infty([0,T],C^{0,1}(\mathbb{R}^3)\cap L^2(\mathbb{R}^3))$, $\partial_t\mathrm{v}\in L^\infty([0,T],L^\infty(\mathbb{R}^3)\cap L^2(\mathbb{R}^3))$ *and* $\operatorname{curl}\mathrm{v}(t)\in L^\infty([0,T],L^1(\mathbb{R}^3))$. *There exists a* p *such that* (1.17) *holds and such that for any* $0<\mathrm{T}<\infty$, $\mathrm{p}\in L^\infty([0,\mathrm{T}],C^{0,1}(\mathbb{R}^3)\cap L^2(\mathbb{R}^3))$ *and* $\partial_t\mathrm{p},\nabla\mathrm{p}\in L^\infty([0,\mathrm{T}],L^2(\mathbb{R}^3))$.

Let v *be a solution to* (1.13) *in* $L^\infty([0,\mathrm{T}],C^{1,\gamma}(\mathbb{R}^2))$, $0<\mathrm{T}<\infty$, *such that* $\mathrm{v}(0)-\langle\nabla U_1,iU_1\rangle\in L^2(\mathbb{R}^2)$. *We have* $\mathrm{v}-\langle\nabla U_1,iU_1\rangle\in L^\infty([0,\mathrm{T}],L^2(\mathbb{R}^2))$, $\mathrm{v}\in L^\infty([0,\mathrm{T}],L^4(\mathbb{R}^2))$, $\operatorname{div}\mathrm{v}\in L^\infty([0,\mathrm{T}],L^2(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2))$ *and* $\partial_t\mathrm{v},\nabla\operatorname{div}\mathrm{v}\in L^2([0,\mathrm{T}],L^2(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2))$. *Also for any* $t\in[0,\mathrm{T}]$, $\frac{1}{2\pi}\operatorname{curl}\mathrm{v}(t)$ *is a probability measure.*

Proof. Let us start with the case of (1.11). We first observe that such a solution exists, and that it corresponds to the solution belonging to the space $E_{2\pi d}$ in the notation of [Ch1, Definition 1.3.3]. It is also known (cf. [Ch1]) that the solution remains such that $\mathrm{v}(t)-d\frac{x^\perp}{|x|^2}\in L^\infty([0,\mathrm{T}],L^2(\mathbb{R}^2\backslash B(0,1)))$ and thus $\mathrm{v}(t)-\langle\nabla U_1,iU_1\rangle\in L^\infty([0,\mathrm{T}],L^2(\mathbb{R}^2)$ (the case of general α and β works similarly).

Since $\langle\nabla U_1,iU_1\rangle$ decays like $1/|x|$, by boundedness of v and the L^2 character of $\mathrm{v}-d\langle\nabla U_1,iU_1\rangle$, we deduce that $\mathrm{v}\in L^\infty(\mathbb{R}_+,L^4(\mathbb{R}^2))$.

The integrability of p is deduced from that of $|\mathrm{v}|^2$ by using the formula

$$\Delta\mathrm{p}=2\beta\left(\operatorname{div}\operatorname{div}(\mathrm{v}\otimes\mathrm{v})-\frac{1}{2}\Delta|\mathrm{v}|^2\right)+2\alpha\operatorname{div}(\operatorname{div}(\mathrm{v}\otimes\mathrm{v}))^\perp \tag{2.2}$$

which means that $\Delta\mathrm{p}$ is a second order derivative of $\mathrm{v}\otimes\mathrm{v}$. Since $\mathrm{v}\in L^4(\mathbb{R}^2)\cap C^{0,1}(\mathbb{R}^2)$, this allows us to pick a pressure p such that $\mathrm{p}\in L^\infty\cap L^2$ (cf. [Ch1, p.13]) and it is also $C^{0,1}$ by the assumed regularity of v. We also note that $\mathrm{v}\operatorname{curl}\mathrm{v}\in L^2(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$ uniformly in time by boundedness of v, curl v, and the fact that curl v is integrable uniformly in time. On the other hand, we may write

$$\partial_t\mathrm{v}=\nabla^\perp\Delta^{-1}\partial_t\operatorname{curl}\mathrm{v}=\nabla^\perp\Delta^{-1}\operatorname{div}(-2\beta\mathrm{v}\operatorname{curl}\mathrm{v}+2\alpha\mathrm{v}^\perp\operatorname{curl}\mathrm{v})$$

where Δ^{-1} is the convolution with $-\frac{1}{2\pi}\log$, and with the above remark, we may deduce that $\partial_t\mathrm{v}$ remains in $L^2(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$ uniformly in time, and the result for $\nabla\mathrm{p}$ follows by using the equation. The same result follows for $\partial_t\mathrm{p}$ by applying $\partial_t\Delta^{-1}$ to (2.2).

The same arguments apply to prove the results stated for (1.17).

Let us now turn to (1.13). It is proved in Theorem 3 that $\mathrm{v}(t)-\mathrm{v}(0)\in L^\infty([0,\mathrm{T}],L^2(\mathbb{R}^2))$, $\partial_t\mathrm{v}\in L^2([0,\mathrm{T}],L^2(\mathbb{R}^2))$, and $\operatorname{div}\mathrm{v}\in L^\infty([0,\mathrm{T}],L^2(\mathbb{R}^2))$. It also follows immediately that $\mathrm{v}(t)-\langle\nabla U_1,iU_1\rangle\in L^\infty([0,\mathrm{T}],L^2(\mathbb{R}^2))$ and the uniform L^4 character of v follows just as in the Euler case above. The fact that $\nabla\operatorname{div}\mathrm{v}\in L^2([0,\mathrm{T}],L^2(\mathbb{R}^2))$ follows in view of (1.13). The fact that $\frac{1}{2\pi}\operatorname{curl}\mathrm{v}$ remains a probability measure is standard by integrating the equation.

Next, differentiating (1.13) we find that

$$\partial_t(\nabla\operatorname{div}\mathrm{v})=\lambda\Delta(\nabla\operatorname{div}\mathrm{v})-2\nabla\operatorname{div}(\mathrm{v}\operatorname{curl}\mathrm{v}). \tag{2.3}$$

It can be found in [Ch2, Proposition 2.1] that if u solves on $\mathbb{R}^2$ the equation $\partial_t u-\Delta u=f$ on $[0,T]$ with $f\in L^\infty([0,T],C^{-2,\gamma}(\mathbb{R}^2))$ and initial data $u_0\in C^{0,\gamma}(\mathbb{R}^2)$ then

$$\|u\|_{L^\infty([0,T],C^\gamma(\mathbb{R}^2))}\le C_T\left(\|u_0\|_{C^\gamma(\mathbb{R}^2)}+\|f\|_{L^\infty([0,T],C^{-2+\gamma}(\mathbb{R}^2))}\right) \tag{2.4}$$

(it suffices to apply the result there with $\rho=p=\infty$ and $s=-2+\gamma$ and notice that the Besov space B^s_∞ is the same as C^s or $C^{0,s}$). Applying this to (2.3), since the right-hand side is $L^\infty([0,\mathrm{T}],C^{-2+\gamma}(\mathbb{R}^2))$ we obtain that $\nabla\operatorname{div}\mathrm{v}\in$

$L^\infty([0,\mathrm{T}],C^\gamma(\mathbb{R}^2))$ hence $L^\infty([0,\mathrm{T}],L^\infty(\mathbb{R}^2))$. Inserting into (1.13) and using that $\operatorname{curl}\mathrm{v}\in L^1\cap L^\infty$ yields that $\partial_t\mathrm{v}\in L^\infty([0,\mathrm{T}],L^\infty(\mathbb{R}^2))$.

□

2.1.2. *Estimates on the solutions to* (1.10).

Lemma 2.2. *Assume u_ε and $\mathrm{v}(t)$ satisfy the assumptions of Theorem 1 or 2. Let $U_{D_\varepsilon}=1$ in the three-dimensional case of Theorem 1 and $D_\varepsilon=N_\varepsilon$ in the case of Theorem 2. Then for any $\mathrm{T}>0$ we have $\partial_t u_\varepsilon\in L^1([0,\mathrm{T}],L^2(\mathbb{R}^n))$ and for any $t\in[0,\mathrm{T}]$,*

$$\nabla(u_\varepsilon(t)-U_{D_\varepsilon}),1-|u_\varepsilon(t)|^2,\nabla u_\varepsilon(t)-iu_\varepsilon(t)N_\varepsilon\mathrm{v}(t)$$

all belong to $L^2(\mathbb{R}^n)$, $\mathcal{E}_\varepsilon(u_\varepsilon(t),t)$ is finite, and

$$j_\varepsilon(t)-N_\varepsilon\mathrm{v}(t)\in(L^1+L^2)(\mathbb{R}^n).$$

Proof. First let us justify that $\partial_t u_\varepsilon(t)\in L^1([0,\mathrm{T}],L^2(\mathbb{R}^n))$. In the two-dimensional Gross-Pitaevskii case, since we assume $u_\varepsilon^0\in U_{D_\varepsilon}+H^2(\mathbb{R}^2)$, then studying the equation for $w_\varepsilon:=u_\varepsilon-U_{D_\varepsilon}$, we find in [BS, Prop. 3, Lemma 3] that w_ε remains in $H^2(\mathbb{R}^2)$ (by semi-group theory) and thus $\partial_t u_\varepsilon=\partial_t w_\varepsilon\in L^\infty_{loc}(\mathbb{R},L^2(\mathbb{R}^2))$ by the equation.

For the two-dimensional parabolic (or mixed flow case), we assume $u_\varepsilon^0\in U_{D_\varepsilon}+H^1(\mathbb{R}^2)$ and (by decay of the energy for $w_\varepsilon:=u_\varepsilon-U_{D_\varepsilon}$) it shown in [Mi, Theorem 1] that $\partial_t u_\varepsilon\in L^1([0,\mathrm{T}],L^2(\mathbb{R}^2))$.

In the case of the three-dimensional solutions to (1.3), it is shown in [Ge, Sec. 3.3] that when $\Delta u_\varepsilon^0\in L^2(\mathbb{R}^3)$ then the solution belongs for all time to $X^2:=\{u\in L^\infty(\mathbb{R}^3),D^2u\in L^2(\mathbb{R}^3)\}$ and thus in view of the equation $\partial_t u_\varepsilon\in L^\infty_{loc}(\mathbb{R},L^2(\mathbb{R}^3))$.

Let us turn to the two-dimensional cases. Following exactly [BS, Proposition 3] or [Mi, Lemma A.6], we get that

$$\frac{1}{2}\int_{\mathbb{R}^2}|\nabla(u_\varepsilon(t)-U_{D_\varepsilon})|^2+|u_\varepsilon(t)-U_{D_\varepsilon}|^2+(1-|u_\varepsilon(t)|^2)^2\le C(\varepsilon,t)$$

where $C(\varepsilon,t)$ is finite and depends on ε, t and U_{D_ε}. We thus obtain the L^2 character of the first three items.

By Lemma 2.1 and $D_\varepsilon=dN_\varepsilon$ (resp. $D_\varepsilon=N_\varepsilon$), we have that $N_\varepsilon\mathrm{v}(t)-\langle\nabla U_{D_\varepsilon},iU_{D_\varepsilon}\rangle\in L^2(\mathbb{R}^2)$. We may then write

$$|\nabla u_\varepsilon-iu_\varepsilon N_\varepsilon\mathrm{v}|\le|\nabla(u_\varepsilon-U_{D_\varepsilon})|+N_\varepsilon|\mathrm{v}||u_\varepsilon-U_{D_\varepsilon}|+|\nabla U_{D_\varepsilon}-iU_{D_\varepsilon}N_\varepsilon\mathrm{v}|.$$

The first two expressions in the right-hand side are in $L^2(\mathbb{R}^2)$ by what precedes and the boundedness of v. For the third quantity, we have that $|\nabla U_{D_\varepsilon}-iU_{D_\varepsilon}N_\varepsilon\mathrm{v}|=|\langle\nabla U_{D_\varepsilon},iU_{D_\varepsilon}\rangle-N_\varepsilon\mathrm{v}|$ outside of $B(0,1)$ and is bounded in $B(0,1)$, by definition of U_D, hence is in L^2 by Lemma 2.1. We conclude by Lemma 2.1 that this term is also in $L^2(\mathbb{R}^2)$. The finiteness of $\mathcal{E}_\varepsilon(u_\varepsilon(t),t)$ is then an immediate consequence of what precedes, the fact that $\mathrm{v}\in L^4$ from Lemma 2.1, and the Cauchy-Schwarz inequality.

Lastly, for $j_\varepsilon-N_\varepsilon\mathrm{v}$ we write

$$|j_\varepsilon-N_\varepsilon\mathrm{v}|\le|\langle\nabla U_{D_\varepsilon},iU_{D_\varepsilon}\rangle-N_\varepsilon\mathrm{v}|+|\nabla(U_{D_\varepsilon}-u_\varepsilon)||u_\varepsilon|+|\nabla U_{D_\varepsilon}||u_\varepsilon-U_{D_\varepsilon}|$$

and we conclude by the previous observations, writing $|u_\varepsilon| = 1 + (|u_\varepsilon| - 1)$ and using that $(1-|u_\varepsilon|)^2 \le (1-|u_\varepsilon|^2)^2$, that $j_\varepsilon - N_\varepsilon \mathrm{v} \in L^1 + L^2(\mathbb{R}^n)$.

□

2.1.3. *Coerciveness of the modulated energy.* We check that the modulated energy $\mathcal{E}_\varepsilon$ does control the quantities we are interested in. We have

Lemma 2.3. *The functional* $\mathcal{E}_\varepsilon$ *being as in* (1.18) *and* ψ *as in* (1.19), *we have for any* u_ε, $R \ge 1$, $1 < p < \infty$,

$$\frac{1}{2}\int_{\mathbb{R}^n} |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}|^2 + \frac{(1-|u_\varepsilon|^2)^2}{4\varepsilon^2} \le \mathcal{E}_\varepsilon(u_\varepsilon) + \varepsilon^2 N_\varepsilon^4 \|\psi\|_{L^2}^2, \tag{2.5}$$

$$\int_{B_R} |j_\varepsilon - N_\varepsilon \mathrm{v}| \le C_{R,p} \|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}\|_{L^p(B_R)} \tag{2.6}$$
$$+ C_{R,p}(\varepsilon^2 N_\varepsilon^2 \|\psi\|_{L^2} + N_\varepsilon \varepsilon \|\mathrm{v}\|_{L^\infty}) \mathcal{E}_\varepsilon(u_\varepsilon)^{\frac{1}{2}} + C\varepsilon \mathcal{E}_\varepsilon(u_\varepsilon) + C_R \varepsilon^2 N_\varepsilon^3 \|\mathrm{v}\|_{L^\infty} \|\psi\|_{L^2},$$

and

$$\int_{\mathbb{R}^n} |j_\varepsilon - N_\varepsilon \mathrm{v}|^2 \le C(\|u_\varepsilon\|_{L^\infty}^2 + \varepsilon^2 N_\varepsilon^2 \|\mathrm{v}\|_{L^\infty}^2)(\mathcal{E}_\varepsilon(u_\varepsilon) + \varepsilon^2 N_\varepsilon^4 \|\psi\|_{L^2}^2), \tag{2.7}$$

where C *is universal,* C_R *depends only on* R *and* $C_{R,p}$ *on* R *and* p.

In view of assumptions (1.23) and (1.27) and Lemma 2.1, the second term on the right-hand sides of (2.5) and (2.7) will always be bounded by $o(N_\varepsilon^2)$. Also, if $\|u_\varepsilon\|_{L^\infty} \le C$, the upper bound in (2.7) is by $C\mathcal{E}_\varepsilon(u_\varepsilon) + o(1)$ and that in (2.6) is by $C_{R,p}\mathcal{E}_\varepsilon(u_\varepsilon)^{\frac{1}{2}} + C\varepsilon\mathcal{E}_\varepsilon(u_\varepsilon)$.

Proof. We observe that

$$\frac{(1-|u_\varepsilon|^2)^2}{2\varepsilon^2} + N_\varepsilon^2 \psi (1-|u_\varepsilon|^2) = \frac{(1-|u_\varepsilon|^2 + \varepsilon^2 N_\varepsilon^2 \psi)^2}{2\varepsilon^2} - \varepsilon^2 N_\varepsilon^4 \psi^2 \ge -\varepsilon^2 N_\varepsilon^4 \psi^2.$$

Thus, using that $\psi \in L^2$ by Lemma 2.1, we have

$$\frac{1}{2}\int_{\mathbb{R}^n} |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}(t)|^2 + \frac{(1-|u_\varepsilon|^2)^2}{4\varepsilon^2} \le \mathcal{E}_\varepsilon(u_\varepsilon) + \frac{1}{2}\int_{\mathbb{R}^n} \varepsilon^2 N_\varepsilon^4 \psi^2.$$

For (2.7), we write that

(2.8)
$$\begin{aligned}|j_\varepsilon - N_\varepsilon \mathrm{v}| \le |j_\varepsilon - |u_\varepsilon|^2 N_\varepsilon \mathrm{v}| + N_\varepsilon |1-|u_\varepsilon|^2||\mathrm{v}| &= |\langle iu_\varepsilon, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}\rangle| + N_\varepsilon |1-|u_\varepsilon|^2||\mathrm{v}| \\ &\le |u_\varepsilon||\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}| + N_\varepsilon|1-|u_\varepsilon|^2||\mathrm{v}| \\ &= |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}| + (|u_\varepsilon|-1)|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}| + N_\varepsilon |1-|u_\varepsilon|^2||\mathrm{v}|.\end{aligned}$$

For (2.6) we integrate this relation over B_R and use Hölder's inequality to get that for any $1 < p < \infty$,

$$\int_{B_R} |j_\varepsilon - N_\varepsilon \mathrm{v}| \le C_{R,p} \left(\int_{B_R} |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}|^p\right)^{\frac{1}{p}}$$
$$+ \left(\int_{B_R} (1-|u_\varepsilon|)^2\right)^{\frac{1}{2}} \left(\int_{B_R} |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}|^2\right)^{\frac{1}{2}} + C_R \|\mathrm{v}\|_{L^\infty} N_\varepsilon \left(\int_{B_R} (1-|u_\varepsilon|^2)^2\right)^{\frac{1}{2}}$$

and using that $(1-|u|)^2 \le (1-|u|^2)^2$ we are led to

$$\int_{B_R} |j_\varepsilon - N_\varepsilon \mathrm{v}| \le C_{R,p} \|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}\|_{L^p(B_R)} + C\varepsilon(\mathcal{E}_\varepsilon(u_\varepsilon) + N_\varepsilon)\left(\mathcal{E}_\varepsilon(u_\varepsilon) + \varepsilon^2 N_\varepsilon^4 \|\psi\|_{L^2}^2\right)^{\frac{1}{2}}$$

hence (2.6) follows. The proof of (2.7) is a straightforward consequence of (2.8). ☐

2.2. **Identities.** In this section, we present important standard and less standard identities that will be used throughout the paper. In all that follows v is a vector field, which implicitly depends on time and solves one of our limiting equations.

2.2.1. *Current and velocity.* We recall that for a family $\{u_\varepsilon\}_\varepsilon$, the supercurrent and vorticity (or Jacobian) are defined as

$$j_\varepsilon = \langle \nabla u_\varepsilon, iu_\varepsilon \rangle \qquad \mu_\varepsilon = \operatorname{curl} j_\varepsilon.$$

Following [SS3], we also define the velocity

$$V_\varepsilon := 2\langle i\partial_t u_\varepsilon, \nabla u_\varepsilon \rangle \tag{2.9}$$

and we have the identity

$$\partial_t j_\varepsilon = \nabla \langle iu_\varepsilon, \partial_t u_\varepsilon \rangle + V_\varepsilon. \tag{2.10}$$

Taking the curl of this relation yields that $\partial_t \mu_\varepsilon = \operatorname{curl} V_\varepsilon$. (In dimension 2, this means that the vorticity is transported by $V_\varepsilon^\perp$, hence the name velocity). We also define the modulated vorticity

$$\tilde{\mu}_\varepsilon := \operatorname{curl}\left(\langle \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}, iu_\varepsilon \rangle + N_\varepsilon \mathrm{v}\right), \tag{2.11}$$

and the modulated velocity

$$\tilde{V}_\varepsilon := 2\langle i(\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \phi), \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}\rangle = V_\varepsilon - N_\varepsilon \mathrm{v}\partial_t |u_\varepsilon|^2 + N_\varepsilon \phi \nabla |u_\varepsilon|^2, \tag{2.12}$$

with ϕ as in (1.20).

We will use the fact that for u_ε solution of (1.10) (resp. (1.24)) we have the relation

$$\operatorname{div} j_\varepsilon = N_\varepsilon \langle (\frac{\alpha}{|\log \varepsilon|} + i\beta)\partial_t u_\varepsilon, iu_\varepsilon \rangle, \tag{2.13}$$

(resp. with $\alpha = 0$ and $\beta = 1$) which is obtained by taking the inner product of (1.10) or (1.24) with iu_ε.

2.2.2. *Stress-energy tensor.* We next introduce the stress-energy tensor associated to a function u: it is the $n \times n$ tensor defined by

$$(S_\varepsilon(u))_{kl} := \langle \partial_k u, \partial_l u \rangle - \frac{1}{2}\left(|\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2\right)\delta_{kl}. \tag{2.14}$$

A direct computation shows that if u is sufficiently regular, we have

$$\operatorname{div} S_\varepsilon(u) := \sum_{l=1}^n \partial_l (S_\varepsilon(u))_{kl} = \langle \nabla u, \Delta u + \frac{1}{\varepsilon^2} u(1-|u|^2)\rangle$$

so if u_ε solves (1.10) or (1.24), we have

$$\operatorname{div} S_\varepsilon(u_\varepsilon) = N_\varepsilon\langle(\frac{\alpha}{|\log\varepsilon|} + i\beta)\partial_t u_\varepsilon, \nabla u_\varepsilon\rangle. \tag{2.15}$$

We next introduce the modulated stress-energy tensor :

$$(\tilde S_\varepsilon(u))_{kl} := \langle \partial_k u - iuN_\varepsilon \mathrm{v}_k, \partial_l u - iuN_\varepsilon \mathrm{v}_l\rangle + N_\varepsilon^2(1-|u|^2)\mathrm{v}_k\mathrm{v}_l \\ -\frac{1}{2}\left(|\nabla u - iuN_\varepsilon \mathrm{v}|^2 + (1-|u|^2)N_\varepsilon^2|\mathrm{v}|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2\right)\delta_{kl}, \tag{2.16}$$

where δ_{kl} is 1 if $k=l$ and 0 otherwise. One can observe that

$$|\tilde S_\varepsilon(u)| \le |\nabla u - iuN_\varepsilon \mathrm{v}|^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2 + N_\varepsilon^2|1-|u|^2||\mathrm{v}|^2 \tag{2.17}$$

and thus with Lemma 2.3 and the Cauchy-Schwarz inequality, we may write

$$\int_{\mathbb{R}^n} |\tilde S_\varepsilon(u)| \le 2\mathcal{E}_\varepsilon(u) + N_\varepsilon^2\|1-|u|^2\|_{L^2}\|\mathrm{v}\|_{L^4}^2 + 2\varepsilon^2 N_\varepsilon^4\|\psi\|_{L^2}^2. \tag{2.18}$$

For simplicity, we will also denote S_ε for $S_\varepsilon(u_\varepsilon)$ and $\tilde S_\varepsilon$ for $\tilde S_\varepsilon(u_\varepsilon)$, as well as

$$S_\mathrm{v} := \mathrm{v}\otimes\mathrm{v} - \frac{1}{2}|\mathrm{v}|^2 I. \tag{2.19}$$

Lemma 2.4. *Let u_ε solve* (1.10) *or* (1.24) *and* v *and ϕ be as above. Then we have*

$$\operatorname{div}\tilde S_\varepsilon(u_\varepsilon) = \frac{N_\varepsilon\alpha}{|\log\varepsilon|}\langle\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\phi, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}\rangle + \frac{\beta}{2}N_\varepsilon V_\varepsilon - \beta N_\varepsilon^2\mathrm{v}\langle\partial_t u_\varepsilon, u_\varepsilon\rangle \\ - N_\varepsilon^2\frac{N_\varepsilon\alpha}{|\log\varepsilon|}|u_\varepsilon|^2\mathrm{v}\phi + N_\varepsilon j_\varepsilon(\frac{N_\varepsilon\alpha}{|\log\varepsilon|}\phi - \operatorname{div}\mathrm{v}) \\ + N_\varepsilon^2\operatorname{div} S_\mathrm{v} - N_\varepsilon\left((\mathrm{v}\cdot\nabla)j_\varepsilon + (j_\varepsilon\cdot\nabla)\mathrm{v} - \nabla(j_\varepsilon\cdot\mathrm{v})\right), \tag{2.20}$$

which in dimension $n=2$ can be rewritten

$$\operatorname{div}\tilde S_\varepsilon(u_\varepsilon) = \frac{N_\varepsilon\alpha}{|\log\varepsilon|}\langle\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\phi, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}\rangle + \frac{\beta}{2}N_\varepsilon V_\varepsilon - \beta N_\varepsilon^2\mathrm{v}\langle\partial_t u_\varepsilon, u_\varepsilon\rangle \\ + N_\varepsilon^2\mathrm{v}^\perp\operatorname{curl}\mathrm{v} - N_\varepsilon j_\varepsilon^\perp\operatorname{curl}\mathrm{v} - N_\varepsilon\mathrm{v}^\perp\mu_\varepsilon \\ + N_\varepsilon^2\mathrm{v}(\operatorname{div}\mathrm{v} - \frac{N_\varepsilon\alpha}{|\log\varepsilon|}|u_\varepsilon|^2\phi) + N_\varepsilon j_\varepsilon(\frac{N_\varepsilon\alpha}{|\log\varepsilon|}\phi - \operatorname{div}\mathrm{v}). \tag{2.21}$$

