MEAN FIELD LIMIT FOR COULOMB FLOWS

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Abstract. We establish the mean-field convergence for systems of points evolving along the gradient-flow of their interaction energy when the interaction is the Coulomb potential or a super-coulombic Riesz potential, for the first time in arbitrary dimension. The proof is based on a modulated energy method using a Coulomb or Riesz distance, assumes that the solutions of the limiting equation are regular enough and exploits a weak-strong stability property for them. The method applies as well to conservative and mixed flows.

1. Introduction

1.1. **Problem and background.** We study here the large N limit of gradient flow evolutions

$$\begin{cases} \dot{x}_i = -\frac{1}{N}\nabla_{x_i}\mathcal{H}_N(x_1,\dots,x_N), & i=1,\dots,N \\ x_i(0)=x_i^0 \end{cases} \tag{1.1}$$

or conservative evolutions of the form

$$\begin{cases} \dot{x}_i = -\frac{1}{N}\mathbb{J}\nabla_{x_i}\mathcal{H}_N(x_1,\dots,x_N) & i=1,\dots,N \\ x_i(0)=x_i^0 \end{cases} \tag{1.2}$$

where $\mathbb{J}$ is an antisymmetric matrix. The points x_i evolve in the whole space $\mathbb{R}^\mathsf{d}$ and their energy $\mathcal{H}_N$ is given by

$$\mathcal{H}_N(x_1,\dots,x_N)=\sum_{i\neq j}\mathsf{g}(x_i-x_j) \tag{1.3}$$

where g is a repulsive interaction kernel which assumed to be logarithmic, Coulomb, or Riesz; more precisely of the form

$$\mathsf{g}(x)=|x|^{-\mathsf{s}} \quad \max(\mathsf{d}-2,0)\le \mathsf{s}<\mathsf{d} \qquad \text{for any } \mathsf{d}\ge 1 \tag{1.4}$$

or

$$\mathsf{g}(x)=-\log|x| \qquad \text{for } \mathsf{d}=1 \text{ or } 2. \tag{1.5}$$

In the case (1.4) with $\mathsf{s}=\mathsf{d}-2$ and $\mathsf{d}\ge 3$, or (1.5) and $\mathsf{d}=2$, g is exactly (a multiple of) the Coulomb kernel. In the other cases of (1.4) it is called a Riesz kernel.

Mixed flows of the form

$$\dot{x}_i=-\frac{1}{N}(\alpha I+\beta\mathbb{J})\nabla_{x_i}\mathcal{H}_N(x_1,\dots,x_N) \qquad \alpha>0$$

can be treated with exactly the same proof, and the case of the same dynamics with an additional forcing $\frac{1}{N}\sum_{i=1}^N F(x_i)$ (with F Lipschitz) as well. These generalizations are left to the reader.

The limiting evolutions as $N \to \infty$ of the systems (1.1) or (1.2), or their mean-field limit, is a well-known and natural question : from a physical point of view they can model interacting particles, from a numerical point of view they correspond to particle approximations of the limiting PDEs or can serve to approximate their equilibrium states. More general interaction kernels including attractive terms are also of interest as aggregation and swarming models, see for instance [CCH]. In dimension 2, choosing (1.5) and $\mathbb{J}$ the rotation by $\pi/2$ in (1.2) corresponds to the so-called *point vortex system* which is well-known in fluid mechanics (cf. for instance [MP]), and its mean-field convergence to the Euler equation in vorticity form was already established [Scho2].

Consider the *empirical measure*

$$\mu_N^t := \frac{1}{N}\sum_{i=1}^N \delta_{x_i^t} \tag{1.6}$$

associated to a solution $X_N^t := (x_1^t, \ldots, x_N^t)$ of the flow (1.1) or (1.2). If the points x_i^0, which themselves depend on N, are such that μ_N^0 converges to some regular measure μ^0, then a formal derivation leads to expecting that for $t > 0$, μ_N^t converges to the solution of the Cauchy problem with initial data μ^0 for the limiting evolution equation

$$\partial_t \mu = \mathrm{div}\,(\nabla(\mathsf{g} * \mu)\mu) \tag{1.7}$$

in the dissipative case (1.1) or

$$\partial_t \mu = \mathrm{div}\,(\mathbb{J}\nabla(\mathsf{g} * \mu)\mu) \tag{1.8}$$

in the conservative case (1.2).

These equations should be understood in a weak sense. Equation (1.7) is sometimes called the *fractional porous medium equation*. The two-dimensional Coulomb version also arises as a model for the evolution of vortices in superconductors. The construction of solutions, their regularity and basic properties, are addressed in [LZ, DZ, AS, SV] for the Coulomb case of (1.7), in [CSV, CV, XZ] for the case $\mathsf{d} - 2 < \mathsf{s} < \mathsf{d}$ of (1.7), and [De, Yu, Scho1] for the two-dimensional Coulomb case of (1.2).