Proof. First, a direct computation yields

$$\tilde S_\varepsilon(u_\varepsilon) = S_\varepsilon(u_\varepsilon) + N_\varepsilon^2 S_\mathrm{v} - N_\varepsilon(\mathrm{v}\otimes j_\varepsilon + j_\varepsilon\otimes\mathrm{v} - (j_\varepsilon\cdot\mathrm{v})I).$$

Since we have the following relations for general vector fields v and j:

$$\operatorname{div}(\mathrm{v}\otimes j) = j\operatorname{div}\mathrm{v} + (\mathrm{v}\cdot\nabla)j, \tag{2.22}$$

we deduce that

$$\operatorname{div}\tilde S_\varepsilon(u_\varepsilon) = \operatorname{div} S_\varepsilon(u_\varepsilon) + N_\varepsilon^2\operatorname{div} S_\mathrm{v} \\ - N_\varepsilon\left(j_\varepsilon\operatorname{div}\mathrm{v} + \mathrm{v}\operatorname{div} j_\varepsilon + (\mathrm{v}\cdot\nabla)j_\varepsilon + (j_\varepsilon\cdot\nabla)\mathrm{v} - \nabla(j_\varepsilon\cdot\mathrm{v})\right). \tag{2.23}$$

On the other hand, writing $\partial_t u_\varepsilon = \partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \phi + iu_\varepsilon N_\varepsilon \phi$ and $\nabla u_\varepsilon = \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v} + iu_\varepsilon N_\varepsilon \mathrm{v}$ yields

$$\begin{aligned}\langle \partial_t u_\varepsilon, \nabla u_\varepsilon\rangle = \langle \partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\phi, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}\rangle + N_\varepsilon j_\varepsilon \phi \\ + N_\varepsilon \mathrm{v}\langle \partial_t u_\varepsilon, iu_\varepsilon\rangle - N_\varepsilon^2 |u_\varepsilon|^2 \mathrm{v}\phi\end{aligned}$$

and combining with (2.9), (2.13) and (2.15), we find

$$\begin{aligned}\operatorname{div} S_\varepsilon(u_\varepsilon) = \frac{N_\varepsilon \alpha}{|\log\varepsilon|}\Big(\langle \partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\phi, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}\rangle + N_\varepsilon j_\varepsilon\phi - N_\varepsilon^2|u_\varepsilon|^2\mathrm{v}\phi\Big) \\ + N_\varepsilon \mathrm{v}\operatorname{div} j_\varepsilon - \beta N_\varepsilon^2 \mathrm{v}\langle\partial_t u_\varepsilon, u_\varepsilon\rangle + N_\varepsilon \frac{\beta}{2} V_\varepsilon.\end{aligned}$$

Inserting into (2.23) yields (2.20). In the two-dimensional case, we notice that we have the identities

$$\operatorname{div} S_{\mathrm{v}} = \mathrm{v}\operatorname{div}\mathrm{v} + \mathrm{v}^\perp \operatorname{curl}\mathrm{v}$$

and

$$(\mathrm{v}\cdot\nabla)j + (j\cdot\nabla)\mathrm{v} - \nabla(j\cdot\mathrm{v}) = j^\perp\operatorname{curl}\mathrm{v} + \mathrm{v}^\perp\operatorname{curl} j,$$

so (2.21) follows. ☐

2.2.3. *Time derivative of the energy.* Given a Lipschitz compactly supported function $\chi(x)$, and a sufficiently regular function $\psi(x,t)$ let us define

(2.24)
$$\hat{\mathcal{E}}_\varepsilon(u,t) = \frac{1}{2}\int_{\mathbb{R}^n}\chi\left(|\nabla u - iuN_\varepsilon \mathrm{v}(t)|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2} + N_\varepsilon^2(1-|u|^2)\psi(x,t)\right).$$

For simplicity we will most often write $\hat{\mathcal{E}}_\varepsilon(u_\varepsilon)$ for $\hat{\mathcal{E}}_\varepsilon(u_\varepsilon(t),t)$.

Lemma 2.5. *Let u_ε solve* (1.10) *or* (1.24) *and* v *satisfy the results of Lemma 2.1. Then we have*

(2.25)
$$\begin{aligned}\frac{d}{dt}\hat{\mathcal{E}}_\varepsilon(u_\varepsilon) = &-\int_{\mathbb{R}^n}\chi\frac{N_\varepsilon\alpha}{|\log\varepsilon|}|\partial_t u_\varepsilon|^2 + \nabla\chi\cdot\langle\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}, \partial_t u_\varepsilon\rangle \\ &+ \int_{\mathbb{R}^n}\chi\left(N_\varepsilon^2\mathrm{v}\cdot\partial_t\mathrm{v} - N_\varepsilon j_\varepsilon\cdot\partial_t\mathrm{v} + N_\varepsilon\langle\partial_t u_\varepsilon, iu_\varepsilon\rangle\operatorname{div}\mathrm{v} - N_\varepsilon V_\varepsilon\cdot\mathrm{v}\right) \\ &+ \frac{1}{2}\int_{\mathbb{R}^n}\chi N_\varepsilon^2\partial_t\left((1-|u_\varepsilon|^2)(\psi - |\mathrm{v}|^2)\right).\end{aligned}$$

Proof. Since the solution u_ε is smooth and χ is compactly supported, expanding the square, we may first rewrite $\hat{\mathcal{E}}_\varepsilon$ as

$$\begin{aligned}\hat{\mathcal{E}}_\varepsilon(u_\varepsilon) = &\frac{1}{2}\int_{\mathbb{R}^n}\chi\left(|\nabla u_\varepsilon|^2 + \frac{(1-|u_\varepsilon|^2)^2}{2\varepsilon^2}\right) \\ &+ \frac{1}{2}\int_{\mathbb{R}^n}\chi\left(N_\varepsilon^2|\mathrm{v}|^2 + N_\varepsilon^2(1-|u_\varepsilon|^2)(\psi-|\mathrm{v}|^2)\right) - \int_{\mathbb{R}^n}\chi N_\varepsilon j_\varepsilon\cdot\mathrm{v}.\end{aligned} \tag{2.26}$$

We then differentiate in time and obtain

$$\frac{d}{dt}\hat{\mathcal{E}}_\varepsilon(u_\varepsilon) = -\int_{\mathbb{R}^n} \chi(\langle \partial_t u_\varepsilon, \Delta u_\varepsilon + \frac{1}{\varepsilon^2} u_\varepsilon(1-|u_\varepsilon|^2)\rangle + \nabla\chi\cdot\langle \nabla u_\varepsilon, \partial_t u_\varepsilon\rangle$$
$$+\int_{\mathbb{R}^n} \chi(N_\varepsilon^2 \mathrm{v}\cdot\partial_t \mathrm{v} - N_\varepsilon j_\varepsilon\cdot\partial_t\mathrm{v} - N_\varepsilon \mathrm{v}\cdot\partial_t j_\varepsilon) + \frac{1}{2}\int_{\mathbb{R}^n} \chi N_\varepsilon^2 \partial_t \left((1-|u_\varepsilon|^2)(\psi - |\mathrm{v}|^2)\right).$$

Inserting (1.10) and (2.10) and also writing $\nabla u_\varepsilon = \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v} + iu_\varepsilon N_\varepsilon \mathrm{v}$, this becomes

$$\begin{aligned}\partial_t \hat{\mathcal{E}}_\varepsilon(u_\varepsilon) = &-\int_{\mathbb{R}^n} \chi \frac{N_\varepsilon \alpha}{|\log\varepsilon|}|\partial_t u_\varepsilon|^2 + \nabla\chi\cdot(\langle \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}, \partial_t u_\varepsilon\rangle + N_\varepsilon \mathrm{v}\langle \partial_t u_\varepsilon, iu_\varepsilon\rangle)\\ &+\int_{\mathbb{R}^n} \chi\left(N_\varepsilon^2\mathrm{v}\cdot\partial_t\mathrm{v} - N_\varepsilon j_\varepsilon\cdot\partial_t \mathrm{v} - N_\varepsilon\mathrm{v}\cdot\nabla\langle\partial_t u_\varepsilon, iu_\varepsilon\rangle - N_\varepsilon V_\varepsilon\cdot\mathrm{v}\right)\\ &+\frac{1}{2}\int_{\mathbb{R}^n}\chi N_\varepsilon^2 \partial_t\left((1-|u_\varepsilon|^2)(\psi-|\mathrm{v}|^2)\right).\end{aligned}$$

With an integration by parts, we find that two terms simplify and we obtain the result. □

3. The Gross-Pitaevskii case : proof of Theorem 1

In this section, we consider the Gross-Pitaevskii cases, in which v solves (1.11) with $\alpha = 0$ and $\beta = 1$ or (1.17), and for which $\operatorname{div}\mathrm{v} = 0$, $\phi = \mathrm{p}$ and $\psi = \mathrm{p} - |\mathrm{v}|^2$. Below, we apply the result of Lemma 2.5 with these choices. First, we insert the equation solved by v and the result of Lemma 2.4 to obtain the crucial step in our proof, where all the algebra combines. We note that in view of (1.16), (1.11) and (1.17) can both be written as

$$\partial_t \mathrm{v} = 2\operatorname{div} S_\mathrm{v} + \nabla \mathrm{p} \tag{3.1}$$

with the notation (2.19).

Lemma 3.1. *Let* u_ε *solve* (1.24) *and* v *solve* (1.11) *or* (1.17) *according to the dimension. Then we have*

$$\begin{aligned}\partial_t\hat{\mathcal{E}}_\varepsilon(u_\varepsilon) = &\int_{\mathbb{R}^n}\chi(2\tilde{S}_\varepsilon(u_\varepsilon):\nabla\mathrm{v} - N_\varepsilon(1-|u_\varepsilon|^2)\partial_t(|\mathrm{v}|^2 - \frac{\mathrm{p}}{2})\\ &-\int_{\mathbb{R}^n}\nabla\chi\cdot\Big(\langle\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}, \partial_t u_\varepsilon\rangle + N_\varepsilon(N_\varepsilon\mathrm{v} - j_\varepsilon)\mathrm{p} - 2\tilde{S}_\varepsilon\mathrm{v}\Big),\end{aligned} \tag{3.2}$$

where for two 2×2 *matrices* A *and* B, $A:B$ *denotes* $\sum_{kl} A_{kl}B_{kl}$.

Proof. Starting from the result of Lemma 2.5, the first step is to insert (3.1), which yields

$$\int_{\mathbb{R}^n}\chi\left(N_\varepsilon^2\mathrm{v}\cdot\partial_t\mathrm{v} - N_\varepsilon j_\varepsilon\cdot\partial_t\mathrm{v}\right) = \int_{\mathbb{R}^n}\chi N_\varepsilon(N_\varepsilon\mathrm{v} - j_\varepsilon)\cdot(2\operatorname{div} S_\mathrm{v} + \nabla\mathrm{p}). \tag{3.3}$$

In order to transform the (first order in the error) term $(N_\varepsilon \mathrm{v} - j_\varepsilon)\cdot \operatorname{div} S_{\mathrm{v}}$ into a quadratic term, we multiply the result of Lemma 2.4 by $2\chi\mathrm{v}$, which yields

$$(3.4)\quad -\int_{\mathbb{R}^n} \chi N_\varepsilon V_\varepsilon\cdot \mathrm{v} = -2\int_{\mathbb{R}^n}\chi \operatorname{div}\tilde S_\varepsilon(u_\varepsilon)\cdot \mathrm{v} + 2\int_{\mathbb{R}^n}\chi N_\varepsilon^2 \operatorname{div} S_{\mathrm{v}}\cdot \mathrm{v}$$
$$- 2N_\varepsilon \int_{\mathbb{R}^n}\chi\left((\mathrm{v}\cdot\nabla)j_\varepsilon + (j_\varepsilon\cdot\nabla)\mathrm{v} - \nabla(j_\varepsilon\cdot \mathrm{v})\right)\cdot \mathrm{v} - \int_{\mathbb{R}^n}\chi N_\varepsilon^2|\mathrm{v}|^2\partial_t|u_\varepsilon|^2.$$

But by direct computation, one may check that for any vector fields v and j we have

$$(3.5)\qquad ((\mathrm{v}\cdot\nabla)j + (j\cdot\nabla)\mathrm{v} - \nabla(j\cdot \mathrm{v}))\cdot \mathrm{v} = -\operatorname{div} S_{\mathrm{v}}\cdot j$$

and applying to v and v, we easily deduce that

$$\operatorname{div} S_{\mathrm{v}}\cdot \mathrm{v} = 0.$$

Therefore, inserting (3.5) applied with v and j_ε, (3.3) and (3.4) into the result of Lemma 2.5, and noticing several cancellations, we obtain

$$\frac{d}{dt}\hat{\mathcal{E}}_\varepsilon(u_\varepsilon) = -\int_{\mathbb{R}^n}\nabla\chi\cdot\langle\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}, \partial_t u_\varepsilon\rangle$$
$$+\int_{\mathbb{R}^n}\chi(N_\varepsilon(N_\varepsilon \mathrm{v} - j_\varepsilon)\cdot\nabla \mathrm{p} - 2\operatorname{div}\tilde S_\varepsilon(u_\varepsilon)\cdot \mathrm{v})$$
$$+\int_{\mathbb{R}^n}\chi\left(N_\varepsilon^2(|\mathrm{v}|^2\partial_t(1-|u_\varepsilon|^2) + \frac12 N_\varepsilon^2\partial_t\left((1-|u_\varepsilon|^2)(\psi - |\mathrm{v}|^2)\right)\right).$$

Integrating by parts and using (2.13), we have

$$\int_{\mathbb{R}^n}\chi N_\varepsilon(N_\varepsilon \mathrm{v} - j_\varepsilon)\cdot\nabla \mathrm{p} = -\int_{\mathbb{R}^n} N_\varepsilon\nabla\chi\cdot(N_\varepsilon \mathrm{v} - j_\varepsilon)\mathrm{p} + \int_{\mathbb{R}^n} N_\varepsilon^2\chi\langle\partial_t u_\varepsilon, u_\varepsilon\rangle \mathrm{p}.$$

Inserting into the previous relation and collecting terms, we are led to

$$\frac{d}{dt}\hat{\mathcal{E}}_\varepsilon(u_\varepsilon) = -\int_{\mathbb{R}^n}\nabla\chi\cdot\left(\langle\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}, \partial_t u_\varepsilon\rangle + N_\varepsilon(N_\varepsilon \mathrm{v} - j_\varepsilon)\mathrm{p}\right)$$
$$-2\int_{\mathbb{R}^n}\chi\operatorname{div}\tilde S_\varepsilon(u_\varepsilon)\cdot \mathrm{v}$$
$$+\int_{\mathbb{R}^n}\chi N_\varepsilon^2\left(\partial_t(1-|u_\varepsilon|^2)\left(|\mathrm{v}|^2 - \frac12 \mathrm{p}\right) + \frac12\partial_t\left((1-|u_\varepsilon|^2)(\psi - |\mathrm{v}|^2)\right)\right).$$

Since we have chosen $\psi = \mathrm{p} - |\mathrm{v}|^2$ we see that the terms involving $\partial_t(1-|u_\varepsilon|^2)$ cancel and we get the conclusion. □

We may check that all the terms in factor of χ or $\nabla\chi$ in (3.2) are in $L^1([0,t]\times\mathbb{R}^n)$, thanks to (2.17), Lemmas 2.1 and 2.2. Inserting for χ in (3.2) a sequence $\{\chi_k\}_k$ of functions bounded in $C^1(\mathbb{R}^n)$ such that $\chi_k\to 1$ and $\|\nabla\chi_k\|_{L^\infty(\mathbb{R}^n)}\to 0$, we may thus integrate (3.2) in time and, using the finiteness of $\mathcal{E}_\varepsilon(u_\varepsilon(t),t)$ given by Lemma 2.2, let $k\to\infty$ to obtain by dominated convergence,

$$\mathcal{E}_\varepsilon(u_\varepsilon(t),t) - \mathcal{E}_\varepsilon(u_\varepsilon^0,0) = \int_0^t\int_{\mathbb{R}^n} 2\tilde S_\varepsilon(u_\varepsilon) : \nabla \mathrm{v} - N_\varepsilon^2(1-|u_\varepsilon|^2)\partial_t(|\mathrm{v}|^2 - \frac{\mathrm{p}}{2}).$$

Inserting (2.18), the bounds given by Lemma 2.1, and using the Cauchy-Schwarz inequality to control $(1-|u_\varepsilon|^2)$ by $\mathcal{E}_\varepsilon$ in view of (2.5), we arrive at

$$\begin{aligned}&\mathcal{E}_\varepsilon(u_\varepsilon(t),t)-\mathcal{E}_\varepsilon(u_\varepsilon^0,0)\\&\quad\le C\int_0^t\mathcal{E}_\varepsilon(u_\varepsilon)+CN_\varepsilon^2\int_0^t\left(N_\varepsilon^2\varepsilon^2+\varepsilon\sqrt{\mathcal{E}_\varepsilon(u_\varepsilon)+C\varepsilon^2N_\varepsilon^4}\right)\\&\quad\le C\int_0^t\mathcal{E}_\varepsilon(u_\varepsilon)+Ct\varepsilon^2N_\varepsilon^4,\end{aligned}$$

where C depends only on the bounds on v. Applying Gronwall's lemma and using (1.23), we finally obtain that

$$\mathcal{E}_\varepsilon(u_\varepsilon(t),t)\le C_t\left(\mathcal{E}_\varepsilon(u_\varepsilon^0,0)+o(N_\varepsilon^2)\right)$$

which is $o(N_\varepsilon^2)$ with the initial data assumption (1.25). The conclusions of the theorem follow in view of either (2.6) or (2.7).

4. The parabolic case

In this section, we turn to the proof of Theorem 2, which only concerns the dimension $n=2$. Throughout we assume that $\alpha=1$, $\beta=0$ and (1.27) holds.

In the rest of the paper, we let χ_R be

$$\chi_R(x)=\begin{cases}\frac{\log|x|-\log R^2}{\log R-\log R^2}\ \text{for}\ R\le|x|\le R^2\\ \chi_R(x)=1\ \text{for}\ |x|\le R\\ \chi_R(x)=0\ \text{for}\ |x|\ge R^2.\end{cases}\tag{4.1}$$

With this choice we have

$$\|\nabla\chi_R\|_{L^\infty(\mathbb{R}^2)}\to0\qquad\|\nabla\chi_R\|_{L^2(\mathbb{R}^2)}\to0\quad\text{as}\quad R\to\infty.\tag{4.2}$$

4.1. **A priori bound on the velocity.** We define T_ε as the maximum time $t\le\min(1,\mathrm{T})$ (where $\mathrm{T}>0$ is the time of existence of the solution to the limiting equation) such that

$$\mathcal{E}_\varepsilon(u_\varepsilon(t))\le\pi N_\varepsilon|\log\varepsilon|+N_\varepsilon^2\qquad\text{for all}\ \ t\le T_\varepsilon.\tag{4.3}$$

Our goal is to show that $\mathcal{E}_\varepsilon(u_\varepsilon(t))\le\pi N_\varepsilon|\log\varepsilon|+o(N_\varepsilon^2)$ for all $t\le T_\varepsilon$, which will imply that $T_\varepsilon=\min(1,\mathrm{T})$.

Let us start with a crude a priori bound on the time derivative of u_ε.

Lemma 4.1. *Assume* (1.27) *and* $\mathcal{E}_\varepsilon(u_\varepsilon^0)\le\pi N_\varepsilon|\log\varepsilon|+o(N_\varepsilon^2)$. *Assume* v *satisfies the results of Lemma 2.1. Then*

$$\int_0^{T_\varepsilon}\int_{\mathbb{R}^2}|\partial_tu_\varepsilon|^2\le CN_\varepsilon^3|\log\varepsilon|^3,$$

where C depends only on the bounds on v.

Proof. Let us return to the result of Lemma 2.5 with the choice $\psi = -|\mathrm{v}|^2$ and $\chi = \chi_R$ as in (4.1). By (2.9), we have $V_\varepsilon = 2\langle i\partial_t u_\varepsilon, \nabla u_\varepsilon\rangle = 2\langle i\partial_t u_\varepsilon, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}\rangle + 2N_\varepsilon \mathrm{v}\langle \partial_t u_\varepsilon, u_\varepsilon\rangle$, and inserting into the result of Lemma 2.5, we obtain

$$\begin{aligned}\partial_t \hat{\mathcal{E}}_\varepsilon(u_\varepsilon) = &-\int_{\mathbb{R}^2} \chi_R \frac{N_\varepsilon}{|\log\varepsilon|}|\partial_t u_\varepsilon|^2 - \int_{\mathbb{R}^2} \nabla\chi_R \cdot \langle \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}, \partial_t u_\varepsilon\rangle \\ &+ \int_{\mathbb{R}^2} \chi_R \left(-N_\varepsilon^2(1-|u_\varepsilon|^2)\partial_t|\mathrm{v}|^2 + N_\varepsilon\langle \partial_t u_\varepsilon, iu_\varepsilon\rangle \operatorname{div}\mathrm{v}\right. \\ &\qquad \left. + N_\varepsilon(N_\varepsilon \mathrm{v} - j_\varepsilon)\cdot \partial_t \mathrm{v} + 2N_\varepsilon \langle i\partial_t u_\varepsilon, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}\rangle \cdot \mathrm{v}\right).\end{aligned}$$

We next observe that again all the terms in factor of χ_R or $\nabla\chi_R$ are in $L^1(\mathbb{R}^2)$ thanks to Lemmas 2.1 and 2.2. We may then integrate in time and let $R \to \infty$ to obtain

$$\begin{aligned}&\int_0^{T_\varepsilon}\int_{\mathbb{R}^2} \frac{N_\varepsilon}{|\log\varepsilon|}|\partial_t u_\varepsilon|^2 \\ &\quad = \mathcal{E}_\varepsilon(u_\varepsilon^0) - \mathcal{E}_\varepsilon(u_\varepsilon(T_\varepsilon)) + \int_0^{T_\varepsilon}\int_{\mathbb{R}^2} N_\varepsilon \langle \partial_t u_\varepsilon, iu_\varepsilon\rangle \operatorname{div}\mathrm{v} + N_\varepsilon(N_\varepsilon\mathrm{v} - j_\varepsilon)\cdot\partial_t \mathrm{v} \\ &\qquad + 2N_\varepsilon\langle i\partial_t u_\varepsilon, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}\rangle \cdot \mathrm{v} + o(1),\end{aligned}$$

where we also used (1.27) and (4.3) to control all the terms containing $(1-|u_\varepsilon|^2)$. Next we insert (2.8) to obtain

$$\begin{aligned}&\int_0^{T_\varepsilon}\int_{\mathbb{R}^2} \frac{N_\varepsilon}{|\log\varepsilon|}|\partial_t u_\varepsilon|^2 \le \mathcal{E}_\varepsilon(u_\varepsilon^0) \\ &+ \int_0^{T_\varepsilon}\int_{\mathbb{R}^2} \Big(N_\varepsilon|\partial_t u_\varepsilon||\operatorname{div}\mathrm{v}| + |1-|u_\varepsilon|||\partial_t u_\varepsilon||\operatorname{div}\mathrm{v}| + N_\varepsilon|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}||\partial_t \mathrm{v}| \\ &\quad + N_\varepsilon|1-|u_\varepsilon|||\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}||\partial_t\mathrm{v}| + N_\varepsilon^2|1-|u_\varepsilon|^2||\mathrm{v}||\partial_t \mathrm{v}| \\ &\quad + 2N_\varepsilon|\partial_t u_\varepsilon||\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}||\mathrm{v}|\Big) + o(1).\end{aligned}$$

Using $|1-|u_\varepsilon|| \le |1-|u_\varepsilon|^2|$, the Cauchy-Schwarz inequality, (1.28), (4.3), the $L^\infty \cap L^2$ character of $\operatorname{div}\mathrm{v}$ and $\partial_t \mathrm{v}$ given by Lemma 2.1, the boundedness of v and Lemma 2.3, we deduce that

$$\begin{aligned}&\int_0^{T_\varepsilon}\int_{\mathbb{R}^2} \frac{N_\varepsilon}{|\log\varepsilon|}|\partial_t u_\varepsilon|^2 \\ &\qquad \le \pi N_\varepsilon|\log\varepsilon| + CN_\varepsilon \int_0^{T_\varepsilon} (1+\|\partial_t u_\varepsilon\|_{L^2})(1+\mathcal{E}_\varepsilon(u_\varepsilon))\,dt + o(N_\varepsilon^2).\end{aligned}$$

Using (4.3), bounding T_ε by 1 and using (1.27), we easily deduce the result. □

4.2. **Preliminaries: ball construction and product estimate.** The proof in the parabolic case is more involved than in the Gross-Pitaevskii case, and in particular it requires all the machinery to study vortices which has been developed over the years. Indeed, as explained in the introduction, we will need to subtract off the (now leading order) contribution of the vortices to the energy. This will be done via the ball-construction method (introduced in [Sa, J1]) coupled with the “Jacobian estimates” [JS1] (with precursors in

[BR1, SS1]). Here we will need a lower bound with smaller errors, as in [SS5, Theorems 4.1 and 6.1], coupled to an improvement due to [ST1].

4.2.1. *Vorticity to modulated vorticity.* Before doing so, we will need the following result which connects the vorticity to the modulated vorticity.

Lemma 4.2. *Assume* v *satisfies the results of Lemma 2.1 and* u_ε *is such that* $\mathcal{E}_\varepsilon(u_\varepsilon) < \infty$. *Let* $\mu_\varepsilon = \operatorname{curl} j_\varepsilon$ *and* $\tilde\mu_\varepsilon = \operatorname{curl}(\langle \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}, iu_\varepsilon\rangle + N_\varepsilon \mathrm{v})$ *as in* (2.11). *We have that* $\tilde\mu_\varepsilon \in L^1(\mathbb{R}^2)$ *with*

$$|\tilde\mu_\varepsilon| \le 2|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}|^2 + |1-|u_\varepsilon|^2||\operatorname{curl}\mathrm{v}| \tag{4.4}$$

and

$$\int_{\mathbb{R}^2} \tilde\mu_\varepsilon = 2\pi N_\varepsilon. \tag{4.5}$$

Moreover, for any $\xi \in H^1(\mathbb{R}^2)$, *we have*

$$\left|\int_{\mathbb{R}^2} \xi(\mu_\varepsilon - \tilde\mu_\varepsilon)\right| \le C\|\nabla \xi\|_{L^2(\mathbb{R}^2)} N_\varepsilon(\varepsilon\sqrt{\mathcal{E}_\varepsilon(u_\varepsilon)} + C\varepsilon^2 N_\varepsilon^2). \tag{4.6}$$

Proof. First, a direct computation gives that (4.4) holds, and it follows immediately with Lemmas 2.1 and 2.2 that $\tilde\mu_\varepsilon \in L^1(\mathbb{R}^2)$. We may then write, with χ_R as in (4.1)

$$\int_{\mathbb{R}^2}\tilde\mu_\varepsilon = \lim_{R\to\infty}\int_{\mathbb{R}^2}\chi_R\tilde\mu_\varepsilon$$
$$= -\lim_{R\to\infty}\int_{\mathbb{R}^2}\nabla^\perp\chi_R\cdot(\langle\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}, iu_\varepsilon\rangle + N_\varepsilon(\mathrm{v} - \langle\nabla U_1, iU_1\rangle + \langle \nabla U_1, iU_1\rangle)).$$

Since $\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}$ and $\mathrm{v} - \langle\nabla U_1, iU_1\rangle$ are in L^2 by Lemmas 2.1 and 2.2, the corresponding terms tend to 0 as $R\to\infty$. There remains

$$\int_{\mathbb{R}^2}\tilde\mu_\varepsilon = -\lim_{R\to\infty}\int_{\mathbb{R}^2} N_\varepsilon\nabla^\perp\chi_R\cdot\langle\nabla U_1, iU_1\rangle$$
$$= \lim_{R\to\infty}\int_{\mathbb{R}^2} N_\varepsilon \chi_R \operatorname{curl}\langle \nabla U_1, iU_1\rangle = 2\pi N_\varepsilon$$

by choice of U_1. For (4.6) we observe that by a direct computation, we have $\tilde\mu_\varepsilon - \mu_\varepsilon = \operatorname{curl}((1-|u_\varepsilon|^2)N_\varepsilon \mathrm{v})$. Thus, for any $\xi\in H^1(\mathbb{R}^2)$ we have

$$\left|\int_{\mathbb{R}^2}\xi(\mu_\varepsilon - \tilde\mu_\varepsilon)\right| = \left|N_\varepsilon\int_{\mathbb{R}^2}(1-|u_\varepsilon|^2)\nabla^\perp\xi\cdot\mathrm{v}\right|$$
$$\le N_\varepsilon\|\nabla\xi\|_{L^2(\mathbb{R}^2)}(C\varepsilon\sqrt{\mathcal{E}_\varepsilon(u_\varepsilon) + \varepsilon^2 N_\varepsilon^4}),$$

where we used Lemma 2.3. □

4.2.2. *Jacobian estimate for unbounded domains.* The next lemma is an infinite domain version of the estimate of [SS5, Theorem 6.1]. For this lemma, we temporarily use the notation μ_ε with a slightly different meaning.

Lemma 4.3. *Let Ω be an open subset of $\mathbb{R}^2$, and let $\Omega^\varepsilon = \{x \in \Omega, dist\,(x, \partial\Omega) > \varepsilon\}$. Let $u_\varepsilon : \Omega \to \mathbb{C}$ and $A_\varepsilon : \Omega \to \mathbb{R}^2$. Assume that $\{B_i\}_i$ is a finite collection of disjoint closed balls of centers a_i and radii r_i covering $\{||u_\varepsilon| - 1| \geq \frac{1}{2}\} \cap \Omega^\varepsilon$, and let $d_i = \deg(u_\varepsilon, \partial B_i)$ if $B_i \subset \Omega^\varepsilon$ and $d_i = 0$ otherwise. Then, setting*

$$\mu_\varepsilon = \operatorname{curl}\left(\langle \nabla u_\varepsilon - iA_\varepsilon u_\varepsilon, iu_\varepsilon\rangle + A_\varepsilon\right),$$

we have, for any $\xi \in C_c^{0,1}(\Omega)$,

$$\left|\int_\Omega \xi(\mu_\varepsilon - 2\pi \sum_i d_i \delta_{a_i})\right| \tag{4.7}$$

$$\leq C\Big(\sum_i r_i + \varepsilon\Big)\|\xi\|_{C^{0,1}(\Omega)} \int_\Omega 2|\nabla u_\varepsilon - iu_\varepsilon A_\varepsilon|^2 + |1 - |u_\varepsilon|^2||\operatorname{curl} A_\varepsilon| + \frac{(1-|u_\varepsilon|^2)^2}{2\varepsilon^2},$$

where C is universal.

Proof. As in [SS5, Chap. 6], we set $\chi : \mathbb{R}_+ \to \mathbb{R}_+$ to be defined by

$$\begin{cases} \chi(x) = 2x & \text{if } x \in [0, \frac{1}{2}] \\ \chi(x) = 1 & \text{if } x \in [\frac{1}{2}, \frac{3}{2}] \\ \chi(x) = 1 + 2(x - 3/2) & \text{if } x \in [\frac{3}{2}, 2] \\ \chi(x) = x & \text{if } x \geq 2. \end{cases}$$

We then let $v_\varepsilon(x) = \frac{\chi(|u_\varepsilon|)}{|u_\varepsilon|} u_\varepsilon$. This is clearly well-defined, with $|v_\varepsilon| = 1$ outside of $\cup_i B_i$ and from [SS5, Chap. 6] or direct calculations, we know that

$$\|\langle \nabla u_\varepsilon - iA_\varepsilon u_\varepsilon, iu_\varepsilon\rangle - \langle \nabla v_\varepsilon - iA_\varepsilon v_\varepsilon, iv_\varepsilon\rangle\|_{L^1(\Omega)} \leq 3\|1 - |u_\varepsilon|\|_{L^2(\Omega)} \|\nabla u_\varepsilon - iA_\varepsilon u_\varepsilon\|_{L^2(\Omega)}, \tag{4.8}$$

and

$$|1 - |v_\varepsilon|| \leq |1 - |u_\varepsilon||, \qquad |\nabla v_\varepsilon - iA_\varepsilon v_\varepsilon| \leq 2|\nabla u_\varepsilon - iA_\varepsilon u_\varepsilon|. \tag{4.9}$$

Letting $\hat\mu_\varepsilon = \operatorname{curl}\left(\langle \nabla v_\varepsilon - iA_\varepsilon v_\varepsilon, iv_\varepsilon\rangle + A_\varepsilon\right)$, we have

$$\left|\int_\Omega \xi(\mu_\varepsilon - \hat\mu_\varepsilon)\right| = \left|\int_\Omega \nabla^\perp \xi \cdot \left(\langle \nabla u_\varepsilon - iu_\varepsilon A_\varepsilon, iu_\varepsilon\rangle - \langle \nabla v_\varepsilon - iA_\varepsilon v_\varepsilon, iv_\varepsilon\rangle\right)\right| \tag{4.10}$$

$$\leq 3\|\nabla\xi\|_{L^\infty(\Omega)} \|1 - |u_\varepsilon|^2\|_{L^2(\Omega)} \|\nabla u_\varepsilon - iA_\varepsilon u_\varepsilon\|_{L^2(\Omega)}.$$

Next, we note that $\hat\mu_\varepsilon$ vanishes wherever $|v_\varepsilon| = 1$, and thus as soon as $||u_\varepsilon| - 1| \leq \frac{1}{2}$. Thus, by property of the balls, we have $\operatorname{Supp} \hat\mu_\varepsilon \cap \Omega^\varepsilon \subset \cup_i B_i$ (recall the definition of Ω^ε in the statement of the lemma). We also have that whenever $B_i \subset \Omega^\varepsilon$, it holds that $\int_{B_i} \hat\mu_\varepsilon = 2\pi d_i$ (see [SS5, Lemma 6.3]). Writing ξ as $\xi(a_i) + O(r_i)\|\xi\|_{C^{0,1}}$ in each B_i, we conclude, exactly as in the proof of [SS5, Theorem 6.1]. The only point that is a bit different is we need an analogue of [SS5, Lemma 6.4] to bound $\sum_i r_i \int_{B_i} |\mu_\varepsilon|$ which works on an unbounded domain. For that, we may check by direct computation that

$$\mu_\varepsilon = 2\langle \nabla u_\varepsilon - iA_\varepsilon u_\varepsilon, i(\nabla u_\varepsilon - iA_\varepsilon u_\varepsilon)\rangle + (1 - |u_\varepsilon|^2) \operatorname{curl} A_\varepsilon, \tag{4.11}$$

so

$$\int_\Omega |\mu_\varepsilon| \leq 2 \int_\Omega 2|\nabla u_\varepsilon - iA_\varepsilon u_\varepsilon|^2 + |1 - |u_\varepsilon|^2||\operatorname{curl} A_\varepsilon|$$

and the same holds for $\hat\mu_\varepsilon$ with a factor 4 in front, in view of (4.9). We thus obtain that

$$\left|\int_\Omega \xi(\mu_\varepsilon - 2\pi\sum_i d_i\delta_{a_i})\right|$$
$$\le C(\sum_i r_i+\varepsilon)\|\xi\|_{C^{0,1}}\int_{\cup_i B_i\cup(\Omega\setminus\Omega^\varepsilon)} 2|\nabla u_\varepsilon - iA_\varepsilon u_\varepsilon|^2 + |1-|u_\varepsilon|^2||\operatorname{curl} A_\varepsilon|$$
$$+ C\|\xi\|_{C^{0,1}}\|1-|u_\varepsilon|^2\|_{L^2(\mathbb{R}^2)}\|\nabla u_\varepsilon - iA_\varepsilon u_\varepsilon\|_{L^2(\mathbb{R}^2)},$$

where C is universal, and we easily deduce the result. □

4.2.3. *Ball construction lower bound, sharp version.* In the result below, we use the Lorentz space $L^{2,\infty}$ as in [ST2], which can be defined by

$$\|f\|_{L^{2,\infty}(\mathbb{R}^2)} = \sup_{|E|<\infty}\int_E |f| \tag{4.12}$$

where $|E|$ denotes the Lebesgue measure of E.

Proposition 4.4. *Assume $\mathcal{E}_\varepsilon(u_\varepsilon)\le \pi N_\varepsilon|\log\varepsilon| + N_\varepsilon^2$ where N_ε satisfies (1.27), and $\mathrm{v}\in C^{1,\gamma}(\mathbb{R}^2)$ with $\operatorname{curl}\mathrm{v}\in L^2(\mathbb{R}^2)$. Then there exists ε_0 such that for any $\varepsilon<\varepsilon_0$, the following holds. There exists a finite collection of disjoint closed balls $\{B_i = B(a_i,r_i)\}_i$ such that, letting $d_i = \deg(u_\varepsilon,\partial B_i)$, the following holds*

(1) $\sum_i r_i \le e^{-\sqrt{N_\varepsilon}}$.
(2) $\{x, ||u_\varepsilon|-1|\ge \frac12\}\subset \cup_i B(a_i,r_i)$.
(3)
$$\frac12\int_{\cup_i B_i}|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}|^2 + \frac{(1-|u_\varepsilon|^2)^2}{2\varepsilon^2}\ge \pi\sum_i |d_i||\log\varepsilon| - o(N_\varepsilon^2).$$
(4)
$$\|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}\|^2_{L^{2,\infty}(\mathbb{R}^2)}\le C\left(\mathcal{E}_\varepsilon(u_\varepsilon) - \pi\sum_i|d_i||\log\varepsilon| + \sum_i |d_i|^2\right) + o(N_\varepsilon^2).$$
(5) *Letting $\tilde\mu_\varepsilon$ be as in (2.11), for any $\xi\in C_c^{0,\gamma}(\mathbb{R}^2)$, we have*

$$\left|\int \xi(2\pi\sum_i d_i\delta_{a_i} - \tilde\mu_\varepsilon)\right|\le o(1)\|\xi\|_{C^{0,\gamma}}. \tag{4.13}$$

Proof. The result of [SS5, Theorem 4.1] applied to u_ε, $A_\varepsilon = N_\varepsilon\mathrm{v}$ and $\alpha = 3/4$, provides for ε small enough, for any $\varepsilon^{1/4}\le r<1$ a collection of disjoint closed balls $\mathcal{B}(r)$ covering $\{x, ||u_\varepsilon|-1|\ge\varepsilon^{3/16}\}$, such that the sum of the radii of the balls in the collection is r, and such that denoting $D := \sum_{B\in\mathcal{B}(r)}|d_B|$, we have

$$\frac12\int_{\cup_{B\in\mathcal{B}(r)}}|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}|^2 + r^2|N_\varepsilon\operatorname{curl}\mathrm{v}|^2 + \frac{(1-|u_\varepsilon|^2)^2}{2\varepsilon^2}\ge \pi D\left(\log\frac{r}{D\varepsilon} - C\right).$$

One notes that the fact that we are in an unbounded domain does not create any problem. Indeed, since $\mathcal{E}_\varepsilon(u_\varepsilon)<\infty$, this implies (cf. e.g. [J1, Lemma 2.3] or [HL, Lemma 3.5])that there exists a radius R_ε such that $|u_\varepsilon| \geq \frac{1}{2}$ outside of $B(0, R_\varepsilon)$ and

$$\int_{B(0,R_\varepsilon)^c} |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}|^2 + \frac{(1-|u_\varepsilon|^2)^2}{2\varepsilon^2} + N_\varepsilon^2(1-|u_\varepsilon|^2)\psi < 1$$

i.e. the remaining energy is smaller than the desired error. This way the construction can be applied in $B(0, R_\varepsilon + 1)$ only, yielding always a finite collection of balls covering $\{|u_\varepsilon| \leq \frac{1}{2}\}$.

We first apply this result with the choice $r = r' = \varepsilon^{1/4}$ to obtain a collection of balls $\{B'_j\}$ with centers a'_j, radii r'_j and degrees d'_j, covering $\{x, ||u_\varepsilon| - 1| \geq \frac{1}{2}\}$ and satisfying $\sum_j r'_j \leq \varepsilon^{1/4}$. The estimates of [SS5] also yield that

$$\sum_j |d'_j| \leq \frac{C}{|\log \varepsilon|} \int_{\cup_j B'_j} |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}|^2 + \varepsilon^{1/2} N_\varepsilon^2 |\operatorname{curl} \mathrm{v}|^2 + \frac{(1-|u_\varepsilon|^2)^2}{2\varepsilon^2}.$$

Using that $\operatorname{curl} \mathrm{v}$ is bounded, the fact that $\sum r'_i \leq \varepsilon^{1/4}$, the upper bound on $\mathcal{E}_\varepsilon(u_\varepsilon)$ and (1.27), we deduce from this relation that $\sum_j |d'_j| \leq CN_\varepsilon$.

We next apply the above result with the choice $r = e^{-\sqrt{N_\varepsilon}}$. This gives a collection of balls $\{B_i\} = \{B(a_i, r_i)\}$ of degrees d_i, satisfying items 1 and 2 of the proposition and

$$\frac{1}{2} \int_{\cup_i B_i} |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}|^2 + e^{-2\sqrt{N_\varepsilon}} |N_\varepsilon \operatorname{curl} \mathrm{v}|^2 + \frac{(1-|u_\varepsilon|^2)^2}{2\varepsilon^2} \\ \geq \pi \sum_i |d_i| \left(\log \frac{e^{-\sqrt{N_\varepsilon}}}{\sum_i |d_i| \varepsilon} - C \right).$$

It is part of the statements of [SS4, Theorem 4.1] that the family $\mathcal{B}(r)$ is increasing in r, i.e. here that the B_i's cover the B'_j's. By additivity of the degree, we thus have $\sum_i |d_i| \leq \sum_j |d'_j| \leq CN_\varepsilon$, and using $\sum_i r_i \leq e^{-\sqrt{N_\varepsilon}}$ and the boundedness of $\operatorname{curl} \mathrm{v}$, the desired estimate of item 3 follows.

For item 4, we use [ST1, Corollary 1.2] which yields that

$$\|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}\|^2_{L^{2,\infty}(\mathbb{R}^2)} \\ \leq C \left(\mathcal{E}_\varepsilon(u_\varepsilon) - \pi \sum_i |d_i| \left(\log \frac{r}{\varepsilon \sum_i |d_i|} - C \right) + \pi \sum_i |d_i|^2 \right) + o(N_\varepsilon^2). \tag{4.14}$$

This is essentially a strengthened version of the result of item 3, in which the difference between the two sides of the inequality is shown to bound from above $\|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}\|^2_{L^{2,\infty}(\mathbb{R}^2)}$.

Let us now turn to item 5, which is an adaptation to an infinite setting of the Jacobian estimates, for instance as in [SS5, Theorem 6.1]. The reason we needed a two-step construction above is that the total radius of the second set of balls, $e^{-\sqrt{N_\varepsilon}}$, (which have to be chosen this large so that they contain enough energy) is not very small compared to $|\log \varepsilon|$ when N_ε is not very large, and thus the Jacobian estimate applied directly on these large balls would give too large of an error.

Letting ξ be a smooth test-function, we may write

$$\int_{\mathbb{R}^2} \Big(\sum_i d_i\delta_{a_i} - \sum_j d'_j\delta_{a'_j}\Big)\xi = \sum_i \Big(d_i\xi(a_i) - \sum_{j, B'_j\subset B_i} d'_j\xi(a'_j)\Big).$$

Since $\sum_{j,B'_j\subset B_i} d'_j = d_i$ and $\sum_j |d'_j| \le CN_\varepsilon$, we may write

$$\left|\int_{\mathbb{R}^2} \Big(\sum_i d_i\delta_{a_i} - \sum_j d'_j\delta_{a'_j}\Big)\xi\right| \le \|\xi\|_{C^{0,1}} \sum_i r_i^\gamma \Big(\sum_{j,B'_j\subset B_i} |d'_j|\Big) \tag{4.15}$$
$$\le \|\xi\|_{C^{0,1}} \Big(\sum_i r_i\Big) \sum_i \sum_{j,B'_j\subset B_i} |d'_j| \le C\|\xi\|_{C^{0,1}} e^{-\gamma\sqrt{N_\varepsilon}} N_\varepsilon = o(1)\|\xi\|_{C^{0,1}}.$$

Applying Lemma 4.3 with $A_\varepsilon = N_\varepsilon \mathrm{v}$ on $\mathbb{R}^2$, and it yields that

$$\left|\int_{\mathbb{R}^2} \xi\Big(2\pi\sum_j d'_j\delta_{a'_j} - \tilde\mu_\varepsilon\Big)\right|$$
$$\le C\varepsilon^{1/4}\|\xi\|_{C^{0,1}} \int_{\mathbb{R}^2} |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}|^2 + N_\varepsilon |1-|u_\varepsilon|^2| |\operatorname{curl}\mathrm{v}| + \frac{(1-|u_\varepsilon|^2)^2}{2\varepsilon^2}.$$

Using that $\operatorname{curl}\mathrm{v} \in L^1(\mathbb{R}^2)$, the upper bound on $\mathcal{E}_\varepsilon(u_\varepsilon)$, and combining with (4.15), we obtain the desired result for $\gamma = 1$. The result for $\gamma < 1$ follows by interpolation as in [JS1], using that $\sum_j |d'_j| \le CN_\varepsilon$ and $\int_{\mathbb{R}^2} |\tilde\mu_\varepsilon| \le CN_\varepsilon|\log\varepsilon|$ hence $\|2\pi\sum_j d'_j\delta_{a'_j} - \tilde\mu_\varepsilon\|_{(C^0)^*} \le CN_\varepsilon$. □

4.2.4. *Ball construction lower bound, localized version.* We will also need a less precise but localizable version of the ball construction. This can be borrowed directly from [J1, Sa, SS1], and combined with the Jacobian estimate of Lemma 4.3, so we omit the proof.

Lemma 4.5. *Under the assumptions of Proposition 4.4, there exists ε_0 such that for all $\varepsilon < \varepsilon_0$, there exists a finite collection of disjoint closed balls $\{B_i\}_i = \{B(a_i, r_i)\}_i$ such that, letting $d_i = \deg(u_\varepsilon, \partial B_i)$, the following holds*

(1) $\sum_i r_i \le e^{-\sqrt{|\log\varepsilon|}}$.

(2)
$$\forall i, \quad \frac12 \int_{B_i} |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}|^2 + \frac{(1-|u_\varepsilon|^2)^2}{2\varepsilon^2} \ge \pi|d_i||\log\varepsilon|(1-o(1)).$$

(3) *For any $0 < \gamma \le 1$ and any $\xi \in C_c^{0,\gamma}(\mathbb{R}^2)$, we have*
$$\left|\int_{\mathbb{R}^2} \xi \left(2\pi\sum_i d_i\delta_{a_i} - \tilde\mu_\varepsilon\right)\right| \le o(1)\|\xi\|_{C^{0,\gamma}}.$$

We emphasize that these balls are not necessarily the same as those obtained by Proposition 4.4.

4.2.5. *Consequences on the energy excess.* We next show how the energy excess $\mathcal{E}_\varepsilon - \pi N_\varepsilon |\log \varepsilon|$ controls various quantities, including the energy outside of the balls.

Corollary 4.6. *For any $t \le T_\varepsilon$ (where T_ε is as in* (4.3)*), letting $\{B_i\}_i$ (depending on ε, t) be the collection of balls given by Proposition 4.4, we have*

$$\frac{1}{2}\int_{\mathbb{R}^2 \setminus \cup_i B_i} |\nabla u_\varepsilon - i u_\varepsilon N_\varepsilon \mathrm{v}|^2 + \frac{(1-|u_\varepsilon|^2)^2}{2\varepsilon^2} \le \mathcal{E}_\varepsilon(u_\varepsilon(t), t) - \pi N_\varepsilon |\log \varepsilon| + o(N_\varepsilon^2), \tag{4.16}$$

$$\text{for } \varepsilon \text{ small enough} \qquad N_\varepsilon \le \sum_i |d_i| \le C N_\varepsilon, \tag{4.17}$$

$$\|\nabla u_\varepsilon - i u_\varepsilon N_\varepsilon \mathrm{v}\|_{L^{2,\infty}(\mathbb{R}^2)} \le C N_\varepsilon, \tag{4.18}$$

and for any nonnegative $\xi \in C^{0,\gamma}(\mathbb{R}^2)$,

$$\frac{1}{2}\int_{\mathbb{R}^2} \xi \left(\frac{1}{|\log \varepsilon|} |\nabla u_\varepsilon - i u_\varepsilon N_\varepsilon \mathrm{v}|^2 - \tilde{\mu}_\varepsilon \right) \le \|\xi\|_{L^\infty} \frac{1}{|\log \varepsilon|} \left(\mathcal{E}_\varepsilon(u_\varepsilon(t), t) - \pi N_\varepsilon |\log \varepsilon| \right) + o(N_\varepsilon) \|\xi\|_{C^{0,\gamma}}. \tag{4.19}$$

Proof. First, applying item 5 of Proposition 4.4 with $\xi = \chi_R$ as in (4.1) and letting $R \to \infty$, we must have $\int_{\mathbb{R}^2} (2\pi \sum_i d_i \delta_{a_i} - \tilde{\mu}_\varepsilon) = o_\varepsilon(1)$. Comparing with (4.5), we deduce that $\sum_i d_i = N_\varepsilon$ for ε small enough. Subtracting the result of item 3 of Proposition 4.4 from $\mathcal{E}_\varepsilon(u_\varepsilon(t), t)$ and using Lemma 2.3, we then obtain (4.16). The upper bound in (4.17) was proved in the course of the proof of Proposition 4.4, the lower bound is an obvious consequence of $\sum_i d_i = N_\varepsilon$.