Establishing the convergence of the empirical measures to solutions of the limiting equations is nontrivial because of the nonlinear terms in the equation and the singularity of the interaction g. In fact, because of the strength of the singularity, treating the case of Coulomb interactions in dimension $\mathsf{d} \ge 3$ (and even more so that of super-coulombic interactions) had remained an open question for a long time, see for instance the introduction of [JW2] and the review [Jab]. It was not even completely clear that the result was true without expressing it in some statistical sense (with respect to the initial data).

In [JW1, JW2], Jabin and Wang introduced a new approach for the related problem of the mean-field convergence of the solutions of Newton's second order system of ODEs to the Vlasov equation, which allowed them to treat all interactions kernels with bounded gradients, but still not Coulomb interactions. The same problem has been addressed in [LP, CCS] with results that still require a cutoff of the Coulomb interactions. Our method already allows to unlock the case of Coulomb interaction for monokinetic data [DS].

The previously known results on the problems we are addressing were the following:

- Schochet [Scho2] proved the convergence of the conservative flow (1.2) to (1.8) in the two-dimensional logarithmic case, and his proof can be readapted to treat (1.1) as well in that case. Results of similar nature were also obtained in [GHL].

- Hauray [Hau] treated the case of all sub-Coulombic interactions ($\mathsf{s} < \mathsf{d} - 2$) for (a possibly higher-dimensional generalization of) (1.2), where particles can have positive and negative charges and thus can attract as well as repel. His proof, which relies on the stability in ∞-Wasserstein distance of the limiting solution, cannot be adapted to $\mathsf{s} \geq \mathsf{d} - 2$.
- In dimension 1, Berman and Onnheim [BO] proved the unconditional convergence for all $0 < \mathsf{s} < 1$ using the framework of Wasserstein gradient flows but their method, based on the convexity of the interaction in dimension 1, does not extend to higher dimensions.
- Duerinckx [Du], inspired by the *modulated energy* method of [Sy] for Ginzburg-Landau equations (where vortices also interact like Coulomb particles in dimension 2), was able to prove the result for $\mathsf{d} = 1$ and $\mathsf{d} = 2$ with $\mathsf{s} < 1$, conditional to the regularity of the limiting solution, as we have here. It was the first paper to address super-coulombic interactions.

In this paper, we extend Duerinckx's result to all cases of (1.4) or (1.5). We are limited to $\mathsf{s} < \mathsf{d}$ and this is natural since for $\mathsf{s} \geq \mathsf{d}$ the interaction kernel g is no longer integrable and the limiting equation is expected to be a different one.

As in [Du], our proof is a modulated energy argument inspired from [Sy], which is a way of exploiting a weak-strong uniqueness principle for the limiting equation. As mentioned above, looking for a stability principle in some Wasserstein distance fails at the Coulomb singularity. Instead we use a distance which is built as a Coulomb (or Riesz) metric, associated to the norm

$$\|\mu\|^2 = \iint \mathsf{g}(x-y)d\mu(x)d\mu(y). \tag{1.9}$$

We are able to show by a Gronwall argument on this metric that the equations (1.7) and (1.8) satisfy a weak-strong uniqueness principle, and this can be translated into a proof of stability and convergence to 0 of the norm of $\mu_N^t - \mu^t$ (if it is initially small, it remains small for all further times).

The proof is self-contained and quantitative. It does not require understanding any qualitative property of the trajectories of the particles, such as for instance their minimal distances along the flow.

1.2. **Main result.** Let X_N denote $(x_1, \dots, x_N)$ and let us define for any probability measure μ,

$$F_N(X_N, \mu) = \iint_{\mathbb{R}^\mathsf{d} \times \mathbb{R}^\mathsf{d} \setminus \triangle} \mathsf{g}(x-y)d\Big(\sum_{i=1}^N \delta_{x_i} - N\mu\Big)(x)d\Big(\sum_{i=1}^N \delta_{x_i} - N\mu\Big)(y) \tag{1.10}$$

where $\triangle$ denotes the diagonal in $\mathbb{R}^\mathsf{d} \times \mathbb{R}^\mathsf{d}$.

We choose for "modulated energy"

$$F_N(X_N^t, \mu^t)$$

where $X_N^t = (x_1^t, \dots, x_N^t)$ are the solutions to (1.1) or (1.2), μ^t solves the expected limiting PDE. It turns out that F_N is a good notion of distance from μ_N^t to μ^t and metrizes at least weak convergence, as described in Proposition 3.5.