The relation (4.18) is a direct consequence of item 4 of Proposition 4.4, (4.3) and (4.17), writing $\sum_i |d_i|^2 \le (\sum_i |d_i|)^2$.

Finally, for (4.19) we use instead the balls given by Lemma 4.5. From item 2 of Lemma 4.5 we may write

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^2 \setminus \cup_i B_i} |\nabla u_\varepsilon - i u_\varepsilon N_\varepsilon \mathrm{v}|^2 + \frac{(1-|u_\varepsilon|^2)^2}{2\varepsilon^2} \\
&+ \sum_i \frac{1}{2}\int_{B_i} |\nabla u_\varepsilon - i u_\varepsilon N_\varepsilon \mathrm{v}|^2 + \frac{(1-|u_\varepsilon|^2)^2}{2\varepsilon^2} - \pi |d_i| |\log \varepsilon| (1 - o(1)) \\
&\le \mathcal{E}_\varepsilon(u_\varepsilon(t), t) - \pi \sum_i |d_i| |\log \varepsilon| (1 + o(1)).
\end{aligned}$$

Moreover, from the same argument as before we have $\sum_i d_i = N_\varepsilon$ and $\sum_i |d_i| \le C N_\varepsilon$ for these balls, hence since the terms of the above sum are all nonnegative, adding to both sides $\pi\sum_i(|d_i|-d_i)|\log\varepsilon|$, we obtain

$$\begin{aligned}&\frac12\int_{\mathbb{R}^2\setminus\cup_i B_i}|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}|^2+\frac{(1-|u_\varepsilon|^2)^2}{2\varepsilon^2}\\&+\sum_i\left|\frac12\int_{B_i}|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}|^2+\frac{(1-|u_\varepsilon|^2)^2}{2\varepsilon^2}-\pi d_i|\log\varepsilon|(1-o(1))\right|\\&\le \mathcal{E}_\varepsilon(u_\varepsilon(t),t)-\pi N_\varepsilon|\log\varepsilon|(1+o(1)).\end{aligned}$$

Next, separating the integral between $\cup_i B_i$ and the complement, and using item 1 of Lemma 4.5, we deduce that

$$\begin{aligned}\frac12\int_{\mathbb{R}^2}\xi|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}|^2+\xi\frac{(1-|u_\varepsilon|^2)^2}{2\varepsilon^2}&\le\pi\sum_i d_i\xi(a_i)|\log\varepsilon|(1+o(1))\\&+o(1)\|\xi\|_{C^{0,\gamma}}+\|\xi\|_{L^\infty}\left(\mathcal{E}_\varepsilon(u_\varepsilon(t),t)-\pi N_\varepsilon|\log\varepsilon|(1+o(1))\right).\end{aligned}\tag{4.20}$$

Combining this with item 3 of Lemma 4.5, we deduce that (4.19) holds. □

4.2.6. *Approximation of* v. We will need an approximation of $\mathrm{v}(t)$ based on the balls constructed via Proposition 4.4.

Lemma 4.7. *Let* v *satisfy the assumptions of Theorem 2 and* u_ε *satisfy* (4.3). *For each* $t\le T_\varepsilon$, *letting* $\{B_i\}_i$ *be the collection of balls constructed in Proposition 4.4, there exists a vector field* $\bar{\mathrm{v}}:\mathbb{R}^2\to\mathbb{R}^2$ *(depending on* ε *and* t*) such that*

(1) $\bar{\mathrm{v}}$ *is constant in each* B_i,
(2) *for every* $\gamma\in[0,1]$, $\|\bar{\mathrm{v}}-\mathrm{v}\|_{C^{0,\gamma}}\le C(\sum_i r_i)^{1-\gamma}\le Ce^{-(1-\gamma)\sqrt{N_\varepsilon}}$ *where* C *depends only on* v *and* γ,
(3) $\bar{\mathrm{v}}-\mathrm{v}$ *has compact support.*

Proof. It is an adaptation of Proposition 9.6 of [SS5], which we can apply to each component of v. We note that since the collection of balls is finite, we may replace $\bar{\mathrm{v}}$ by $\mathrm{v}+\chi(\bar{\mathrm{v}}-\mathrm{v})$ where χ is a smooth positive cut-off function which is equal to 1 on a large enough ball containing $\cup_i B_i$ and vanishes outside of a large enough ball. This makes $\bar{\mathrm{v}}-\mathrm{v}$ compactly supported without affecting the other properties. □

By continuity of u_ε (for fixed ε) and of v, one may check that we may make $\bar{\mathrm{v}}(t)$ measurable in t. While we have a good control on $\nabla\bar{\mathrm{v}}$, we have no control on $\partial_t\bar{\mathrm{v}}$, and this is what prevents us from applying this method in the regime $N_\varepsilon\le O(|\log\varepsilon|)$ in the Schrödinger case.

4.2.7. *Product estimate.* Finally, to control the velocity of the vortices, we also need the following result, whose proof is postponed to Appendix A, and which is an ε-quantitative version of the "product estimate" of [SS3].

We let M_ε be a quantity such that

$$\forall q>0,\ \lim_{\varepsilon\to0}M_\varepsilon\varepsilon^q=0,\ \lim_{\varepsilon\to0}\frac{|\log\varepsilon|}{M_\varepsilon^q}=0,\ \lim_{\varepsilon\to0}\frac{\log M_\varepsilon}{|\log\varepsilon|}=0.\tag{4.21}$$

For example $M_\varepsilon=e^{\sqrt{|\log\varepsilon|}}$ will do. In the statement below we do not aim at optimality, however we state the result in a way that would allow to go beyond the range $N_\varepsilon \le O(|\log \varepsilon|)$ that we are considering here (for example up to arbitrary powers of $|\log \varepsilon|$ or quantities that satisfy the properties (4.21)).

Proposition 4.8. *Let $u_\varepsilon : [0,\tau] \times \mathbb{R}^2 \to \mathbb{C}$. Let* v *be a solution to* (1.11) *or* (1.13) *and ϕ be as in* (1.20). *Let $\tilde{V}_\varepsilon$ be as in* (2.12). *Let $X \in C^{0,1}([0,\tau]\times\mathbb{R}^2, \mathbb{R}^2)$ be a spatial vector field. Set*

$$F_\varepsilon = \int_0^\tau \left(\int_{\mathbb{R}^2} |\partial_t u_\varepsilon - i u_\varepsilon N_\varepsilon \phi|^2 + \mathcal{E}_\varepsilon(t) \right) dt,$$

and assume $F_\varepsilon \le M_\varepsilon$. Then for any $\Lambda \ge 1$, we have, as $\varepsilon \to 0$,

$$\begin{aligned} (4.22) \quad & \left| \int_0^\tau \int_{\mathbb{R}^2} \tilde{V}_\varepsilon \cdot X \right| \\ & \le \frac{1 + C\frac{\log M_\varepsilon}{|\log \varepsilon|}}{|\log \varepsilon|} \left(\frac{1}{\Lambda} \int_0^\tau \int_{\mathbb{R}^2} |\partial_t u_\varepsilon - i u_\varepsilon N_\varepsilon \phi|^2 + \Lambda \int_0^\tau \int_{\mathbb{R}^2} |(\nabla u_\varepsilon - i u_\varepsilon N_\varepsilon \mathrm{v}) \cdot X|^2 \right) \\ & + C\|X\|_{L^\infty} \left(\left(\Lambda^3 (1 + \|X\|_{C^{0,1}}) M_\varepsilon^{-1/8} + \varepsilon N_\varepsilon \right) \left(F_\varepsilon + N_\varepsilon^2 \right) + \sqrt{F_\varepsilon \sup_{t\in[0,\tau]} \mathcal{E}_\varepsilon(t) M_\varepsilon^{-1/8}} \right), \end{aligned}$$

where C depends only on the bounds on ϕ and v.

The second line in the right-hand side is $o(1)$, so is $\log M_\varepsilon / |\log \varepsilon|$. One can see that optimizing over Λ by taking

$$\Lambda = \left(\frac{\int_0^\tau \int_{\mathbb{R}^2} |\partial_t u_\varepsilon - i u_\varepsilon N_\varepsilon \phi|^2}{\int_0^\tau \int_{\mathbb{R}^2} |(\nabla u_\varepsilon - i u_\varepsilon N_\varepsilon \mathrm{v}) \cdot X|^2} \right)^{\frac{1}{2}}$$

yields a right-hand side in the form of a product (plus error terms), hence the name “product estimate”.

4.3. **Proof of Theorem 2.**

We now present the main proof.

4.3.1. *Evolution of the modulated energy.*

In the next result, we take as before $\psi = -|\mathrm{v}|^2$ in the definition (2.24), and insert the equation solved by v and (2.21) into (2.25) to obtain the crucial computation.

Lemma 4.9. *Let u_ε solve* (1.10) *and* v *solve* (1.11) *or* (1.13). *Assume that* (1.27) *and* (4.3) *hold. Then, for any $t \le T_\varepsilon$, we have*

$$(4.23) \qquad \mathcal{E}_\varepsilon(u_\varepsilon(t), t) - \mathcal{E}_\varepsilon(u_\varepsilon^0, 0) = I_S + I_V + I_E + I_D + I_d + I_m + I_v + o(1)$$

where

$$I_S = 2\int_0^t \int_{\mathbb{R}^2} \tilde{S}_\varepsilon : \nabla \bar{\mathrm{v}}^\perp$$

$$I_V = -\int_0^t \int_{\mathbb{R}^2} N_\varepsilon \tilde{V}_\varepsilon \cdot \mathrm{v}$$

$$I_E = -\int_0^t \int_{\mathbb{R}^2} 2N_\varepsilon |\mathrm{v}|^2 \tilde{\mu}_\varepsilon$$

$$I_D = -\int_0^t \int_{\mathbb{R}^2} \frac{N_\varepsilon}{|\log\varepsilon|} |\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \phi|^2 - N_\varepsilon \langle \partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \phi, iu_\varepsilon\rangle (\operatorname{div} \mathrm{v} - \frac{N_\varepsilon}{|\log\varepsilon|}\phi) - 2\frac{N_\varepsilon}{|\log\varepsilon|}\langle \partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\phi, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}\rangle \cdot \bar{\mathrm{v}}^\perp$$

$$I_d = \int_0^t \int_{\mathbb{R}^2} 2N_\varepsilon \bar{\mathrm{v}}^\perp \cdot (j_\varepsilon - N_\varepsilon \mathrm{v})(\frac{N_\varepsilon}{|\log\varepsilon|}\phi - \operatorname{div}\mathrm{v})$$

$$I_m = \int_0^t \int_{\mathbb{R}^2} N_\varepsilon^2 (1-|u_\varepsilon|^2)\left(-\partial_t |\mathrm{v}|^2 + \mathrm{v}\cdot\nabla\phi + 2\frac{N_\varepsilon}{|\log\varepsilon|}\phi \mathrm{v}\cdot \bar{\mathrm{v}}^\perp\right)$$

$$I_v = \int_0^t \int_{\mathbb{R}^2} 2N_\varepsilon (j_\varepsilon - N_\varepsilon \mathrm{v})\cdot(\mathrm{v}-\bar{\mathrm{v}}) \operatorname{curl}\mathrm{v} + 2N_\varepsilon(\bar{\mathrm{v}}-\mathrm{v})\cdot \mathrm{v}\tilde{\mu}_\varepsilon.$$

Proof. Again, we start from the result of Lemma 2.5 applied with $\psi = -|\mathrm{v}|^2$. The first step is to write

$$(4.24)\quad -\int_{\mathbb{R}^2} \chi \frac{N_\varepsilon}{|\log\varepsilon|}|\partial_t u_\varepsilon|^2 = -\int_{\mathbb{R}^2} \chi\frac{N_\varepsilon}{|\log\varepsilon|}\left(|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\phi|^2 - N_\varepsilon^2 |u_\varepsilon|^2\phi^2 + 2N_\varepsilon \phi\langle\partial_t u_\varepsilon, iu_\varepsilon\rangle\right).$$

The second step is to use (2.1) to write

$$(4.25)\quad \int_{\mathbb{R}^2} \chi\left(N_\varepsilon^2 \mathrm{v}\cdot\partial_t \mathrm{v} - N_\varepsilon j_\varepsilon \cdot \partial_t \mathrm{v}\right) = \int_{\mathbb{R}^2} \chi N_\varepsilon^2(-2|\mathrm{v}|^2 \operatorname{curl}\mathrm{v} + \mathrm{v}\cdot\nabla\phi) - \int_{\mathbb{R}^2} \chi N_\varepsilon j_\varepsilon \cdot(-2\mathrm{v}\operatorname{curl}\mathrm{v} + \nabla\phi).$$

On the other hand, integrating by parts and using (2.13), we have

$$(4.26)\quad \int_{\mathbb{R}^2} \chi N_\varepsilon (N_\varepsilon \mathrm{v} - j_\varepsilon)\cdot\nabla\phi = -\int_{\mathbb{R}^2} N_\varepsilon \nabla\chi\cdot(N_\varepsilon \mathrm{v} - j_\varepsilon)\phi - \int_{\mathbb{R}^2} N_\varepsilon^2 \chi\left(\operatorname{div}\mathrm{v} - \frac{1}{|\log\varepsilon|}\langle\partial_t u_\varepsilon, iu_\varepsilon\rangle\right)\phi.$$

Next, we would like to transform the linear term $\int_{\mathbb{R}^2}\chi(-2N_\varepsilon^2|\mathrm{v}|^2\operatorname{curl}\mathrm{v} + 2N_\varepsilon j_\varepsilon\cdot \mathrm{v}\operatorname{curl}\mathrm{v})$ into a quadratic term plus error terms. For that, we would like to multiply the result of Lemma 2.4 by $2\mathrm{v}^\perp$, which after integration by parts leads to terms in $\int_{\mathbb{R}^2}\tilde{S}_\varepsilon : \nabla \mathrm{v}^\perp$. Using $\bar{\mathrm{v}}$ rather than v leads instead to integrals that live only *outside of the balls* since $\nabla\bar{\mathrm{v}} = 0$ there by item 1 of Lemma 4.7. These terms will thus be controlled by the energy outside of the balls, i.e. the excess energy, as desired. This creates an additional set of error terms in $\bar{\mathrm{v}} - \mathrm{v}$, which we will control thanks to item 2 in Lemma 4.7.

So as explained, let us multiply the result of Lemma 2.4 by $2\chi\bar{\mathrm{v}}^\perp$, where $\bar{\mathrm{v}}$ is given, for each time t, by the result of Lemma 4.7. This yields

$$\begin{aligned}
&\int_{\mathbb{R}^2} 2\chi\bar{\mathrm{v}}^\perp \cdot \operatorname{div}\tilde{S}_\varepsilon(u_\varepsilon)\\
&= \int_{\mathbb{R}^2} \chi\left(2\frac{N_\varepsilon}{|\log\varepsilon|}\langle \partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\phi, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}\rangle\right)\cdot \bar{\mathrm{v}}^\perp\\
&+ \int_{\mathbb{R}^2} \chi\left(2N_\varepsilon^2|\mathrm{v}|^2 \operatorname{curl}\mathrm{v} - 2N_\varepsilon j_\varepsilon\cdot \mathrm{v})\operatorname{curl}\mathrm{v} - 2N_\varepsilon|\mathrm{v}|^2\mu_\varepsilon\right)\\
&+ \int_{\mathbb{R}^2} \chi\left(2N_\varepsilon^2(\bar{\mathrm{v}}-\mathrm{v})\cdot \mathrm{v}\operatorname{curl}\mathrm{v} - 2N_\varepsilon j_\varepsilon\cdot(\bar{\mathrm{v}}-\mathrm{v})\operatorname{curl}\mathrm{v}\right.\\
&\qquad \left. -2N_\varepsilon(\bar{\mathrm{v}}-\mathrm{v})\cdot \mathrm{v}\mu_\varepsilon\right)\\
&+ \int_{\mathbb{R}^2} \chi\left(2N_\varepsilon^2 \mathrm{v}\cdot\bar{\mathrm{v}}^\perp(\operatorname{div}\mathrm{v} - \frac{N_\varepsilon}{|\log\varepsilon|}|u_\varepsilon|^2\phi) + 2N_\varepsilon j_\varepsilon\cdot\bar{\mathrm{v}}^\perp(\frac{N_\varepsilon}{|\log\varepsilon|}\phi - \operatorname{div}\mathrm{v})\right).
\end{aligned}$$

Inserting (2.12), (4.24), (4.25) and (4.26) into the result of Lemma 2.5 applied with $\psi = -|\mathrm{v}|^2$, taking advantage of the cancellations and using one integration by parts, we obtain

(4.27)

$$\begin{aligned}
&\frac{d}{dt}\hat{\mathcal{E}}_\varepsilon(u_\varepsilon) = -\int_{\mathbb{R}^2}\chi\frac{N_\varepsilon}{|\log\varepsilon|}\left(|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\phi|^2 - N_\varepsilon^2|u_\varepsilon|^2|\phi|^2\right)\\
&- \int_{\mathbb{R}^2}\nabla\chi\cdot\left(\langle\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}, \partial_t u_\varepsilon\rangle + N_\varepsilon(N_\varepsilon\mathrm{v} - j_\varepsilon)\phi - 2\tilde{S}_\varepsilon\bar{\mathrm{v}}^\perp\right)\\
&+ \int_{\mathbb{R}^2}\chi\left(N_\varepsilon\langle\partial_t u_\varepsilon, iu_\varepsilon\rangle\Big(\operatorname{div}\mathrm{v} - \frac{N_\varepsilon}{|\log\varepsilon|}\phi\Big) - N_\varepsilon^2\phi\operatorname{div}\mathrm{v}\right)\\
&+ \int_{\mathbb{R}^2}\chi\left(N_\varepsilon(-\tilde{V}_\varepsilon - N_\varepsilon\mathrm{v}\partial_t|u_\varepsilon|^2 + N_\varepsilon\phi\nabla|u_\varepsilon|^2)\cdot\mathrm{v} + 2\tilde{S}_\varepsilon : \nabla\bar{\mathrm{v}}^\perp\right)\\
&+ \int_{\mathbb{R}^2}\chi\left(2\frac{N_\varepsilon}{|\log\varepsilon|}\langle\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\phi, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}\rangle\cdot\bar{\mathrm{v}}^\perp - 2N_\varepsilon|\mathrm{v}|^2\mu_\varepsilon\right)\\
&+ \int_{\mathbb{R}^2}\chi\left(2N_\varepsilon^2(\bar{\mathrm{v}}-\mathrm{v})\cdot\mathrm{v}\operatorname{curl}\mathrm{v} - 2N_\varepsilon j_\varepsilon\cdot(\bar{\mathrm{v}}-\mathrm{v})\operatorname{curl}\mathrm{v} - 2N_\varepsilon(\bar{\mathrm{v}}-\mathrm{v})\cdot\mathrm{v}\mu_\varepsilon\right)\\
&+ \int_{\mathbb{R}^2}\chi 2N_\varepsilon\bar{\mathrm{v}}^\perp\cdot\left((j_\varepsilon - N_\varepsilon\mathrm{v})(\frac{N_\varepsilon}{|\log\varepsilon|}\phi - \operatorname{div}\mathrm{v}) + \frac{N_\varepsilon}{|\log\varepsilon|}N_\varepsilon(1-|u_\varepsilon|^2)\phi\mathrm{v}\right)\\
&- \int_{\mathbb{R}^2}\chi N_\varepsilon^2\partial_t((1-|u_\varepsilon|^2)|\mathrm{v}|^2).
\end{aligned}$$

Let us make three transformations to this expression. First, let us single out the terms

$$\begin{aligned}
&\int_{\mathbb{R}^2}\chi N_\varepsilon^2(-|\mathrm{v}|^2\partial_t|u_\varepsilon|^2 + \phi\mathrm{v}\cdot\nabla|u_\varepsilon|^2 - \partial_t((1-|u_\varepsilon|^2)|\mathrm{v}|^2)\\
&= \int_{\mathbb{R}^2}\chi N_\varepsilon^2(1-|u_\varepsilon|^2)(-\partial_t|\mathrm{v}|^2 + \nabla\phi\cdot\mathrm{v} + \phi\operatorname{div}\mathrm{v}) + \int_{\mathbb{R}^2}N_\varepsilon^2\nabla\chi\cdot\mathrm{v}(1-|u_\varepsilon|^2)\phi
\end{aligned}$$

(where we have used an integration by parts).

Second, let us replace $\langle \partial_t u_\varepsilon, iu_\varepsilon\rangle$ by $\langle \partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \phi, iu_\varepsilon\rangle + N_\varepsilon |u_\varepsilon|^2\phi$ in (4.27), this leads to a cancellation of the terms in $(1-|u_\varepsilon|^2)\phi \operatorname{div} \mathrm{v}$ and in $|u_\varepsilon|^2\phi^2$.

Third, owing to (4.6) let us replace $-2\int_{\mathbb{R}^2} \chi N_\varepsilon |\mathrm{v}|^2 \mu_\varepsilon$ by

$$-2\int_{\mathbb{R}^2} \chi N_\varepsilon |\mathrm{v}|^2 \tilde{\mu}_\varepsilon + O\left((\|\nabla \chi\|_{L^2} + \|\nabla \mathrm{v}\|_{L^\infty}\|\mathrm{v}\|_{L^2}) N_\varepsilon (\varepsilon \sqrt{\mathcal{E}_\varepsilon(u_\varepsilon)} + \varepsilon^4 N_\varepsilon^2)\right)$$

and the same for $-2\int_{\mathbb{R}^2} N_\varepsilon (\bar{\mathrm{v}} - \mathrm{v}) \cdot \mathrm{v}\mu_\varepsilon$. Both give rise to $o(1)$ error terms by (1.27) and (4.3).

After these substitutions, let us integrate in time and take $\chi = \chi_R$. We may check via Lemmas 2.2 and 2.1, and the fact that $\bar{\mathrm{v}} - \mathrm{v}$ is compactly supported, that all integrands in factor of χ_R or $\nabla \chi_R$ are in L^1. Letting $R \to \infty$, we then get the conclusion. □

We next turn to studying these terms one by one. We will show that I_d, I_m, I_v are all negligible terms, while I_V, I_E and I_D recombine algebraically thanks to the product estimate, to give a term bounded by the energy outside the balls, as does I_S.

4.3.2. *The negligible terms.* Let us start with I_d. In the case $N_\varepsilon \ll |\log \varepsilon|$, from (1.20) we have $\phi = \mathrm{p}$ and we also have $\operatorname{div} \mathrm{v} = 0$. Then, in view of (4.3) and (2.7) we may bound

$$|I_d| \le 2\int_{\mathbb{R}^2} N_\varepsilon \frac{N_\varepsilon}{|\log \varepsilon|} |\bar{\mathrm{v}}||j_\varepsilon - N_\varepsilon \mathrm{v}||\mathrm{p}| \\ \le C\|\bar{\mathrm{v}}\|_{L^\infty}\|\mathrm{p}\|_{L^2} \sqrt{\pi N_\varepsilon |\log \varepsilon| + N_\varepsilon^2}\, \frac{N_\varepsilon^2}{|\log \varepsilon|} = o(N_\varepsilon^2),$$

where we used the boundedness of v hence of $\bar{\mathrm{v}}$ and the L^2 character of p (see Lemma 2.1). In the case $\frac{|\log \varepsilon|}{N_\varepsilon} \to \lambda > 0$ finite (see (1.12)), we have $\phi = \lambda \operatorname{div} \mathrm{v}$ and

$$\frac{N_\varepsilon}{|\log \varepsilon|}\phi - \operatorname{div} \mathrm{v} = (\frac{\lambda N_\varepsilon}{|\log \varepsilon|} - 1) \operatorname{div} \mathrm{v} = o(1)|\operatorname{div} \mathrm{v}|. \tag{4.28}$$

We again conclude easily that $|I_d| \le o(N_\varepsilon^2)$ in that regime too.

The term I_m is easily seen to be $o(N_\varepsilon^2)$, using the Cauchy-Schwarz inequality, (4.3) and Lemma 2.3 and the integrability of $\partial_t \mathrm{v}$, p, ∇p, $\operatorname{div} \mathrm{v}$ and $\nabla \operatorname{div} \mathrm{v}$ provided by Lemma 2.1.