Our main result is a Gronwall inequality on the time-derivative of $F_N(X_N^t, \mu^t)$, which implies a quantitative rate of convergence of μ_N^t to μ^t in that metric.

Throughout the paper, s should be understood as equal to 0 in the formulae in the cases (1.5), and $(\cdot)_+$ denotes the positive part of a number.

Theorem 1. *Assume that* g *is of the form* (1.4) *or* (1.5)*. Assume* (1.7)*, respectively* (1.8)*, admits a solution* μ^t *such that, for some* $T > 0$*,*

$$\begin{cases} \mu^t \in L^\infty([0,T], L^\infty(\mathbb{R}^\mathsf{d})), \ \text{and } \nabla^2 \mathsf{g} * \mu^t \in L^\infty([0,T], L^\infty(\mathbb{R}^\mathsf{d})) \quad \text{if } \ \mathsf{s} < \mathsf{d}-1 \\ \mu^t \in L^\infty([0,T], C^\sigma(\mathbb{R}^\mathsf{d})) \text{ with } \sigma > \mathsf{s}-\mathsf{d}+1, \ \text{and } \nabla^2 \mathsf{g} * \mu^t \in L^\infty([0,T], L^\infty(\mathbb{R}^\mathsf{d})) \text{ if } \ \mathsf{s} \ge \mathsf{d}-1. \end{cases} \tag{1.11}$$

Let X_N^t *solve* (1.1)*, respectively* (1.2)*. Then there exist constants* C_1, C_2 *depending only on the norms of* μ^t *controlled by* (1.11) *and an exponent* $\beta < 2$ *depending only on* $\mathsf{d}, \mathsf{s}, \sigma$*, such that for every* $t \in [0,T]$ *we have*

$$F_N(X_N^t, \mu^t) \le \left(F_N(X_N^0, \mu^0) + C_1 N^\beta \right) e^{C_2 t}. \tag{1.12}$$

In particular, using the notation (1.6)*, if* $\mu_N^0 \rightharpoonup \mu^0$ *and is such that*

$$\lim_{N\to\infty} \frac{1}{N^2} F_N(X_N^0, \mu^0) = 0,$$

then the same is true for every $t \in [0,T]$ *and*

$$\mu_N^t \rightharpoonup \mu^t \tag{1.13}$$

in the weak sense.

Establishing the convergence of the empirical measures is essentially equivalent to proving *propagation of molecular chaos* (see [Go,HM,Jab] and references therein) which means showing that if $f_N^0(x_1, \dots, x_N)$ is the initial probability density of seeing particles at $(x_1, \dots, x_N)$ and if f_N^0 converges to some factorized state $\mu^0 \otimes \dots \otimes \mu^0$, then the k-point marginals $f_{N,k}^t$ converge for all time to $(\mu^t)^{\otimes k}$. With Remark 3.6, our result implies a convergence of this type as well.

1.3. **Comments on the assumptions.** Let us now comment on the regularity assumption made in (1.11). First of all, one can check (see Lemma 3.1) that the assumption (1.11) is implied by

$$\mu \in L^\infty([0,T], C^\theta(\mathbb{R}^\mathsf{d})) \quad \text{for some } \theta > \mathsf{s}-\mathsf{d}+2, \tag{1.14}$$

which coincides with the assumption made in [Du] and is a bit stronger. This weakening of the assumption allows to include for instance the case of measures which are (a regular function times) the characteristic function of some regular set, such as in the situation of vortex patches for the Euler equation in vorticity form, corresponding to (1.2) in the two-dimensional logarithmic case. These vortex patches were first studied in [Ch2,BC,Si] where it was shown that if the patch initially has a $C^{1,\alpha}$ boundary this remains the case over time, and our second assumption that the velocity $\nabla \mathsf{g} * \mu^t$ be Lipschitz holds as well (see also [BK]). It is not too difficult to check that in all dimensions this second condition holds any time μ is C^σ with $\sigma > 0$ away from a finite number of $C^{1,\alpha}$ hypersurfaces. More generally, such situations with patches can be expected to naturally arise in all the Coulomb cases. For instance, in the dissipative Coulomb case (1.1), in any dimension, a self-similar solution in the form of (a constant multiple of) the characteristic function of an expanding ball was exhibited in [SV] and shown to be an attractor of the dynamics. For the non-Coulomb dissipative cases, the corresponding self-similar solutions, called *Barenblatt solutions*, are of the form

$$t^{-\frac{\mathsf{d}}{2+\mathsf{s}}} (a - b x^2 t^{-\frac{2}{2+\mathsf{s}}})_+^{\frac{\mathsf{s}-\mathsf{d}+2}{2}}$$

as shown in [BIK, CV] (and this formula retrieves the solution of [SV] for $\mathsf{s} = \mathsf{d} - 2$).