To bound the first terms of I_v we first use (2.8) and the fact that $\mathrm{v} \in C^{1,\gamma}$ to get

(4.29)
$$\left|\int_{\mathbb{R}^2} (j_\varepsilon - N_\varepsilon \mathrm{v}) \cdot (\mathrm{v} - \bar{\mathrm{v}}) \operatorname{curl} \mathrm{v}\right| \le C\|\mathrm{v} - \bar{\mathrm{v}}\|_{L^\infty} \Big(\int_{\mathbb{R}^2} |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}||\operatorname{curl} \mathrm{v}| \\ + |1 - |u_\varepsilon|||\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}| + N_\varepsilon^2 |1 - |u_\varepsilon|^2||\operatorname{curl} \mathrm{v}|\Big)$$

Using the Cauchy-Schwarz inequality, (2.5), (4.3), (1.27), and the L^2 character of $\operatorname{curl} \mathrm{v}$ (by Lemma 2.1, $\operatorname{curl} \mathrm{v} \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$), we find that the last two terms in the right-hand side integral give $o(1)$ terms. To bound the contribution of the first term, we split the integral over $\mathbb{R}^2$ using the balls B_i given by Proposition 4.4 as follows :

$$\int_{\mathbb{R}^2} |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}||\operatorname{curl}\mathrm{v}|$$
$$\leq \int_{\cup_i B_i} |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}||\operatorname{curl}\mathrm{v}| + \int_{\mathbb{R}^2\setminus\cup_i B_i} |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}||\operatorname{curl}\mathrm{v}|$$
$$\leq C\|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}\|_{L^{2,\infty}}|\cup_i B_i| + \left(\int_{\mathbb{R}^2\setminus\cup_i B_i} |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}|^2\right)^{\frac12} \|\operatorname{curl}\mathrm{v}\|_{L^2(\mathbb{R}^2)},$$

where we used (4.12) and the boundedness of $\operatorname{curl}\mathrm{v}$. In view of (4.16), (4.18), (4.3) and item 1 of Proposition 4.4, we deduce that

$$\int_{\mathbb{R}^2} |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}||\operatorname{curl}\mathrm{v}| \leq Ce^{-2\sqrt{N_\varepsilon}}N_\varepsilon + CN_\varepsilon$$

and inserting into (4.29) and using item 2 of Lemma 4.7, we deduce that

$$\left|\int_{\mathbb{R}^2} (j_\varepsilon - N_\varepsilon \mathrm{v})\cdot(\mathrm{v}-\bar{\mathrm{v}})\operatorname{curl}\mathrm{v}\right| \leq o(1).$$

For the second term of I_v we apply item 3 of Lemma 4.5 with $\xi = \mathrm{v} - \bar{\mathrm{v}}$ to obtain

$$\int_{\mathbb{R}^2} (\mathrm{v}-\bar{\mathrm{v}})\tilde{\mu}_\varepsilon = 2\pi\sum_i d_i(\mathrm{v}-\bar{\mathrm{v}})(a_i) + o(1)\|\mathrm{v}-\bar{\mathrm{v}}\|_{C^{0,\gamma}}$$
$$\leq 2\pi\sum_i |d_i|\|\mathrm{v}-\bar{\mathrm{v}}\|_{L^\infty} + o(1)\|\mathrm{v}-\bar{\mathrm{v}}\|_{C^{0,\gamma}} = o(1),$$

where we used item 2 of Lemma 4.7 and the fact that $\sum_i |d_i| \leq CN_\varepsilon$ for these balls. We deduce that $I_v = o(1)$ and conclude that $I_d + I_m + I_v = o(N_\varepsilon^2)$.

4.3.3. *The dominant terms.* The term I_S can easily be treated with the help of (2.17). Using that from Lemma 4.7, $\nabla\bar{\mathrm{v}}$ vanishes outside of the balls B_i given by Proposition 4.4 and is bounded otherwise by a constant depending on v, and using (2.17), (4.16) and Lemma 2.3, we may write for each $t \leq T_\varepsilon$,

$$\left|\int_{\mathbb{R}^2} 2\tilde{S}_\varepsilon : \nabla\bar{\mathrm{v}}^\perp\right|$$
$$\leq 2\|\nabla\bar{\mathrm{v}}\|_{L^\infty}\int_{\mathbb{R}^2\setminus\cup_i B_i} |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}|^2 + \frac{1}{4\varepsilon^2}(1-|u_\varepsilon|^2)^2 + N_\varepsilon^2|\mathrm{v}|^2|1-|u_\varepsilon|^2|$$
$$\leq C(\mathcal{E}_\varepsilon(u_\varepsilon(t)) - \pi N_\varepsilon|\log\varepsilon|) + o(N_\varepsilon^2).$$

We thus conclude that

$$|I_S| \leq C\left(\int_0^t \mathcal{E}_\varepsilon(u_\varepsilon(s)) - \pi N_\varepsilon|\log\varepsilon|\right) + o(N_\varepsilon^2). \tag{4.30}$$

For the term I_D we replace $\bar{\mathrm{v}}^\perp$ by $\mathrm{v}^\perp+(\bar{\mathrm{v}}-\mathrm{v})^\perp$, and using Young's inequality, we write

$$
\begin{aligned}
I_D \le \frac{N_\varepsilon}{|\log\varepsilon|}&\left(-\frac12\int_0^t\int_{\mathbb{R}^2}|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\phi|^2 + 2\int_0^t\int_{\mathbb{R}^2}\left|(\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v})\cdot \mathrm{v}^\perp\right|^2\right)\\
&+\|\bar{\mathrm{v}}-\mathrm{v}\|_{L^\infty}\frac{N_\varepsilon}{|\log\varepsilon|}\left(\int_0^t\int_{\mathbb{R}^2}|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\phi|^2+\int_0^t\int_{\mathbb{R}^2}|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}|^2\right)\\
&\qquad+\int_0^t\int_{\mathbb{R}^2}N_\varepsilon|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\phi||\operatorname{div}\mathrm{v}-\frac{N_\varepsilon}{|\log\varepsilon|}\phi|.
\end{aligned}\tag{4.31}
$$

We claim that

$$
\begin{aligned}
\int_0^t\int_{\mathbb{R}^2}N_\varepsilon|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\phi||\operatorname{div}\mathrm{v}-\frac{N_\varepsilon}{|\log\varepsilon|}\phi|&\\
\le o\left(\frac{N_\varepsilon}{|\log\varepsilon|}\right)\int_0^t\int_{\mathbb{R}^2}&|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\phi|^2+o(N_\varepsilon^2).
\end{aligned}\tag{4.32}
$$

Indeed, either $N_\varepsilon \ll |\log\varepsilon|$, in which case $\operatorname{div}\mathrm{v}=0$ and $\phi=\mathrm{p}$ and the result follows from the Cauchy-Schwarz inequality after inserting a factor

$$\frac{1}{\sqrt{N_\varepsilon}}\left(\frac{N_\varepsilon}{|\log\varepsilon|}\right)^{1/4}\sqrt{N_\varepsilon}\left(\frac{|\log\varepsilon|}{N_\varepsilon}\right)^{1/4}$$

and the L^2 character of p; or $|\log\varepsilon|/N_\varepsilon\to\lambda$ and we may use (4.28) and the L^2 character of $\operatorname{div}\mathrm{v}$ to conclude the same. On the other hand, by (4.3) and item 2 of Lemma 4.7, we have

$$
\begin{aligned}
\|\bar{\mathrm{v}}-\mathrm{v}\|_{L^\infty}\frac{N_\varepsilon}{|\log\varepsilon|}&\left(\int_0^t\int_{\mathbb{R}^2}|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\phi|^2+\int_0^t\int_{\mathbb{R}^2}|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}|^2\right)\\
&\le Ce^{-\sqrt{N_\varepsilon}}\frac{N_\varepsilon}{|\log\varepsilon|}\int_0^t\int_{\mathbb{R}^2}|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\phi|^2+o(1).
\end{aligned}\tag{4.33}
$$

We next distinguish two cases:
Case 1: the case where

$$\int_0^t\int_{\mathbb{R}^2}|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\phi|^2\le 20\int_0^t\int_{\mathbb{R}^2}|(\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v})\cdot\mathrm{v}|^2.$$

By (4.3) this implies that

$$\int_0^t\int_{\mathbb{R}^2}|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\phi|^2\le 20\|\mathrm{v}\|_{L^\infty}(\pi N_\varepsilon|\log\varepsilon|+N_\varepsilon^2).\tag{4.34}$$

It follows, together with (4.3) and (1.27) that $F_\varepsilon\le CN_\varepsilon|\log\varepsilon|$ where F_ε is as in Proposition 4.8. From Lemma 2.1 we have that $\partial_t\mathrm{v}$ is uniformly bounded while $\mathrm{v}\in C^{1,\gamma}$ in space, hence v is Lipschitz in space-time, so we may apply

Proposition 4.8 with $M_\varepsilon = e^{\sqrt{|\log \varepsilon|}}$, $\tau = t$, $\Lambda = 2$ and $X = \mathrm{v}$ to obtain

$$I_V \le \frac{N_\varepsilon}{|\log\varepsilon|}\left(\frac{1}{2}\int_0^t\int_{\mathbb{R}^2}|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\phi|^2 + 2\int_0^t\int_{\mathbb{R}^2}|(\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v})\cdot \mathrm{v}|^2\right) + o(N_\varepsilon^2). \tag{4.35}$$

Case 2: the case where

$$\int_0^t\int_{\mathbb{R}^2}|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\phi|^2 \ge 20\int_0^t\int_{\mathbb{R}^2}|(\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v})\cdot \mathrm{v}|^2.$$

We may rewrite that condition as

$$\begin{aligned}\frac{1}{4}\int_0^t\int_{\mathbb{R}^2}|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\phi|^2 + 4\int_0^t\int_{\mathbb{R}^2}|(\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v})\cdot \mathrm{v}|^2 \\ \le \left(\frac{1}{4}+\frac{1}{10}\right)\int_0^t\int_{\mathbb{R}^2}|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\phi|^2 + 2\int_0^t\int_{\mathbb{R}^2}|(\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v})\cdot \mathrm{v}|^2.\end{aligned} \tag{4.36}$$

We note that in that situation, thanks to Lemma 4.1 and the L^2 character of ϕ, we have $F_\varepsilon \le 2\int_0^t\int_{\mathbb{R}^2}|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\phi|^2 \le CN_\varepsilon^3|\log\varepsilon|^3$, where F_ε is as in Proposition 4.8. Choosing $M_\varepsilon = e^{\sqrt{|\log\varepsilon|}}$ we may then apply that proposition with $M_\varepsilon = e^{\sqrt{|\log\varepsilon|}}$, $\tau = t$, $\Lambda = 4$ and $X = \mathrm{v}$, and combining the result with (4.36), we are led to

$$\begin{aligned} I_V \le \frac{N_\varepsilon}{|\log\varepsilon|}\left(\frac{1}{4}+\frac{1}{10}\right)\int_0^t\int_{\mathbb{R}^2}|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\phi|^2 \\ + \frac{2N_\varepsilon}{|\log\varepsilon|}\int_0^t\int_{\mathbb{R}^2}|(\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v})\cdot \mathrm{v}|^2 + o\left(\frac{N_\varepsilon}{|\log\varepsilon|}\right)\left(\int_0^t\int_{\mathbb{R}^2}|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\phi|^2\right) + o(1).\end{aligned} \tag{4.37}$$

This implies that

$$\begin{aligned} I_V \le \frac{N_\varepsilon}{|\log\varepsilon|}\left(\frac{1}{2}\int_0^t\int_{\mathbb{R}^2}|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\phi|^2 + 2\int_0^t\int_{\mathbb{R}^2}|(\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v})\cdot \mathrm{v}|^2\right) \\ - \frac{1}{8}\frac{N_\varepsilon}{|\log\varepsilon|}\int_0^t\int_{\mathbb{R}^2}|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\phi|^2 + o(1).\end{aligned} \tag{4.38}$$

Returning to the general situation, we may now combine in the first case (4.31), (4.32), (4.33), (4.34) and (4.35), and in the second case (4.31), (4.32), (4.33) and (4.38). Noticing an exact recombination of the terms

$$\begin{aligned} -\frac{1}{2}\int_0^t\int_{\mathbb{R}^2}|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\phi|^2 + 2\int_0^t\int_{\mathbb{R}^2}\left|(\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v})\cdot \mathrm{v}^\perp\right|^2 \\ + \frac{1}{2}\int_0^t\int_{\mathbb{R}^2}|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\phi|^2 + 2\int_0^t\int_{\mathbb{R}^2}|(\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v})\cdot \mathrm{v}|^2,\end{aligned}$$

we obtain in both cases that

$$I_D + I_V \le \frac{2N_\varepsilon}{|\log \varepsilon|} \int_0^t \int_{\mathbb{R}^2} |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}|^2 |\mathrm{v}|^2 + o(N_\varepsilon^2). \tag{4.39}$$

On the other hand, from (4.19) applied with $\xi = |\mathrm{v}|^2$, we have

$$2N_\varepsilon \int_{\mathbb{R}^2} \frac{1}{|\log \varepsilon|} |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}|^2 |\mathrm{v}|^2 - |\mathrm{v}|^2 \tilde{\mu}_\varepsilon \le (\|\mathrm{v}\|_{L^\infty}^2 + 1) \frac{N_\varepsilon}{|\log \varepsilon|} (\mathcal{E}_\varepsilon(u_\varepsilon) - \pi N_\varepsilon |\log \varepsilon|) + o(N_\varepsilon^2),$$

so using that $N_\varepsilon \le O(|\log \varepsilon|)$ and combining with (4.39), we obtain that

$$I_V + I_D + I_E \le C \int_0^t (\mathcal{E}_\varepsilon(u_\varepsilon(s)) - \pi N_\varepsilon |\log \varepsilon|)\, ds + o(N_\varepsilon^2). \tag{4.40}$$

Let us point out that this is the only place in the proof where we really are limited to the situation where $N_\varepsilon \le O(|\log \varepsilon|)$.

4.3.4. *The Gronwall argument.* Combining (4.40) with (4.30) and the result on the negligible terms, we are led to

$$\mathcal{E}_\varepsilon(u_\varepsilon(t)) - \mathcal{E}_\varepsilon(u_\varepsilon^0) \le C \int_0^t (\mathcal{E}_\varepsilon(u_\varepsilon(s)) - \pi N_\varepsilon |\log \varepsilon|)\, ds + o(N_\varepsilon^2)$$

and this holds for any $t \le T_\varepsilon$.

In view of the assumption on the initial data, Gronwall's lemma immediately yields that

$$\mathcal{E}_\varepsilon(u_\varepsilon(t), t) \le \pi N_\varepsilon |\log \varepsilon| + o(N_\varepsilon^2) \tag{4.41}$$

for all $t \le T_\varepsilon$. Thus we must have $T_\varepsilon = \min(1, \mathrm{T})$ and we may extend the argument up to time T to obtain that (4.41) holds until T. This proves the first assertion of Theorem 2.

4.3.5. *The convergence result.* To conclude the proof of Theorem 2, there remains to check that this implies that $\frac{\langle \nabla u_\varepsilon, iu_\varepsilon \rangle}{N_\varepsilon} \to \mathrm{v}$ in $L^p_{loc}(\mathbb{R}^2)$ for $p < 2$. In view of (4.16), (4.41) implies that for every $t \le \mathrm{T}$,

$$\int_{\mathbb{R}^2 \setminus \cup_i B_i} |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}|^2 \le o(N_\varepsilon^2), \tag{4.42}$$

hence for any ball B_R centered at the origin and any $1 \le p < 2$, by Hölder's inequality,

$$\int_{B_R \setminus \cup_i B_i} |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}|^p \le o(N_\varepsilon^p). \tag{4.43}$$

Meanwhile, by (4.18) and using the embedding of $L^{2,\infty}(B_R)$ into $L^q(B_R)$ for any $q < 2$, by Hölder's inequality and item 1 of Proposition 4.4, we deduce that for any $p < q < 2$,

$$\int_{\cup_i B_i} |\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}|^p \le o(1).$$

Combining this with (4.43), we conclude that $\frac{1}{N_\varepsilon}(\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}) \to 0$ in $L^p_{loc}(\mathbb{R}^2)$.

In the case $\frac{|\log \varepsilon|}{N_\varepsilon} \to \lambda$, with (2.5) and the upper bound (4.41), we have in addition that $\frac{1}{N_\varepsilon}(\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v})$ is bounded in $L^2(\mathbb{R}^2)$. It thus has a weak limit f, up to extraction of a subsequence ε_k. As in [SS5, p. 151], letting Ω_ε denote $\cup_i B_i$ for each given ε, since $\sum_i r_i \le e^{-\sqrt{N_\varepsilon}}$, we may extract a further subsequence such that $\mathcal{A}_n := \cup_{k\ge n}\Omega_{\varepsilon_k}$ has Lebesgue measure tending to 0 as $n \to \infty$. For any fixed n, by weak convergence we have

$$\liminf_{k\to\infty} \int_{\mathbb{R}^2\setminus\Omega_{\varepsilon_k}} \frac{|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}|^2}{N_\varepsilon^2} \ge \liminf_{k\to\infty} \int_{\mathbb{R}^2\setminus\mathcal{A}_n} \frac{|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}|^2}{N_\varepsilon^2} \ge \int_{\mathbb{R}^2\setminus\mathcal{A}_n} |f|^2,$$

but the left-hand side is equal to 0 by (4.42), so letting $n \to \infty$, we deduce that f must be 0. Since this is true for any subsequence, we conclude that $\frac{1}{N_\varepsilon}(\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v})$ converges weakly in L^2 to 0.

The appropriate convergence of $j_\varepsilon/N_\varepsilon$ is then deduced by (2.6) (2.7) and (2.8), and this concludes the proof of Theorem 2.

Remark 4.10. *In order to treat the mixed flow or complex Ginzburg-Landau case, the computations are very similar and shown in Appendix C below in the gauge case. One should multiply the result of Lemma 2.4 by* $2\beta\mathrm{v} + 2\alpha\mathrm{v}^\perp$ *instead of* $\mathrm{v}^\perp$*, and use* $\phi = \mathrm{p}$ *or* $\frac{\lambda}{\alpha}\operatorname{div}\mathrm{v}$ *(respectively), and* $\psi = \beta\phi - |\mathrm{v}|^2$*. A supplementary error term in* $O(\beta\int_{\mathbb{R}^2}\tilde{V}_\varepsilon\cdot(\mathrm{v}-\bar{\mathrm{v}}))$ *appears, which can be controlled only by an estimate on* $\int_{\mathbb{R}^2}|\tilde{V}_\varepsilon|$ *and leads to the extra condition* $N_\varepsilon \gg \log|\log\varepsilon|$.

Appendix A. Proof of Proposition 4.8

As already mentioned, the result is a quantitative version of the "product estimate" of [SS3]. It also needs to be adapted to the case of an infinite domain, which we do by a localization procedure based on a partition of unity.

As in [SS3] we view things in three dimensions where the first dimension is time and the last two are spatial dimensions. By analogy with a gauge, we introduce the vector-field in three-space

$$A_\varepsilon = N_\varepsilon(\phi, \mathrm{v}), \tag{A.1}$$

whose first coordinate is $N_\varepsilon\phi$ and whose last two coordinates are those of $N_\varepsilon\mathrm{v}$. Equivalently, we can identify A_ε with a 1-form. We also note that

$$\frac{1}{N_\varepsilon}\operatorname{curl} A_\varepsilon = (\operatorname{curl}\mathrm{v}, \partial_t\mathrm{v}_1 - \partial_1\phi, \partial_t\mathrm{v}_2 - \partial_2\phi). \tag{A.2}$$

We then define the 2-form

$$J_\varepsilon = d\left(\langle du_\varepsilon - iu_\varepsilon A_\varepsilon, iu_\varepsilon\rangle + A_\varepsilon\right), \tag{A.3}$$

where d corresponds to the differential in three-space.

Lemma A.1. *Identifying a spatial vector-field* X *with the 2-form* $X_2 dt\wedge dx_1 + X_1 dt\wedge dx_2$*, we have*

$$J_\varepsilon = \tilde{\mu}_\varepsilon dx_1\wedge dx_2 + \tilde{V}_\varepsilon + (1-|u_\varepsilon|^2)N_\varepsilon(\partial_t\mathrm{v} - \nabla\phi), \tag{A.4}$$

where $\tilde{\mu}_\varepsilon$ *is as in* (2.11) *and* $\tilde{V}_\varepsilon$ *as in* (2.12).

Proof. By definition, $J_\varepsilon = J^t_\varepsilon dx_1 \wedge dx_2 + J^2_\varepsilon dt \wedge dx_1 + J^1_\varepsilon dt \wedge dx_2$, where for $k = 1, 2$,

$$\begin{cases} J^t_\varepsilon = \operatorname{curl}(\langle \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}, iu_\varepsilon\rangle + N_\varepsilon \mathrm{v}) = \tilde{\mu}_\varepsilon \\ J^k_\varepsilon = \partial_t(\langle \partial_k u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_k, iu_\varepsilon\rangle + N_\varepsilon \mathrm{v}_k) - \partial_k(\langle \partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \phi, iu_\varepsilon\rangle + N_\varepsilon \phi) \end{cases}$$

To obtain the expression of J_ε, it thus suffices to compute

$$\begin{aligned} &\partial_t(\langle \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}, iu_\varepsilon\rangle + N_\varepsilon \mathrm{v}) - \nabla(\langle \partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \phi, iu_\varepsilon\rangle + N_\varepsilon \phi) \\ &= \langle \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}, i\partial_t u_\varepsilon\rangle + N_\varepsilon \partial_t \mathrm{v} + \langle \partial_t(\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}), iu_\varepsilon\rangle - \langle \nabla \partial_t u_\varepsilon, iu_\varepsilon\rangle \\ &- \langle \partial_t u_\varepsilon, i\nabla u_\varepsilon\rangle + \nabla((|u_\varepsilon|^2 - 1)N_\varepsilon \phi) \\ &= 2\langle \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}, i\partial_t u_\varepsilon\rangle + N_\varepsilon \mathrm{v}\langle u_\varepsilon, \partial_t u_\varepsilon\rangle + (1 - |u_\varepsilon|^2)N_\varepsilon \partial_t \mathrm{v} - N_\varepsilon \mathrm{v}\langle \partial_t u_\varepsilon, u_\varepsilon\rangle \\ &+ N_\varepsilon \nabla((|u_\varepsilon|^2 - 1)\phi) \\ &= \tilde{V}_\varepsilon - 2N_\varepsilon \phi\langle \nabla u_\varepsilon, u_\varepsilon\rangle + (1 - |u_\varepsilon|^2)N_\varepsilon \partial_t \mathrm{v} + N_\varepsilon \nabla((|u_\varepsilon|^2 - 1)\phi) \end{aligned}$$

where we used (2.12), and this yields the result. □

We work in the space-time slab $[0, \tau] \times \mathbb{R}^2$. We consider X (here a spatial vector field, depending on time) and Y (here $Y = e_t$ the unit vector of the time coordinate) two vector fields on $[0, \tau] \times \mathbb{R}^2$. In order to reduce ourselves to the situation where X is locally constant, we use a partition of unity at a small scale: let M_ε be as in (4.21) and let us consider a covering of $[0, \tau] \times \mathbb{R}^2$ by balls of radius $2M_\varepsilon^{-1/4}$ centered at points of $M_\varepsilon^{-1/4}\mathbb{Z}^3$, and let $\{D_k\}_{k\in\mathbb{N}}$ be an indexation of this sequence of balls and $\{\chi_k\}_{k\in\mathbb{N}}$ a partition of unity associated to this covering (which we observe has bounded overlap) such that $\sum_{k\in\mathbb{N}} \chi_k = 1$ and $\|\nabla \chi_k\|_{L^\infty} \le M_\varepsilon^{1/4}$. For each $k \in \mathbb{N}$, let then X_k be the average of X in D_k. Then, working only in D_k, without loss of generality, we can assume that X_k is aligned with the first space coordinate vector e_1, with (e_t, e_1, e_2) forming an orthonormal frame and the coordinates in that frame being denoted by (t, w, σ). We will assume first that $X_k \neq 0$. Let us define for each k, σ the set

$$\Omega_{k,\sigma} = \{(t, w) | (t, w, \sigma) \in D_k\},$$

which is a slice of D_k (hence a two-dimensional ball). Let us write $J_{\varepsilon,k,\sigma}$ for $J_\varepsilon(e_1, e_t)$ restricted to $\Omega_{k,\sigma}$. In other words, by (A.3), if ξ is a smooth test-function on $\Omega_{\sigma,k}$, we have

$$\int_{\Omega_{k,\sigma}} \xi \wedge J_{\varepsilon,k,\sigma} = -\int_{\Omega_{k,\sigma}} d\xi \wedge (\langle du_\varepsilon - iu_\varepsilon A_\varepsilon, iu_\varepsilon\rangle + A_\varepsilon) \tag{A.5}$$

where d denotes the differential in the slice $\Omega_{k,\sigma}$.

We let g_k be the constant metric on $\Omega_{k,\sigma}$ defined by $g_k(e_1, e_1) = \Lambda/|X_k|$, $g_k(e_t, e_t) = 1/\Lambda$ and $g_k(e_1, e_t) = 0$ with $\Lambda \ge 1$ given.

We then apply the ball construction method in each set $\Omega_{k,\sigma}$. Instead of constructing balls for the flat metric, we construct geodesic balls for the metric associated to g_k, i.e. here, ellipses.

Lemma A.2. *Let $\Omega_{k,\sigma} \subset \mathbb{R}^2$ be as above and denote $\Omega^\varepsilon_{k,\sigma} = \{x \in \Omega_{k,\sigma} | dist\,(x, \partial\Omega_{k,\sigma}) > \varepsilon\}$. Assume that*

$$F_{\varepsilon,k,\sigma} := \frac{1}{2}\int_{\Omega_{k,\sigma}} \Big(|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \phi|^2 + |\partial_1 u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_1|^2 + \frac{1}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 + N_\varepsilon^2(|\nabla\phi|^2 + |\partial_t \mathrm{v}|^2)\Big) \le M_\varepsilon$$

with M_ε as in (4.21). Then if ε is small enough, there exists a finite collection of disjoint closed balls $\{B_i\}$ for the metric g_k of centers a_i and radii r_i such that

(1) $\sum_i r_i \le \Lambda M_\varepsilon^{-1}$
(2) $\cup_i B_i$ *covers* $\{||u_\varepsilon(x)|-1| \ge \frac{1}{2}\} \cap \Omega^\varepsilon_{k,\sigma}$.
(3) *Writing* $d_i = \deg(u_\varepsilon, \partial B_i)$ *if* $B_i \subset \Omega^\varepsilon_{k,\sigma}$ *and* $d_i = 0$ *otherwise, we have for each* i,

$$\frac{1}{2}\int_{B_i} \frac{1}{|X_k|}\Big(\Lambda|X_k|^2|\partial_1 u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_1|^2 + \frac{1}{\Lambda}|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\phi|^2 + 2M_\varepsilon^{-2}N_\varepsilon^2(\Lambda|X_k|^2|\nabla\phi|^2 + \frac{1}{\Lambda}|\partial_t \mathrm{v}|^2)\Big) \ge \pi|d_i|\;\;(|\log\varepsilon| - C\log M_\varepsilon)\,. \tag{A.6}$$

(4) *Letting* $\mu_{\varepsilon,k,\sigma} = 2\pi\sum_i d_i\delta_{a_i}$, *we have for any* $0 < \gamma \le 1$, *and any* $\xi \in C_c^{0,\gamma}(\Omega_{k,\sigma})$,

$$\left|\int \xi \wedge J_{\varepsilon,k,\sigma} - \xi\mu_{\varepsilon,k,\sigma}\right| \le C\|\xi\|_{C^{0,\gamma}}(\Lambda M_\varepsilon^{-2})^\gamma F_{\varepsilon,k,\sigma}.$$

Proof. The first 3 items are a rewriting of [SS3, Proposition IV. 2], itself based on the ball construction, that needs to be adapted to the case of the nonstandard metric. As in [SS3, Proposition IV. 2] we start by noting, via the co-area formula, that there exists m_ε with $\frac{1}{2}M_\varepsilon^{-1} \le m_\varepsilon \le M_\varepsilon^{-1}$ such that setting $\omega := \{|u_\varepsilon| \le 1 - m_\varepsilon\}$ has perimeter for the g_k metric bounded by $C\varepsilon M_\varepsilon^3$. We may then apply [SS5, Proposition 4.3] outside that set to $v_\varepsilon(t,w) = \frac{u_\varepsilon}{|u_\varepsilon|}(\sqrt{\Lambda}|X_k|we_1 + \frac{1}{\sqrt{\Lambda}}te_t)$ and $\tilde{A}_\varepsilon(t,w) = A_\varepsilon(\sqrt{\Lambda}|X_k|we_1 + \frac{1}{\sqrt{\Lambda}}te_t)$ restricted to the slice, with initial radius $r_0 = C\varepsilon M_\varepsilon^3$ and final radius $r_1 = M_\varepsilon^{-1}$. This yields a collection of disjoint closed balls $\bar{B}_i$ with sum of radii bounded by M_ε^{-1} and such that

$$\frac{1}{2}\int_{\cup_i \bar{B}_i \setminus \omega} |\nabla v_\varepsilon|^2 + M_\varepsilon^{-2}|\operatorname{curl}\tilde{A}_\varepsilon|^2 \ge \pi\sum_i |d_i|(|\log\varepsilon| - C\log M_\varepsilon).$$

Making the change of variables $x = \sqrt{\Lambda}|X_k|w$ and $s = \frac{t}{\sqrt{\Lambda}}$, we obtain balls B_i, the images of the $\bar{B}_i$'s by the change of variable, which are geodesic balls for the metric g_k and whose sum of the radii is bounded by $\Lambda M_\varepsilon^{-1}$ (since $\Lambda \ge 1$); and inserting (A.2) and using (4.21) we obtain (A.6). We note that the fact that the domain size also depends on ε does not create any problem in applying that proof.

Item 4 is a consequence of Lemma 4.3 adapted to the present setting with differential forms, replacing $\nabla u_\varepsilon - iu_\varepsilon A_\varepsilon$ by $du_\varepsilon - iu_\varepsilon A_\varepsilon$ and using again (A.2).

□

We now proceed as in the proof of [SS3]. We set $\nu_{\varepsilon,k,\sigma}$ to be the $\mu_{\varepsilon,k,\sigma}$ of Lemma A.2 (item 4) if the assumption $F_{\varepsilon,k,\sigma} \le M_\varepsilon$ is verified, and 0 if not. We note that

$$\|J_{\varepsilon,k,\sigma} - \nu_{\varepsilon,k,\sigma}\|_{(C^{0,1}_c(\Omega_{k,\sigma}))'} \le C\Lambda M_\varepsilon^{-1/2} F_{\varepsilon,k,\sigma} \tag{A.7}$$

is true in all cases. Indeed, either $F_{\varepsilon,k,\sigma} \le M_\varepsilon$ in which case the result is true by item 4 of Lemma A.2 since $M_\varepsilon^{-1/2} \ge M_\varepsilon^{-2}$, or $\nu_{\varepsilon,k,\sigma} = 0$ in which case, for any $\xi \in C^{0,1}_c(\Omega_{k,\sigma})$, starting from (A.5) and writing $|\langle \nabla u_\varepsilon - iA_\varepsilon u_\varepsilon, iu_\varepsilon\rangle| \le |\nabla u_\varepsilon - iA_\varepsilon u_\varepsilon| + ||u_\varepsilon| - 1||\nabla u_\varepsilon - iA_\varepsilon u_\varepsilon|$, we obtain with the Cauchy-Schwarz inequality, using the boundedness of $\Omega_{k,\sigma}$,

$$\begin{aligned}\left|\int \xi \wedge J_{\varepsilon,k,\sigma}\right| &\le C\|\nabla\xi\|_{L^\infty} \int_{\Omega_{k,\sigma}} |\nabla u_\varepsilon - iu_\varepsilon A_\varepsilon| + |A_\varepsilon| + |1 - |u_\varepsilon|^2| \\ &\le C\|\nabla\xi\|_{L^\infty} \sqrt{F_{\varepsilon,k,\sigma}} + \varepsilon F_{\varepsilon,k,\sigma} \le M_\varepsilon^{-1/2} F_{\varepsilon,k,\sigma},\end{aligned}$$

for ε small enough. But since $F_{\varepsilon,k,\sigma} \ge M_\varepsilon$, we have $\sqrt{F_{\varepsilon,k,\sigma}} + \varepsilon F_{\varepsilon,k,\sigma} \le 2M_\varepsilon^{-1/2} F_{\varepsilon,k,\sigma}$ and thus we find that (A.7) holds as well. By (A.6), we also have that

$$\int |\nu_{\varepsilon,k,\sigma}| \le \frac{C}{|\log\varepsilon|} \max\left(\frac{1}{\Lambda|X_k|}, \Lambda|X_k|\right) F_{\varepsilon,k,\sigma}. \tag{A.8}$$

Next, we choose η a function depending only on time, vanishing at 0 and τ, such that $\eta = 1$ in $[M_\varepsilon^{-1/4}, \tau - M_\varepsilon^{-1/4}]$ if that interval is not empty, and affine otherwise. By construction

$$\|\eta\|_{C^{0,1}} \le CM_\varepsilon^{1/4}, \quad \|\eta\chi_k\|_{C^{0,1}} \le CM_\varepsilon^{1/4}. \tag{A.9}$$

We may now write that

$$\begin{aligned}\int_{\Omega_{k,\sigma}} \frac{\eta\chi_k}{|X_k|} \Big(\Lambda|X_k|^2|\partial_1 u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_1|^2 + \frac{1}{\Lambda}|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \phi|^2 \\ + 2M_\varepsilon^{-2} N_\varepsilon^2 (\Lambda|X_k|^2|\nabla\phi|^2 + \frac{1}{\Lambda}|\partial_t \mathrm{v}|^2)\Big) \\ \ge (|\log\varepsilon| - C\log M_\varepsilon) \int_{\Omega_{k,\sigma}} (\eta\chi_k - C\Lambda^2 M_\varepsilon^{1/4-1})\nu_{\varepsilon,k,\sigma}.\end{aligned} \tag{A.10}$$

Indeed, if we are in a slice where $\nu_{\varepsilon,k,\sigma} = 0$, this is trivially true. If not, we apply (A.6) and obtain

$$\begin{aligned}\frac{1}{2}\int_{\Omega^\varepsilon_{k,\sigma}\cap(\cup_i B_i)} \frac{\eta\chi_k}{|X_k|} \Big(\Lambda|X_k|^2|\partial_1 u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_1|^2 + \frac{1}{\Lambda}|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \phi|^2 \\ + 2M_\varepsilon^{-2} N_\varepsilon^2 (\Lambda|X_k|^2|\nabla\phi|^2 + \frac{1}{\Lambda}|\partial_t \mathrm{v}|^2)\Big) \\ \ge 2\pi \sum_i |d_i| \min_{B_i}(\eta\chi_k)(|\log\varepsilon| - C\log M_\varepsilon)).\end{aligned}$$

Inserting then $\min_{B_i} \eta\chi_k \ge (\eta\chi_k)(a_i) - C\Lambda r_i \|\eta\chi_k\|_{C^{0,1}}$ and (A.9), and using item 1 of Lemma A.2, yields (A.10).

Next, we integrate (A.10) with respect to σ, and combine with (A.7) and (A.8) to obtain

$$\begin{aligned}
&\int_{D_k} \frac{\eta\chi_k}{|X_k|}\Big(\Lambda|X_k|^2|\partial_1 u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_1|^2 + \frac{1}{\Lambda}|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\phi|^2\\
&+ 2M_\varepsilon^{-2}N_\varepsilon^2(\Lambda|X_k|^2|\nabla\phi|^2 + \frac{1}{\Lambda}|\partial_t \mathrm{v}|^2)\Big)\\
&\geq (|\log\varepsilon| - C\log M_\varepsilon)\int_{\mathbb{R}^2\times[0,\tau]} \eta\chi_k J_{\varepsilon,k,\sigma}\\
&- C\max(\frac{1}{\Lambda|X_k|}, \Lambda|X_k|)\Lambda^2 M_\varepsilon^{1/4-1}F_{\varepsilon,k} + CM_\varepsilon^{1/4}|\log\varepsilon|\Lambda M_\varepsilon^{-1/2}F_{\varepsilon,k},
\end{aligned} \tag{A.11}$$

with $= F_{\varepsilon,k} := \int F_{\varepsilon,k,\sigma}d\sigma$. Moreover, by (A.4), (since we assumed X_k is along the direction e_1) we have

$$J_{\varepsilon,k,\sigma} = J_\varepsilon(e_1, e_t) = \frac{1}{|X_k|}\Big(\tilde{V}_\varepsilon\cdot X_k + N_\varepsilon(1-|u_\varepsilon|^2)(\partial_t \mathrm{v} - \nabla\phi)\cdot X_k\Big)$$

so we may bound

$$\begin{aligned}
&\int_{D_k} \eta\chi_k N_\varepsilon(1-|u_\varepsilon|^2)(\partial_t \mathrm{v} - \nabla\phi)\cdot X_k\\
&\qquad\leq C\|X\|_{L^\infty}N_\varepsilon\|1-|u_\varepsilon|^2\|_{L^2(D_k)}\|\partial_t \mathrm{v} - \nabla\phi\|_{L^2(D_k)}\\
&\qquad\qquad\leq C\|X\|_{L^\infty}N_\varepsilon\varepsilon F_{\varepsilon,k}.
\end{aligned}$$

Inserting into (A.11) and multiplying by $|X_k|$, we may write

$$\begin{aligned}
&\int_{D_k} \eta\chi_k\Big(\Lambda|X_k|^2|\partial_x u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}|^2 + \frac{1}{\Lambda}|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\phi|^2\\
&+ 2M_\varepsilon^{-2}N_\varepsilon^2(\Lambda|X_k|^2|\nabla\phi|^2 + \frac{1}{\Lambda}|\partial_t \mathrm{v}|^2)\Big)\\
&\geq (|\log\varepsilon| - C\log M_\varepsilon)\int_{D_k} \eta\chi_k\tilde{V}_\varepsilon\cdot X_k\\
&- \max(\frac{1}{\Lambda}, \Lambda|X_k|^2)\Lambda^2 M_\varepsilon^{-3/4} + (\Lambda M_\varepsilon^{-1/4}|\log\varepsilon| + \varepsilon N_\varepsilon)\|X\|_{L^\infty}F_{\varepsilon,k},
\end{aligned}$$

and we note that this holds as well if $X_k = 0$. We may next replace X_k by X in the left-hand side and the $\int \tilde{V}_\varepsilon\cdot X_k$ term, and using that $|X - X_k| \leq CM_\varepsilon^{-1/4}\|X\|_{C^{0,1}}$ in D_k, the error thus created is bounded above by

$$M_\varepsilon^{-1/4}\Lambda|\log\varepsilon|\|X\|_{L^\infty}(1+\|X\|_{C^{0,1}})F_{\varepsilon,k}$$

where we have used that by definition of $\tilde{V}_\varepsilon$, we have $\int_{D_k}|\tilde{V}_\varepsilon|\le F_{\varepsilon,k}$. We may thus absorb this error into the others and write (since $\Lambda\ge 1$)

$$\begin{aligned}&\int_{D_k}\eta\chi_k\Big(\Lambda|X|^2|\partial_x u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}|^2+\frac{1}{\Lambda}|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\phi|^2\Big)\\&+2M_\varepsilon^{-2}N_\varepsilon^2(\Lambda|X|^2|\nabla\phi|^2+\frac{1}{\Lambda}|\partial_t \mathrm{v}|^2)\Big)\\&\ge(|\log\varepsilon|-C\log M_\varepsilon)\int_{D_k}\eta\chi_k\tilde{V}_\varepsilon\cdot X\\&-C\|X\|_{L^\infty}\left(\Lambda^3(1+\|X\|_{C^{0,1}})M_\varepsilon^{-1/8}+\varepsilon N_\varepsilon\right)F_{\varepsilon,k}.\end{aligned}$$

Summing over k, using that $\sum_k\chi_k=1$ in $[0,\tau]\times\mathbb{R}^2$, the finite overlap of the covering, the fact that $\partial_t\mathrm{v}$ and $\nabla\phi\in L^2([0,\tau],L^2(\mathbb{R}^2))$ by Lemma 2.1 and (2.3), we are led to

$$\begin{aligned}\int_0^\tau\int_{\mathbb{R}^2}\eta\left(\Lambda|X|^2|\partial_x u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}|^2+\frac{1}{\Lambda}|\partial_t u_\varepsilon - iu_\varepsilon\phi|^2\right)&\\ \ge(|\log\varepsilon|-C\log M_\varepsilon)\int_0^\tau\int_{\mathbb{R}^2}\eta\tilde{V}_\varepsilon\cdot X&\\ -C\|X\|_{L^\infty}\left(\Lambda^3(1+\|X\|_{C^{0,1}})M_\varepsilon^{-1/8}+\varepsilon N_\varepsilon\right)\left(F_\varepsilon+N_\varepsilon^2\right).&\end{aligned}$$

Moreover, by choice of η and definition of $\tilde{V}_\varepsilon$, we have

$$\begin{aligned}&\left|\int_0^\tau\int_{\mathbb{R}^2}(1-\eta)X\cdot\tilde{V}_\varepsilon\right|\\&\le 2\|X\|_{L^\infty}\int_{[0,M_\varepsilon^{-1/4}]\cup[\tau-M_\varepsilon^{-1/4},\tau]}\int_{\mathbb{R}^2}|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\phi||\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon\mathrm{v}|\\&\le C\|X\|_{L^\infty}\sqrt{F_\varepsilon\Big(\sup_{t\in[0,\tau]}\mathcal{E}_\varepsilon(t)+o(1)\Big)}M_\varepsilon^{-1/8}.\end{aligned}$$

where we have used the Cauchy-Schwarz inequality and (2.5). Combining the last two relations, we deduce the desired result.

Appendix B. Existence and uniqueness for (1.14)

We recall that the definitions of Hölder spaces that we use are at the beginning of Section 2. For the sake of generality we study (1.14) with arbitrary $\alpha>0$ and $\beta\ge 0$ such that $\alpha^2+\beta^2=1$, and $\lambda>0$. We denote by $\omega=\operatorname{curl}\mathrm{v}$ and $d=\operatorname{div}\mathrm{v}$. We note that if v solves (1.13) then (ω,d) solve

$$\begin{cases}\partial_t\omega=2\operatorname{div}(\beta\mathrm{v}\omega+\alpha\mathrm{v}^\perp\omega)\\ \partial_t d=\dfrac{\lambda}{\alpha}\Delta d+2\operatorname{div}(\beta\mathrm{v}^\perp\omega-\alpha\mathrm{v}\omega).\end{cases}\tag{B.1}$$

Theorem 3. *Assume* $\lambda>0$, $\alpha>0$ *and* $\alpha^2+\beta^2=1$. *Assume* $\mathrm{v}(0)$ *is such that* $\frac{1}{2\pi}\omega(0)$ *is a probability measure which also belongs to* $C^\gamma(\mathbb{R}^2)$, *and* $d(0)\in C^\gamma\cap L^p(\mathbb{R}^2)$, *for some* $0<\gamma\le 1$ *and some* $1\le p<2$. *Then there exists a unique local in time solution to* (1.13) *on some interval* $[0,T]$, $T>0$, *which is* *such that* $\mathrm{v} \in L^\infty([0,T], C^{1+\gamma'})$ *for any* $\gamma' < \gamma$. *Moreover, we have* $\mathrm{v}(t) - \mathrm{v}(0) \in L^\infty([0,T], L^2(\mathbb{R}^2))$, $\partial_t \mathrm{v} \in L^2([0,T], L^2(\mathbb{R}^2))$, $d \in L^\infty([0,T], L^p(\mathbb{R}^2))$ *and* $\frac{1}{2\pi}\omega(t)$ *is a probability measure for every* $t \in [0,T]$.

We start with some preliminary results.

Lemma B.1. *Let* v *be a vector field in* $L^\infty([0,T], C^{1,\gamma}(\mathbb{R}^2))$, $\omega_0 \in C^\gamma$, $0 < \gamma \le 1$, $f \in L^\infty(\mathbb{R}, C^\gamma)$ *then the equation*

$$\left\{ \begin{array}{l} \partial_t \omega = \operatorname{div}(v\omega) + f \\ \omega(0) = \omega_0 \end{array} \right. \tag{B.2}$$

has a unique solution, and it holds that for some $C > 0, C_0 > 1$,

$$\|\omega(t)\|_{L^1(\mathbb{R}^2)} \le \|\omega_0\|_{L^1(\mathbb{R}^2)} + \exp\left(C \int_0^t \|\nabla v(s)\|_{L^\infty} \, ds \right) \int_0^t \|f(s)\|_{L^\infty} \, ds \tag{B.3}$$

and for any $-1 < \sigma \le \gamma$,

(B.4)

$$\|\omega(t)\|_{C^\sigma} \le C_0 \left(\|\omega_0\|_{C^\sigma} + \int_0^t \|f(s)\|_{C^\sigma} \, ds + \|\omega_0\|_{L^\infty} \int_0^t \|\operatorname{div} v(s)\|_{C^\sigma} \, ds \right) \times \exp\left(C \int_0^t \|\nabla v(s)\|_{L^\infty} \, ds \right).$$

Proof. One may rewrite the equation as

$$\partial_t \omega = v \cdot \nabla \omega + \omega \operatorname{div} v + f.$$

Then, by propagation along characteristics, we obtain first

$$\|\omega(t)\|_{L^1} \le \|\omega_0\|_{L^1} + \left(\int_0^t \|f(s)\|_{L^\infty} \, ds \right) \exp \int_0^t \|\operatorname{div} v(s)\|_{L^\infty} \, ds,$$

second,

$$\|\omega(t)\|_{L^\infty} \le \left(\|\omega_0\|_{L^\infty} + \int_0^t \|f(s)\|_{L^\infty} \, ds \right) \exp \int_0^t \|\operatorname{div} v(s)\|_{L^\infty} \, ds,$$

and third (B.4) for $\sigma = \gamma$ follows by a Gronwall argument. For general $\sigma \le \gamma$, one can proceed as in [BCD, Chap. 3].

□

The next lemma about the regularizing effect of the heat equation can be found in [Ch2, Proposition 2.1] (applied with $p = \infty$, $\rho = \infty$ and noting that B^s_∞ is the same as C^s).

Lemma B.2. *If* $g \in L^\infty([0,T], C^{-1+\gamma} \cap H^{-1}(\mathbb{R}^2))$ *and* $u_0 \in C^\gamma(\mathbb{R}^2)$, *then the equation*

$$\left\{ \begin{array}{l} \partial_t u = \nu \Delta u + g \\ u(0) = u_0 \end{array} \right. \tag{B.5}$$

has a unique solution which is in $L^\infty([0,T], C^\gamma \cap L^2(\mathbb{R}^2))$ *and* $L^2([0,T], H^1(\mathbb{R}^2))$ *and for any* $\sigma \le \gamma$,

$$\|u\|_{L^\infty([0,T],C^\sigma)} \le C_0 \left(\|u_0\|_{C^\sigma} + \sqrt{\frac{T}{\nu}} \|g\|_{L^\infty([0,T],C^{-1+\sigma})} \right). \tag{B.6}$$

We note that the fact that $u \in L^2([0,T], H^1(\mathbb{R}^2))$ comes from the fact that $g \in L^2([0,T], H^{-1}(\mathbb{R}^2))$ and the regularizing effect of the heat equation.

Lemma B.3. *Assume u solves in $[0,T]$*

$$\begin{cases} \partial_t u = \Delta u + \operatorname{div} f \\ u(0) = u_0 \end{cases} \tag{B.7}$$

then, if $q \le p$ and $\frac{1}{p} - \frac{1}{q} + \frac{1}{2} \ge 0$, we have for any $t \in [0,T]$,

$$\|u(t)\|_{L^p(\mathbb{R}^2)} \le \|u_0\|_{L^p} + C_{p,q} t^{\frac{1}{p}-\frac{1}{q}+\frac{1}{2}} \|f\|_{L^\infty([0,T],L^q(\mathbb{R}^2))}.$$

Proof. We follow [Du, Lemma 2.3]. Using Duhamel's formula, we may write

$$u(t,\cdot) = G(t,\cdot) * u_0 + w(t,\cdot)$$

with

$$w(t,\cdot) = \int_0^t \int \Gamma_{t-s}(x-y) f(s,y)\, ds\, dy$$

where $G(t,x) = \frac{e^{\frac{-|x|^2}{4t}}}{4\pi t}$ and $\Gamma_t(x) = -\partial_x G(t,x) = \frac{x}{8\pi t^2} e^{\frac{-|x|^2}{4t}}$. By Young's inequality for convolutions, we have

$$\begin{aligned} \|u(t)\|_{L^p(\mathbb{R}^2)} &\le \|u_0\|_{L^p(\mathbb{R}^2)} \|G(t,\cdot)\|_{L^1(\mathbb{R}^2)} + \|w(t)\|_{L^p(\mathbb{R}^2)} \\ &\le \|u_0\|_{L^p(\mathbb{R}^2)} + \|w(t)\|_{L^p(\mathbb{R}^2)}. \end{aligned}$$

We turn to studying w. We may write with Hölder's inequality,

$$|w(t,x)| \le \int_0^t \left(\int \Gamma_{t-s}(x-y)^{\frac{q'}{2}}\, dy \right)^{\frac{1}{q'}} \left(\int \Gamma_{t-s}(x-y)^{\frac{q}{2}} |f(s,y)|^q\, dy \right)^{\frac{1}{q}} ds$$

with q' such that $1/q + 1/q' = 1$, and hence with Hölder's inequality again, if $q \le p$,

$$\begin{aligned} &\|w(t)\|_{L^p(\mathbb{R}^2)} \\ &\quad \le \int_0^t \left(\int \Gamma_{t-s}^{\frac{q'}{2}} \right)^{\frac{1}{q'}} \left(\int \left(\int \Gamma_{t-s}(x-y)^{\frac{q}{2}} |f(s,y)|^q\, dy \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} ds \\ &\le \int_0^t (\int \Gamma_{t-s}^{\frac{q'}{2}}\, dy)^{\frac{1}{q'}} \left(\int (\int \Gamma_{t-s}(x-y)^{\frac{p}{2}} |f(s,y)|^q\, dy) (\int |f(s,y)|^q\, dy)^{\frac{p}{q}-1} dx \right)^{\frac{1}{p}} ds \\ &\quad \le \int_0^t \left(\int \Gamma_{t-s}^{\frac{q'}{2}} \right)^{\frac{1}{q'}} \left(\int |f(s,y)|^q\, dy \right)^{\frac{1}{q}-\frac{1}{p}} \left(\int \Gamma_{t-s}^{\frac{p}{2}} \int |f(s,y)|^q\, dy \right)^{\frac{1}{p}} ds \\ &\qquad = \int_0^t \left(\int \Gamma_{t-s}^{\frac{q'}{2}} \right)^{\frac{1}{q'}} \left(\int |f(s,y)|^q\, dy \right)^{\frac{1}{q}} \left(\int \Gamma_{t-s}^{\frac{p}{2}} \right)^{\frac{1}{p}} ds \end{aligned}$$

where for the passage from the second to third line we used Young's inequality for convolutions. We may thus write

$$\|w(t)\|_{L^p(\mathbb{R}^2)} \le \|f\|_{L^\infty([0,T],L^q(\mathbb{R}^2))} \int_0^t \|\Gamma_s\|_{L^{q'/2}(\mathbb{R}^2)}^{1/2} \|\Gamma_s\|_{L^{p/2}(\mathbb{R}^2)}^{\frac{1}{2}}\, ds.$$

But computing explicitly, we find $\|\Gamma_s\|_{L^r(\mathbb{R}^2)} = C_r s^{-3/2+1/r}$ and so we deduce that

$$\|w(t)\|_{L^p(\mathbb{R}^2)} \le C_{p,q} \|f\|_{L^\infty([0,T],L^q(\mathbb{R}^2))} t^{\frac{1}{2}+\frac{1}{p}-\frac{1}{q}},$$

and the result follows. □

Finally, we will need the following potential estimates:

Lemma B.4. *Let $u \in C^\gamma \cap L^p(\mathbb{R}^2)$ with $0 < \gamma \le 1$ and $1 \le p < 2$. Then $\nabla\Delta^{-1}u$, where Δ^{-1} is meant as the convolution with $-\frac{1}{2\pi}\log$, is well defined and*

$$\|\nabla\Delta^{-1}u\|_{C^{1,\gamma}(\mathbb{R}^2)} \le C_1(\|u\|_{C^\gamma(\mathbb{R}^2)} + \|u\|_{L^p(\mathbb{R}^2)}). \tag{B.8}$$

Proof. Setting $v = \nabla\Delta^{-1}u$ we may write

$$v(x) = -\frac{1}{2\pi}\int \frac{x-y}{|x-y|^2} u(y)\, dy$$

and one may check that this integral is convergent with

(B.9)

$$|v(x)| \le C\|u\|_{L^\infty(\mathbb{R}^2)} \int_{|y-x|\le 1} \frac{1}{|x-y|}\, dy + \|u\|_{L^p} \left(\int_{|y-x|\ge 1} \frac{1}{|x-y|^{p'}}\right)^{1/p'}$$
$$\le C(\|u\|_{L^\infty(\mathbb{R}^2)} + \|u\|_{L^p(\mathbb{R}^2)})$$

with $1/p + 1/p' = 1$. Let then w be such that $v = \nabla w$, and thus $\Delta w = u$. For any $x \in \mathbb{R}^2$, by Schauder estimates for elliptic equations and (B.9), we have

$$\begin{aligned}\|\nabla w\|_{C^{1,\gamma}(B(x,1))} &\le C(\|\nabla w\|_{L^\infty(B(x,2))} + \|u\|_{C^\gamma(B(x,2))}) \\ &\le C(\|u\|_{L^p(\mathbb{R}^2)} + \|u\|_{C^\gamma(\mathbb{R}^2)}).\end{aligned}$$

Taking the sup over x yields the result. □

We now turn to the proof of Theorem 3 for which we set up an iterative scheme.

Let $\mathrm{v}_0 = \mathrm{v}(0)$ and $\omega_0 = \operatorname{curl} \mathrm{v}_0, d_0 = \operatorname{div} \mathrm{v}_0$. Given v_n and $\omega_n = \operatorname{curl} \mathrm{v}_n, d_n = \operatorname{div} \mathrm{v}_n$, we want to solve

$$\begin{cases} \partial_t \omega_{n+1} = 2\operatorname{div}((\beta \mathrm{v}_n + \alpha \mathrm{v}_n^\perp)\omega_{n+1}) \\ \partial_t d_{n+1} = \frac{\lambda}{\alpha}\Delta d_{n+1} + 2\operatorname{div}((\beta \mathrm{v}_n^\perp - \alpha \mathrm{v}_n)\omega_n) \\ \omega_{n+1}(0) = \omega(0) \\ d_{n+1}(0) = d(0). \end{cases} \tag{B.10}$$

Let

$$t_n = \sup\{t, K_n(t) \le 2C_1C_0K_0\}$$

where

$$K_n(t) = \|\omega_n(t)\|_{C^\gamma} + \|\omega_n(t)\|_{L^1} + \|d_n(t)\|_{C^\gamma} + \|d_n(t)\|_{L^p} + \|\mathrm{v}_n(t)\|_{C^{1,\gamma}}$$

$$K_0 = \|\omega(0)\|_{L^1(\mathbb{R}^2)} + \|\omega(0)\|_{C^\gamma} + \|d(0)\|_{L^p} + \|d(0)\|_{C^\gamma},$$

and C_0 is the maximum of the constants in (B.4), (B.6), and C_1 the maximum of the constant in (B.8) and 1.

Let us show by induction that (B.10) is solvable and there exists $T_0 > 0$ independent of n such that $t_n > T_0$ for all n.

For $n = 0$, this statement is true by assumption. Assume it is true for n. Then in view of Lemma B.1, we have that the equation (B.10) for $n+1$ has a solution $\omega_{n+1} \in L^\infty([0,T], C^\gamma \cap L^1)$, with, if $T \le T_0 < t_n$,

(B.11)

$$\begin{aligned}
&\|\omega_{n+1}\|_{L^\infty([0,T],C^\gamma)} \\
&\quad \le C_0\Big(\|\omega(0)\|_{C^\gamma} + \|\omega(0)\|_{L^\infty} \int_0^T \|\mathrm{v}_n\|_{C^{1,\gamma}}(s)\,ds\Big) \exp\left(C\int_0^T \|\mathrm{v}_n\|_{C^1}(s)\,ds\right) \\
&\quad \le C_0\left(\|\omega(0)\|_{C^\gamma} + \|\omega(0)\|_{L^\infty} 2C_1C_0TK_0\right) \exp\left(2C_1C_0CTK_0\right).
\end{aligned}$$

Also it is straightforward to see by integrating the equation and since $\omega_{n+1} \in L^1$ by (B.3) that $\frac{1}{2\pi}\omega_{n+1}$ remains a probability measure.

Similarly, Lemma B.2 yields the existence of a solution $d_{n+1} \in L^\infty([0,T], C^\gamma)$ with, if $T \le T_0 < t_n$,

$$\text{(B.12)} \quad \begin{aligned} \|d_{n+1}\|_{L^\infty([0,T],C^\gamma)} &\le C_0(\|d(0)\|_{C^\gamma} + C\sqrt{T}\|\mathrm{v}_n\omega_n\|_{C^\gamma}) \\ &\le C_0(\|d(0)\|_{C^\gamma} + C\sqrt{T}C_1C_0^3K_0^2), \end{aligned}$$

where C depends only on α and λ; and by Lemma B.3 applied with $q = 1$,

$$\text{(B.13)} \quad \begin{aligned} \|d_{n+1}\|_{L^\infty([0,T],L^p)} &\le \|d(0)\|_{L^p} + CT^{\frac1p - \frac12}\|\mathrm{v}_n\omega_{n+1}\|_{L^\infty([0,T],L^1)} \\ &\le \|d(0)\|_{L^p} + CT^{\frac1p-\frac12}C_0C_1K_0. \end{aligned}$$

We then let

$$\text{(B.14)} \qquad \mathrm{v}_{n+1} = \nabla\Delta^{-1}d_{n+1} - \nabla^\perp\Delta^{-1}\omega_{n+1}.$$

By Lemma B.4, this is well-defined and

$$\text{(B.15)} \qquad \|\mathrm{v}_{n+1}\|_{C^{1,\gamma}} \le C_1\left(\|d_{n+1}\|_{C^\gamma} + \|d_{n+1}\|_{L^p} + \|\omega_{n+1}\|_{C^\gamma} + \|\omega_{n+1}\|_{L^1}\right).$$

In view of (B.11), (B.12), (B.13), and (B.15) we then deduce that if T_0 is chosen small enough (depending on K_0 and the various constants), then we will find that $t_{n+1} \ge T_0$. The desired result is thus proved by induction.

Let us now show that $\{\omega_n\}$ and $\{d_n\}$ are Cauchy sequences in $C^{-1+\gamma}$. By subtracting the equations for n and $n+1$ we have

(B.16)

$$\begin{cases}
\partial_t(\omega_{n+1}-\omega_n) = 2\,\mathrm{div}\Big((\beta\mathrm{v}_n + \alpha\mathrm{v}_n^\perp)(\omega_{n+1}-\omega_n) \\
\qquad +(\beta(\mathrm{v}_n - \mathrm{v}_{n-1}) + \alpha(\mathrm{v}_n-\mathrm{v}_{n-1})^\perp)\omega_n\Big) \\
\partial_t(d_{n+1}-d_n) = \frac{\lambda}{\alpha}\Delta(d_{n+1}-d_n) \\
+2\,\mathrm{div}\left((\beta\mathrm{v}_n^\perp - \alpha\mathrm{v}_n)(\omega_n-\omega_{n-1}) + (\beta(\mathrm{v}_n-\mathrm{v}_{n-1})^\perp - \alpha(\mathrm{v}_n-\mathrm{v}_{n-1}))\omega_n\right).
\end{cases}$$

Since $\omega_n(0) = \omega(0)$ for all n, applying (B.4) with $\sigma = -1+\gamma$, and the result of the previous step, we have

$$\begin{aligned}
&\|\omega_{n+1} - \omega_n\|_{L^\infty([0,T],C^\sigma)} \\
&\le e^{C\int_0^T \|\nabla \mathrm{v}_n\|_{L^\infty}\,ds} \int_0^T \|\operatorname{div}\left((\beta(\mathrm{v}_n - \mathrm{v}_{n-1}) + \alpha(\mathrm{v}_n - \mathrm{v}_{n-1})^\perp \omega_n\right)\|_{C^\sigma}\,ds \\
&\le Ce^{CT} T \|(\mathrm{v}_n - \mathrm{v}_{n-1})\omega_n\|_{L^\infty([0,T],C^{1+\sigma})} \\
&\le Ce^{CT} T \left(\|\omega_n - \omega_{n-1}\|_{L^\infty([0,T],C^\sigma)} + \|d_n - d_{n-1}\|_{L^\infty([0,T],C^\sigma)}\right).
\end{aligned}$$

Similarly, in view of (B.6),

$$\begin{aligned}
\|d_{n+1} - d_n\|_{L^\infty([0,T],C^\sigma)} \le C\sqrt{T}\Big(&\|\mathrm{v}_n(\omega_n - \omega_{n-1})\|_{L^\infty([0,T],C^\sigma)} \\
&+ \|(\mathrm{v}_n - \mathrm{v}_{n-1})\omega_n\|_{L^\infty([0,T],C^\sigma)}\Big) \\
\le C\sqrt{T}\left(\|\omega_n - \omega_{n-1}\|_{L^\infty([0,T],C^\sigma)} \right.&\left.+ \|d_n - d_{n-1}\|_{L^\infty([0,T],C^\sigma)}\right),
\end{aligned}$$

if $\sigma < -1+\gamma$. We deduce by standard arguments that $\{\omega_n\}$ and $\{d_n\}$ form a Cauchy sequence in $L^\infty([0,T],C^\sigma)$ if T is taken small enough. By interpolation $\{\mathrm{v}_n\}$ is a Cauchy sequence in $C^{1,\gamma'}$, for $\gamma' < \gamma$ (resp. $\{\omega_n\}$ in $C^{\gamma'}$ and $\{d_n\}$ in $C^{\gamma'}$). The limits v, ω, d will obviously solve (B.1), ω will be in $L^\infty([0,T],L^1(\mathbb{R}^2))$, d in $L^\infty([0,T],L^p(\mathbb{R}^2)) \cap L^2([0,T],H^1(\mathbb{R}^2))$ by Lemmas B.2 and B.3, and (from (B.14))

$$\mathrm{v} = \nabla\Delta^{-1}d - \nabla^\perp\Delta^{-1}\omega, \tag{B.17}$$

with $\mathrm{v} \in L^\infty([0,T],C^{1,\gamma'})$. Taking the time derivative of (B.17) and using (B.1), it is standard to deduce that v must solve (1.14). By integrating the equation, using that v is bounded and $\omega \in L^1(\mathbb{R}^2)$, we also have that $\frac{1}{2\pi}\omega$ remains a probability density.

Next, we prove that $\mathrm{v}-\mathrm{v}(0)$ remains in $L^2(\mathbb{R}^2)$. Using (1.14) and integration by parts, for $\zeta(x) = e^{-\eta|x|}$ with $0 < \eta < 1$ we compute

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^2} \zeta|\mathrm{v}(t) - \mathrm{v}(0)|^2 = 2\int_{\mathbb{R}^2} \zeta(\mathrm{v} - \mathrm{v}(0)) \cdot \left(\frac{\lambda}{\alpha}\nabla \operatorname{div}\mathrm{v} + 2\beta \mathrm{v}^\perp \operatorname{curl}\mathrm{v} - 2\alpha \mathrm{v}\operatorname{curl}\mathrm{v}\right) \\
&= -\frac{2\lambda}{\alpha}\int_{\mathbb{R}^2} \zeta(\operatorname{div}\mathrm{v})^2 + \frac{2\lambda}{\alpha}\int_{\mathbb{R}^2} \zeta(\operatorname{div}\mathrm{v})(\operatorname{div}\mathrm{v}(0)) - \frac{2\lambda}{\alpha}\int_{\mathbb{R}^2} \nabla\zeta \cdot (\mathrm{v} - \mathrm{v}(0))\operatorname{div}\mathrm{v} \\
&\qquad - 4\alpha\int_{\mathbb{R}^2} \zeta|\mathrm{v}|^2\omega + \zeta\mathrm{v}\cdot\mathrm{v}(0)\omega - 4\beta\int_{\mathbb{R}^2} \zeta\mathrm{v}^\perp \cdot \mathrm{v}(0)\omega
\end{aligned}$$

and using Young's inequality, the fact that $|\nabla\zeta| \le |\zeta|$ and that $\operatorname{div}\mathrm{v} \in L^2$, we find

$$\frac{d}{dt}\int_{\mathbb{R}^2} \zeta|\mathrm{v} - \mathrm{v}(0)|^2 \le C\left(\int_{\mathbb{R}^2} \zeta|\operatorname{div}\mathrm{v}(0)|^2 + \int_{\mathbb{R}^2} \zeta|\mathrm{v}(0)|^2\omega + \int_{\mathbb{R}^2} \zeta|\mathrm{v} - \mathrm{v}(0)|^2\right)$$

with a constant C depending on α, β and λ. Using Gronwall's lemma, since $\int_{\mathbb{R}^2} |\omega|$ is uniformly bounded and $\mathrm{v}(0)$ is bounded, we deduce that $\int_{\mathbb{R}^2} \zeta|\mathrm{v}(t) - \mathrm{v}(0)|^2 \le C_t$, with C_t independent from η. Letting then $\eta \to 0$, the desired claim follows.

To prove uniqueness, it suffices to apply the same idea of weak-strong uniqueness as used in the proof of the main results: if v_1 and v_2 are two solutions of (1.14), we introduce the "relative" energy $E(t) = \frac{1}{2}\int_{\mathbb{R}^2} |\mathrm{v}_1(t) - \mathrm{v}_2(t)|^2$, and the relative stress-energy tensor

$$T = (\mathrm{v}_1 - \mathrm{v}_2) \otimes (\mathrm{v}_1 - \mathrm{v}_2) - \frac{1}{2}|\mathrm{v}_1 - \mathrm{v}_2|^2 I.$$

By direct computation we have

(B.18)

$$\begin{aligned}\frac{d}{dt}E = \int_{\mathbb{R}^2} (\mathrm{v}_1 - \mathrm{v}_2)\Big(\frac{\lambda}{\alpha}\nabla(\operatorname{div}\mathrm{v}_1 - \operatorname{div}\mathrm{v}_2) + 2\beta(\mathrm{v}_1^\perp \operatorname{curl}\mathrm{v}_1 - \mathrm{v}_2^\perp \operatorname{curl}\mathrm{v}_2) \\ - 2\alpha(\mathrm{v}_1 \operatorname{curl}\mathrm{v}_1 - \mathrm{v}_2 \operatorname{curl}\mathrm{v}_2)\Big) \\ = \int_{\mathbb{R}^2} -\frac{\lambda}{\alpha}|\operatorname{div}(\mathrm{v}_1 - \mathrm{v}_2)|^2 + 2\beta(\mathrm{v}_1 - \mathrm{v}_2)\cdot \mathrm{v}_2^\perp \operatorname{curl}(\mathrm{v}_1 - \mathrm{v}_2) \\ + \int_{\mathbb{R}^2} -2\alpha|\mathrm{v}_1 - \mathrm{v}_2|^2 \operatorname{curl}\mathrm{v}_1 - 2\alpha(\mathrm{v}_1 - \mathrm{v}_2)\cdot \mathrm{v}_2 \operatorname{curl}(\mathrm{v}_1 - \mathrm{v}_2).\end{aligned}$$

and

$$\operatorname{div} T = (\mathrm{v}_1 - \mathrm{v}_2)\operatorname{div}\mathrm{v}_1 + (\mathrm{v}_1 - \mathrm{v}_2)^\perp \operatorname{curl}(\mathrm{v}_1 - \mathrm{v}_2). \quad \text{(B.19)}$$

Multiplying (B.19) by $2\beta\mathrm{v}_2 + 2\alpha\mathrm{v}_2^\perp$ and rearranging terms, we find

$$\begin{aligned}\int_{\mathbb{R}^2} (2\beta\mathrm{v}_2 + 2\alpha\mathrm{v}_2^\perp)\cdot \operatorname{div} T = \int_{\mathbb{R}^2} (2\beta\mathrm{v}_2 + 2\alpha\mathrm{v}_2^\perp)\cdot(\mathrm{v}_1 - \mathrm{v}_2)\operatorname{div}(\mathrm{v}_1 - \mathrm{v}_2) \\ + \int_{\mathbb{R}^2} (2\alpha\mathrm{v}_2 - 2\beta\mathrm{v}_2^\perp)\cdot(\mathrm{v}_1 - \mathrm{v}_2)\operatorname{curl}(\mathrm{v}_1 - \mathrm{v}_2),\end{aligned}$$

and inserting into (B.18) and using one integration by parts, we obtain

$$\begin{aligned}\frac{d}{dt}E \le -\int_{\mathbb{R}^2} \frac{\lambda}{\alpha}|\operatorname{div}(\mathrm{v}_1 - \mathrm{v}_2)|^2 + \int_{\mathbb{R}^2} T : \nabla(2\beta\mathrm{v}_2 + 2\alpha\mathrm{v}_2^\perp) \\ + \int_{\mathbb{R}^2} (2\beta\mathrm{v}_2 + 2\alpha\mathrm{v}_2^\perp)\cdot(\mathrm{v}_1 - \mathrm{v}_2)\operatorname{div}(\mathrm{v}_1 - \mathrm{v}_2).\end{aligned}$$

We may next use the boundedness of v_2 and $\nabla\mathrm{v}_2$, bound $|T|$ by $2E$, and use the Cauchy-Schwarz inequality to absorb the last term into the first negative term plus a constant times E, to finally obtain a differential inequality of the form $\frac{d}{dt}E \le CE$. The integrations by parts can easily be justified by using cut-off functions like χ_R and taking the limit, using the fact that $\mathrm{v}_i - \mathrm{v}(0) \in L^2(\mathbb{R}^2)$ and $\operatorname{div}(\mathrm{v}_1 - \mathrm{v}_2) \in L^2(\mathbb{R}^2)$. We then conclude to uniqueness by Gronwall's lemma.

Appendix C. The gauge case

In this appendix, we present in a formal manner the quantities that should be introduced and the computations which need to be followed to obtain the limiting dynamical laws in the two-dimensional case with gauge mentioned in

Section 1.3.4 of the introduction. We use the notation of that section, in particular the starting point is (1.31). In the rest of this appendix, we will not write down negligible terms, such as terms involving $1-|u_\varepsilon|^2$. We will however present the computations in the general case of the mixed flow (1.31), which can thus be used as a model for studying the mixed flow in the case without gauge.

We start by introducing some notation: we set

$$\partial_\Phi := \partial_t - i\Phi_\varepsilon$$

and

$$\phi := \begin{cases} \mathrm{p} & \text{in cases leading to (1.32)} \\ \frac{\lambda}{\alpha}\operatorname{div}\mathrm{v} & \text{in cases leading to (1.33).}\end{cases} \tag{C.1}$$

We will also denote $\Phi' = \Phi_\varepsilon + N_\varepsilon\phi$ and $\partial_{\Phi'} = \partial_t - i(\Phi_\varepsilon - N_\varepsilon\phi)$, $A' = A_\varepsilon + N_\varepsilon \mathrm{v}$ (dropping the ε) and $h_\varepsilon = \operatorname{curl} A_\varepsilon$. We define the velocity as in (2.43) and Lemma 2.12 in [Ti]:

$$\begin{aligned} V_\varepsilon &:= -\nabla\langle iu_\varepsilon, \partial_{\Phi_\varepsilon} u_\varepsilon\rangle + \partial_t\langle iu_\varepsilon, \nabla_{A_\varepsilon} u_\varepsilon\rangle - E_\varepsilon \\ &= 2\langle i\partial_{\Phi_\varepsilon} u_\varepsilon, \nabla_{A_\varepsilon} u_\varepsilon\rangle + (|u_\varepsilon|^2-1)E_\varepsilon \end{aligned} \tag{C.2}$$

thus

$$\partial_t j_\varepsilon = \nabla\langle iu_\varepsilon, \partial_{\Phi_\varepsilon} u_\varepsilon\rangle + V_\varepsilon + E_\varepsilon. \tag{C.3}$$

The modulated velocity is then defined as

$$\tilde{V}_\varepsilon = 2\langle i\partial_{\Phi'} u_\varepsilon, \nabla_{A'} u_\varepsilon\rangle + (|u_\varepsilon|^2-1)E_\varepsilon. \tag{C.4}$$

The modulated energy is defined as

(C.5)

$$\mathcal{E}_\varepsilon(u_\varepsilon, A_\varepsilon) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla_{A_\varepsilon} u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}|^2 + |\operatorname{curl} A_\varepsilon - N_\varepsilon \mathrm{h}|^2 + \frac{(1-|u_\varepsilon|^2)^2}{2\varepsilon^2} + (1-|u_\varepsilon|^2)\psi$$

with

$$\psi = \beta\phi - |\mathrm{v}|^2. \tag{C.6}$$

We will also use the fact that for u_ε solution of (1.10) we have the relation

$$\operatorname{div} j_\varepsilon = N_\varepsilon\langle(\frac{\alpha}{|\log\varepsilon|} + i\beta)\partial_\Phi u_\varepsilon, iu_\varepsilon\rangle. \tag{C.7}$$

The stress energy tensor is now

(C.8) $(S_\varepsilon(u,A))_{kl} :=$

$$\langle(\partial_k u - iA_k u, \partial_l u - iA_l u\rangle - \frac{1}{2}\left(|\nabla_A u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2 - |\operatorname{curl} A|^2\right)\delta_{kl}.$$

A direct computation shows that if u is sufficiently regular, we have

$$\begin{aligned} \operatorname{div} S_\varepsilon(u,A) &:= \sum_l \partial_l (S_\varepsilon(u,A))_{kl} \\ &= \langle\nabla_A u, \nabla_A^2 u + \frac{1}{\varepsilon^2}u(1-|u|^2)\rangle - \operatorname{curl} A(\nabla^\perp \operatorname{curl} A + \langle iu, \nabla_A u\rangle)^\perp \end{aligned}$$

so if u_ε solves (1.10), we have

$$\operatorname{div} S_\varepsilon(u_\varepsilon, A_\varepsilon) = N_\varepsilon \langle (\frac{\alpha}{|\log\varepsilon|} + i\beta)\partial_\Phi u_\varepsilon, \nabla_{A_\varepsilon} u_\varepsilon\rangle + \sigma E_\varepsilon^\perp \operatorname{curl} A_\varepsilon. \tag{C.9}$$

For simplicity, we will denote it by S_ε. The modulated stress tensor is now defined by

(C.10)

$$(\tilde S_\varepsilon(u,A))_{kl} = \langle \partial_k u - iuA_k - iuN_\varepsilon \mathrm{v}_k, \partial_l u - iuA_l u - iuN_\varepsilon \mathrm{v}_l\rangle + N_\varepsilon^2(1-|u|^2)\mathrm{v}_k \mathrm{v}_l$$
$$-\frac{1}{2}\left(|\nabla_A u - iuN_\varepsilon \mathrm{v}|^2 + (1-|u|^2)N_\varepsilon^2|\mathrm{v}|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2 - |\operatorname{curl} A - N_\varepsilon \mathrm{h}|^2\right)\delta_{kl},$$

and by direct computation

$$\begin{aligned}\operatorname{div} \tilde S_\varepsilon = {} & \operatorname{div} S_\varepsilon + N_\varepsilon^2 \mathrm{v} \operatorname{div} \mathrm{v} + N_\varepsilon^2 \mathrm{v}^\perp \operatorname{curl} \mathrm{v} - N_\varepsilon j_\varepsilon \operatorname{div} \mathrm{v} \\ & - N_\varepsilon j_\varepsilon^\perp \operatorname{curl} \mathrm{v} - N_\varepsilon \mathrm{v} \operatorname{div} j_\varepsilon - N_\varepsilon \mathrm{v}^\perp \operatorname{curl} j_\varepsilon + \nabla(\frac{1}{2}N_\varepsilon^2 \mathrm{h}^2 - N_\varepsilon \mathrm{h} h_\varepsilon).\end{aligned} \tag{C.11}$$

Combining (C.9) and (C.11) and we obtain

(C.12)

$$\operatorname{div} \tilde S_\varepsilon = N_\varepsilon \langle (\frac{\alpha}{|\log\varepsilon|} + i\beta)\partial_\Phi u_\varepsilon, \nabla_{A_\varepsilon} u_\varepsilon\rangle + \sigma h_\varepsilon E_\varepsilon^\perp + N_\varepsilon^2 \mathrm{v} \operatorname{div} \mathrm{v} + N_\varepsilon^2 \mathrm{v}^\perp \operatorname{curl} \mathrm{v}$$
$$- j_\varepsilon N_\varepsilon \operatorname{div} \mathrm{v} - N_\varepsilon j_\varepsilon^\perp \operatorname{curl} \mathrm{v} - N_\varepsilon \mathrm{v} \operatorname{div} j_\varepsilon - N_\varepsilon \mathrm{v}^\perp \operatorname{curl} j_\varepsilon + \nabla(\frac{1}{2}N_\varepsilon^2 \mathrm{h}^2 - N_\varepsilon \mathrm{h} h_\varepsilon).$$

Writing $\partial_\Phi u_\varepsilon = \partial_{\Phi'} u_\varepsilon + iu_\varepsilon N_\varepsilon \phi$ and $\nabla_A u_\varepsilon = \nabla_{A'} u_\varepsilon + iu_\varepsilon N_\varepsilon \mathrm{v}$, and inserting (C.2) and (C.7) into (C.12), we are led to

$$\begin{aligned}\operatorname{div} \tilde S_\varepsilon = {} & \frac{N_\varepsilon \alpha}{|\log\varepsilon|}\langle \partial_{\Phi'} u_\varepsilon, \nabla_{A'} u_\varepsilon\rangle + \sigma h_\varepsilon E_\varepsilon^\perp + \nabla(\frac{1}{2}N_\varepsilon^2 \mathrm{h}^2 - N_\varepsilon \mathrm{h} h_\varepsilon) \\ & + \frac{\beta}{2} N_\varepsilon V_\varepsilon - \frac{\beta}{2} N_\varepsilon^2 \mathrm{v} \partial_t |u_\varepsilon|^2 + N_\varepsilon^2 \mathrm{v}^\perp \operatorname{curl} \mathrm{v} - N_\varepsilon j_\varepsilon^\perp \operatorname{curl} \mathrm{v} - N_\varepsilon \mathrm{v}^\perp \operatorname{curl} j_\varepsilon + o(1).\end{aligned} \tag{C.13}$$

Multiplying relation (C.13) by $2\beta \mathrm{v} + 2\alpha \mathrm{v}^\perp$ yields

$$\begin{aligned} & \int_{\mathbb{R}^2} (2\beta \mathrm{v} + 2\alpha \mathrm{v}^\perp) \cdot \operatorname{div} \tilde S_\varepsilon \\ & = \int_{\mathbb{R}^2} \left(\frac{N_\varepsilon \alpha}{|\log\varepsilon|}\langle \partial_{\Phi'} u_\varepsilon, \nabla_{A'} u_\varepsilon\rangle + \frac{\beta}{2} N_\varepsilon V_\varepsilon - \frac{\beta}{2} N_\varepsilon^2 \mathrm{v}\partial_t |u_\varepsilon|^2\right) \cdot (2\beta \mathrm{v} + 2\alpha \mathrm{v}^\perp) \\ & + \int_{\mathbb{R}^2} 2\alpha N_\varepsilon^2 |\mathrm{v}|^2 \operatorname{curl} \mathrm{v} + N_\varepsilon j_\varepsilon \cdot (2\beta \mathrm{v}^\perp - 2\alpha \mathrm{v}) \operatorname{curl} \mathrm{v} - 2\alpha N_\varepsilon |\mathrm{v}|^2 \operatorname{curl} j_\varepsilon \\ & + \int_{\mathbb{R}^2} \nabla(\frac{1}{2} N_\varepsilon^2 \mathrm{h}^2 - N_\varepsilon \mathrm{h} h_\varepsilon) \cdot (2\beta \mathrm{v} + 2\alpha \mathrm{v}^\perp) - \sigma h_\varepsilon E_\varepsilon (2\beta \mathrm{v}^\perp - 2\alpha \mathrm{v}) + o(1).\end{aligned}$$

This allows to express $\int_{\mathbb{R}^2} N_\varepsilon \beta^2 V_\varepsilon \cdot \mathrm{v}$ so that splitting $V_\varepsilon \cdot \mathrm{v}$ as $\alpha^2 V_\varepsilon \cdot \mathrm{v} + \beta^2 V_\varepsilon \cdot \mathrm{v}$ (since $\alpha^2 + \beta^2 = 1$) and inserting the previous relation we may obtain

(C.14)
$$\begin{aligned}
&-\int_{\mathbb{R}^2} N_\varepsilon V_\varepsilon \cdot \mathrm{v} = \int_{\mathbb{R}^2} N_\varepsilon V_\varepsilon \cdot (-\alpha^2 \mathrm{v} + \alpha\beta \mathrm{v}^\perp) - N_\varepsilon^2 \beta^2 |\mathrm{v}|^2 \partial_t |u_\varepsilon|^2 \\
&+ \int_{\mathbb{R}^2} (2\beta \mathrm{v} + 2\alpha \mathrm{v}^\perp) \cdot \Big(-\operatorname{div} \tilde{S}_\varepsilon(u_\varepsilon) + \frac{N_\varepsilon \alpha}{|\log \varepsilon|} \langle \partial_{\Phi'} u_\varepsilon, \nabla_{A'} u_\varepsilon \rangle \Big) \\
&+ \int_{\mathbb{R}^2} 2\alpha N_\varepsilon^2 |\mathrm{v}|^2 \operatorname{curl} \mathrm{v} + N_\varepsilon j_\varepsilon \cdot (2\beta \mathrm{v}^\perp - 2\alpha \mathrm{v}) \operatorname{curl} \mathrm{v} - 2\alpha N_\varepsilon |\mathrm{v}|^2 \operatorname{curl} j_\varepsilon \\
&+ \int_{\mathbb{R}^2} \nabla (\frac{1}{2} N_\varepsilon^2 \mathrm{h}^2 - N_\varepsilon \mathrm{h} \operatorname{curl} A) \cdot (2\beta \mathrm{v} + 2\alpha \mathrm{v}^\perp) - \sigma h_\varepsilon E_\varepsilon (2\beta \mathrm{v}^\perp - 2\alpha \mathrm{v}) + o(1).
\end{aligned}$$

The analogue of Lemma 2.5 combined with (1.32) or (1.33) is

$$\begin{aligned}
&\frac{d}{dt} \mathcal{E}_\varepsilon(u_\varepsilon) = -\int_{\mathbb{R}^2} \frac{N_\varepsilon \alpha}{|\log \varepsilon|} |\partial_\Phi u_\varepsilon|^2 + \sigma |E_\varepsilon|^2 + \int_{\mathbb{R}^2} N_\varepsilon \langle i u_\varepsilon, \partial_\Phi u_\varepsilon \rangle \operatorname{div} \mathrm{v} \\
&+ \int_{\mathbb{R}^2} N_\varepsilon^2 \mathrm{v} \cdot \partial_t \mathrm{v} - N_\varepsilon j_\varepsilon \cdot \partial_t \mathrm{v} + N_\varepsilon (-V_\varepsilon - E_\varepsilon) \cdot \mathrm{v} + N_\varepsilon \partial_t \mathrm{h} (N_\varepsilon \mathrm{h} - h_\varepsilon) - N_\varepsilon \mathrm{h} \partial_t h_\varepsilon \\
&+ \int_{\mathbb{R}^2} \frac{1}{2} N_\varepsilon^2 \partial_t \left((1 - |u_\varepsilon|^2)(\psi - |\mathrm{v}|^2) \right).
\end{aligned}$$

Using (1.32) or (1.33), we may write

$$\begin{aligned}
&\int_{\mathbb{R}^2} N_\varepsilon^2 \mathrm{v} \cdot \partial_t \mathrm{v} - N_\varepsilon j_\varepsilon \cdot \partial_t \mathrm{v} \\
&\qquad = \int_{\mathbb{R}^2} N_\varepsilon (N_\varepsilon \mathrm{v} - j_\varepsilon) \cdot \Big(\mathrm{E} + (-2\alpha \mathrm{v} + 2\beta \mathrm{v}^\perp)(\operatorname{curl} \mathrm{v} + \mathrm{h}) + \nabla \phi \Big).
\end{aligned}$$

Next, we insert again $\partial_\Phi u_\varepsilon = \partial_{\Phi'} u_\varepsilon + i u_\varepsilon N_\varepsilon \phi$ and write that

$$N_\varepsilon \mathrm{v} - j_\varepsilon = -\nabla^\perp (N_\varepsilon \mathrm{h} - h_\varepsilon) - \sigma (N_\varepsilon \mathrm{E} - E_\varepsilon) \tag{C.15}$$

which holds by subtracting the equations (1.31) and (1.32)–(1.33), and we insert (C.14) and use an integration by parts to obtain

$$\begin{aligned}
&\frac{d}{dt} \mathcal{E}_\varepsilon = \int_{\mathbb{R}^2} \tilde{S}_\varepsilon : \nabla (2\beta \mathrm{v} + 2\alpha \mathrm{v}^\perp) + N_\varepsilon V_\varepsilon \cdot (-\alpha^2 \mathrm{v} + \alpha\beta \mathrm{v}^\perp) - N_\varepsilon^2 \beta^2 |\mathrm{v}|^2 \partial_t |u_\varepsilon|^2 \\
&- \int_{\mathbb{R}^2} 2\alpha N_\varepsilon |\mathrm{v}|^2 \operatorname{curl} j_\varepsilon + N_\varepsilon \phi \operatorname{div} (N_\varepsilon \mathrm{v} - j_\varepsilon) \\
&- \int_{\mathbb{R}^2} \frac{N_\varepsilon \alpha}{|\log \varepsilon|} |\partial_{\Phi'} u|^2 - \frac{N_\varepsilon \alpha}{|\log \varepsilon|} \langle \partial_{\Phi'} u, \nabla_{A'} u \rangle \cdot (2\beta \mathrm{v} + 2\alpha \mathrm{v}^\perp) \\
&+ \int_{\mathbb{R}^2} N_\varepsilon (-\nabla^\perp (N_\varepsilon \mathrm{h} - h_\varepsilon) - \sigma (N_\varepsilon \mathrm{E} - E_\varepsilon)) \cdot ((-2\alpha \mathrm{v} + 2\beta \mathrm{v}^\perp) \mathrm{h} + \mathrm{E}) \\
&+ \int_{\mathbb{R}^2} -\sigma |E_\varepsilon|^2 - N_\varepsilon E_\varepsilon \cdot \mathrm{v} - N_\varepsilon \operatorname{curl} \mathrm{E} (N_\varepsilon \mathrm{h} - h_\varepsilon) + N_\varepsilon \mathrm{h} \operatorname{curl} E_\varepsilon
\end{aligned}$$

$$+\int_{\mathbb{R}^2} \nabla(\frac{1}{2}N_\varepsilon^2 \mathrm{h}^2 - N_\varepsilon \mathrm{h} h_\varepsilon)\cdot(2\beta \mathrm{v} + 2\alpha \mathrm{v}^\perp) - \sigma h_\varepsilon E_\varepsilon (2\beta \mathrm{v}^\perp - 2\alpha \mathrm{v})$$
$$+\int_{\mathbb{R}^2} \frac{1}{2} N_\varepsilon^2 \partial_t \left((1-|u_\varepsilon|^2)(\psi - |\mathrm{v}|^2)\right) + o(1).$$

Next, we replace V_ε by $\tilde V_\varepsilon + N_\varepsilon \mathrm{v}\partial_t |u_\varepsilon|^2 - N_\varepsilon \phi \nabla |u_\varepsilon|^2$ obtained by comparing (C.2) and (C.4), and use (C.7) to reexpress $\operatorname{div} j_\varepsilon$. By choice (C.6), the terms in factor of $\partial_t(1-|u_\varepsilon|^2)$ cancel, and the terms in factor of ϕ, $\operatorname{div}\mathrm{v}$ and $\langle \partial_\Phi u_\varepsilon, i u_\varepsilon\rangle$ formally cancel (up to small errors) as in the parabolic case. Then, we insert

$$\begin{aligned}
&\sigma h_\varepsilon E_\varepsilon \cdot (2\beta \mathrm{v}^\perp - 2\alpha \mathrm{v})\\
&= \sigma h_\varepsilon (E_\varepsilon - N_\varepsilon \mathrm{E})\cdot(2\beta \mathrm{v}^\perp - 2\alpha \mathrm{v}) + N_\varepsilon h_\varepsilon(-\mathrm{v} - \nabla^\perp \mathrm{h})\cdot(2\beta \mathrm{v}^\perp - 2\alpha \mathrm{v})\\
&= \sigma h_\varepsilon (E_\varepsilon - N_\varepsilon \mathrm{E})\cdot(2\beta \mathrm{v}^\perp - 2\alpha \mathrm{v}) + 2\alpha N_\varepsilon |\mathrm{v}|^2 h_\varepsilon - N_\varepsilon h_\varepsilon \nabla^\perp \mathrm{h}\cdot(2\beta \mathrm{v}^\perp - 2\alpha \mathrm{v}).
\end{aligned}$$

The term $2\alpha N_\varepsilon |\mathrm{v}|^2 h_\varepsilon$ gets grouped with $2\alpha N_\varepsilon |\mathrm{v}|^2 \operatorname{curl} j_\varepsilon$ to form $-2\alpha |\mathrm{v}|^2 \mu_\varepsilon$. Combining all these elements, there remains

$$\begin{aligned}
&\frac{d}{dt}\mathcal{E}_\varepsilon = \int_{\mathbb{R}^2} \tilde S_\varepsilon : \nabla(2\beta \mathrm{v} + 2\alpha \mathrm{v}^\perp) + N_\varepsilon \tilde V_\varepsilon \cdot (-\alpha^2 \mathrm{v} + \alpha\beta \mathrm{v}^\perp) - 2\alpha N_\varepsilon |\mathrm{v}|^2 \mu_\varepsilon\\
&\int_{\mathbb{R}^2} -\frac{N_\varepsilon \alpha}{|\log \varepsilon|}|\partial_{\Phi'} u|^2 + \frac{N_\varepsilon \alpha}{|\log\varepsilon|}\langle \partial_{\Phi'} u, \nabla_{A'} u\rangle \cdot (2\beta \mathrm{v} + 2\alpha \mathrm{v}^\perp)\\
&+\int_{\mathbb{R}^2} -N_\varepsilon \nabla^\perp (N_\varepsilon \mathrm{h} - h_\varepsilon)\cdot((-2\alpha \mathrm{v} + 2\beta \mathrm{v}^\perp)\mathrm{h} + \mathrm{E})\\
&+\int_{\mathbb{R}^2} \sigma(2\alpha \mathrm{v} - 2\beta \mathrm{v}^\perp)(N_\varepsilon \mathrm{E} - E_\varepsilon)(N_\varepsilon \mathrm{h} - h_\varepsilon) - N_\varepsilon \operatorname{curl}\mathrm{E}(N_\varepsilon \mathrm{h} - h_\varepsilon)\\
&+\int_{\mathbb{R}^2} -\sigma N_\varepsilon (N_\varepsilon \mathrm{E} - E_\varepsilon) N_\varepsilon \mathrm{E} - \sigma |E_\varepsilon|^2 + N_\varepsilon \mathrm{h} \operatorname{curl} E_\varepsilon - N_\varepsilon E_\varepsilon \cdot \mathrm{v}\\
&+\int_{\mathbb{R}^2} \nabla(\frac{1}{2}N_\varepsilon^2 \mathrm{h}^2 - N_\varepsilon \mathrm{h} h_\varepsilon)\cdot(2\beta \mathrm{v} + 2\alpha \mathrm{v}^\perp) + N_\varepsilon h_\varepsilon \nabla \mathrm{h}\cdot(2\beta \mathrm{v} + 2\alpha \mathrm{v}^\perp) + o(1).
\end{aligned}$$

Keeping only the last three lines, after some simplifications using again (1.32), and some integration by parts, these three lines can be rewritten as

$$\begin{aligned}
&\int_{\mathbb{R}^2} -N_\varepsilon \mathrm{h} \nabla (N_\varepsilon \mathrm{h} - h_\varepsilon)\cdot(2\alpha \mathrm{v}^\perp + 2\beta \mathrm{v}) + \sigma(N_\varepsilon \mathrm{E} - E_\varepsilon)(2\alpha \mathrm{v} - 2\beta \mathrm{v}^\perp)(N_\varepsilon \mathrm{h} - h_\varepsilon)\\
&\int_{\mathbb{R}^2} -\sigma |E_\varepsilon|^2 - N_\varepsilon E_\varepsilon \cdot \mathrm{v} + N_\varepsilon \mathrm{h} \operatorname{curl} E_\varepsilon\\
&+\int_{\mathbb{R}^2} N_\varepsilon \mathrm{h} \nabla (N_\varepsilon \mathrm{h} - h_\varepsilon)\cdot(2\beta \mathrm{v} + 2\alpha \mathrm{v}^\perp) - \sigma N_\varepsilon (N_\varepsilon \mathrm{E} - E_\varepsilon)\mathrm{E} + o(1)\\
&= \int_{\mathbb{R}^2} \sigma (N_\varepsilon \mathrm{E} - E_\varepsilon)(2\alpha \mathrm{v} - 2\beta \mathrm{v}^\perp)(N_\varepsilon \mathrm{h} - h_\varepsilon) + \int_{\mathbb{R}^2} -\sigma |E_\varepsilon - N_\varepsilon \mathrm{E}|^2 + o(1).
\end{aligned}$$

We thus finally obtain

(C.16)

$$\frac{d}{dt}\mathcal{E}_\varepsilon = \int_{\mathbb{R}^2} \tilde{S}_\varepsilon : \nabla(2\beta \mathrm{v} + 2\alpha \mathrm{v}^\perp) + \int_{\mathbb{R}^2} N_\varepsilon \tilde{V}_\varepsilon \cdot (-\alpha^2 \mathrm{v} + \alpha\beta \mathrm{v}^\perp) - \int_{\mathbb{R}^2} 2\alpha N_\varepsilon |\mathrm{v}|^2 \mu_\varepsilon$$

$$\int_{\mathbb{R}^2} -\frac{N_\varepsilon \alpha}{|\log \varepsilon|}|\partial_{\Phi'} u_\varepsilon|^2 + \frac{N_\varepsilon \alpha}{|\log \varepsilon|}\langle \partial_{\Phi'} u_\varepsilon, \nabla_{A'} u_\varepsilon\rangle \cdot (2\beta \mathrm{v} + 2\alpha \mathrm{v}^\perp)$$

$$\int_{\mathbb{R}^2} \sigma(N_\varepsilon \mathrm{E} - E_\varepsilon)(2\alpha \mathrm{v} - 2\beta \mathrm{v}^\perp)(N_\varepsilon \mathrm{h} - h_\varepsilon) + \int_{\mathbb{R}^2} -\sigma |E_\varepsilon - N_\varepsilon \mathrm{E}|^2 + o(1).$$

For the terms involving E we write with Young's inequality

$$\int_{\mathbb{R}^2} \sigma(N_\varepsilon \mathrm{E} - E_\varepsilon)(2\alpha \mathrm{v} - 2\beta \mathrm{v}^\perp)(N_\varepsilon \mathrm{h} - h_\varepsilon) \le \sigma \int_{\mathbb{R}^2} |E_\varepsilon - N_\varepsilon \mathrm{E}|^2 + C\int_{\mathbb{R}^2} |N_\varepsilon \mathrm{h} - h_\varepsilon|^2$$

and we deduce that the last line in (C.16) is bounded above by $C\int_{\mathbb{R}^2} |N_\varepsilon \mathrm{h} - h_\varepsilon|^2$, which we claim it itself bounded by a constant time the energy excess $\mathcal{E}_\varepsilon - \pi N_\varepsilon |\log \varepsilon|$. Indeed, in writing down the analogue of Proposition 4.4 and 4.16, one may replace $\int_{\mathbb{R}^2} |\operatorname{curl} A_\varepsilon - N_\varepsilon \mathrm{h}|^2$ in (C.5) by half of itself and still obtain the same optimal lower bounds for the energy in the balls, so we deduce that $\int_{\mathbb{R}^2} |\operatorname{curl} A_\varepsilon - N_\varepsilon \mathrm{h}|^2$ must be bounded by the order of the energy excess.

To finish, we use Young's inequality again to bound

$$\int_{\mathbb{R}^2} \frac{N_\varepsilon \alpha}{|\log \varepsilon|}\langle \partial_{\Phi'} u_\varepsilon, \nabla_{A'} u_\varepsilon\rangle \cdot (2\beta \mathrm{v} + 2\alpha \mathrm{v}^\perp)$$
$$\le \frac{N_\varepsilon \alpha}{|\log \varepsilon|}\left(\frac{1}{2}\int_{\mathbb{R}^2} |\partial_{\Phi'} u_\varepsilon|^2 + \frac{1}{2}\int_{\mathbb{R}^2} |\nabla_{A'} u_\varepsilon \cdot (\beta \mathrm{v} + \alpha \mathrm{v}^\perp)|^2\right)$$

and the product estimate to formally bound

$$\int_{\mathbb{R}^2} N_\varepsilon \tilde{V}_\varepsilon \cdot (-\alpha^2 \mathrm{v} + \alpha\beta \mathrm{v}^\perp)$$
$$\le \frac{N_\varepsilon \alpha}{|\log \varepsilon|}\left(\frac{1}{2}\int_{\mathbb{R}^2} |\partial_{\Phi'} u_\varepsilon|^2 + 2\int_{\mathbb{R}^2} |\nabla_{A'} u_\varepsilon \cdot (-\alpha \mathrm{v} + \beta \mathrm{v}^\perp)|^2\right)$$

and using that $\alpha \mathrm{v} - \beta \mathrm{v}^\perp = (\beta \mathrm{v} + \alpha \mathrm{v}^\perp)^\perp$ and $|\beta \mathrm{v} + \alpha \mathrm{v}^\perp|^2 = |\mathrm{v}|^2$, we see that these two relations add up to a left-hand side bounded by

$$\frac{N_\varepsilon \alpha}{|\log \varepsilon|}\left(\int_{\mathbb{R}^2} |\partial_{\Phi'} u_\varepsilon|^2 + 2\int_{\mathbb{R}^2} |\nabla_{A'} u_\varepsilon|^2 |\mathrm{v}|^2\right)$$

which will recombine with $-\int_{\mathbb{R}^2} \frac{N_\varepsilon \alpha}{|\log \varepsilon|}|\partial_{\Phi'} u_\varepsilon|^2 + 2\alpha N_\varepsilon |\mathrm{v}|^2 \mu_\varepsilon$ into a term bounded by $C(\mathcal{E}_\varepsilon - \pi N_\varepsilon |\log \varepsilon|)$. Inserting into (C.16) and replacing the use of v by that of $\bar{\mathrm{v}}$ in the case with dissipation, we may then obtain a Gronwall relation $\frac{d}{dt}\mathcal{E}_\varepsilon \le C(\mathcal{E}_\varepsilon - \pi N_\varepsilon |\log \varepsilon|) + o(N_\varepsilon^2)$ as in the case without gauge, and conclude in the same way.

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