If the initial μ^0 is sufficiently regular, the stronger assumption (1.14) is known to hold with $T = \infty$ for the Coulomb case (see [LZ] where the proof works as well in higher dimensions), and it is known up to some $T > 0$ in the case $(\mathsf{d} - 2)_+ < \mathsf{s} \le \mathsf{d} - 1$ [XZ]. As for (1.8), to our knowledge the desired regularity is only known in dimension 2 for the Euler equation in vorticity form (see [Wo, Ch2]), although the arguments of [XZ] written for the dissipative case seem to also apply to the conservative one. Our convergence result thus holds in these cases, under the assumption that the limit μ^0 of μ_N^0 is sufficiently regular and that $F_N(X_N^0, \mu^0) = o(N^2)$. Note that, as shown in [Du], the latter is implied by the convergence of the initial energy

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{i\neq j} \mathsf{g}(x_i^0 - x_j^0) = \iint_{\mathbb{R}^\mathsf{d}\times\mathbb{R}^\mathsf{d}} \mathsf{g}(x-y)d\mu^0(x)d\mu^0(y)$$

which can be viewed as a well-preparedness condition.

For $\mathsf{d} - 1 < \mathsf{s} < \mathsf{d}$, even the local in time propagation of regularity of solutions of (1.7) remains an open problem. Note that the uniqueness of regular enough solutions is always implied by the weak-strong stability argument we use, detailed in [Du], and reproduced in Section 1.4 below.

Requiring some regularity of the solutions to the limiting equation for establishing convergence with relative entropy / modulated entropy / modulated energy methods is fairly common: the same situation appears for instance in [JW1, JW2] or in the derivation of the Euler equations from the Boltzmann equation via the modulated entropy method, see [SR] and references therein.

1.4. **The method.** As mentioned, our method exploits a weak-strong uniqueness principle for the solutions of (1.7), resp. (1.8), which is exactly the same as [Du, Lemma 2.1, Lemma 2.2] (and can be easily readapted to the conservative case) and states that if μ_1^t and μ_2^t are two L^∞ solutions to (1.7) such that $\nabla^2(\mathsf{g} * \mu_2) \in L^1([0,T], L^\infty)$, we have

$$(1.15)\quad \iint_{\mathbb{R}^\mathsf{d}\times\mathbb{R}^\mathsf{d}} \mathsf{g}(x-y)d(\mu_1^t - \mu_2^t)(x)d(\mu_1^t - \mu_2^t)(y) \\ \le e^{C\int_0^t \|\nabla^2(\mathsf{g}*\mu_2^s)\|ds} \iint_{\mathbb{R}^\mathsf{d}\times\mathbb{R}^\mathsf{d}} \mathsf{g}(x-y)d(\mu_1^0 - \mu_2^0)(x)d(\mu_1^0 - \mu_2^0)(y).$$

But the Coulomb or Riesz energy (1.9) is nothing else than the fractional Sobolev $H^{-\alpha}$ norm of μ with $\alpha = \frac{\mathsf{d}-\mathsf{s}}{2}$, hence this is a good metric of convergence and implies the weak-strong uniqueness property.

A crucial ingredient is the use of the stress-energy (or energy-momentum) tensor which naturally appears when taking the inner variations of the energy (1.9) (this is standard in the calculus of variations, see for instance [He, Sec. 1.3.2]). To explain further, let us restrict for now to the Coulomb case, and set $h^\mu = \mathsf{g} * \mu$. In that case, we have

$$(1.16)\quad -\Delta h^\mu = \mathsf{c_d}\mu$$

for some constant $\mathsf{c_d}$ depending only on d. The first key is to reexpress the Coulomb energy (1.9) as a single integral in h^μ, more precisely we easily find via an integration by parts that

$$\iint_{\mathbb{R}^\mathsf{d}\times\mathbb{R}^\mathsf{d}} \mathsf{g}(x-y)d\mu(x)d\mu(y) = \int_{\mathbb{R}^\mathsf{d}} h^\mu d\mu = -\frac{1}{\mathsf{c_d}}\int_{\mathbb{R}^\mathsf{d}} h^\mu \Delta h^\mu = \frac{1}{\mathsf{c_d}}\int_{\mathbb{R}^\mathsf{d}} |\nabla h^\mu|^2.$$

The stress-energy tensor is then defined as the $\mathsf{d}\times\mathsf{d}$ tensor with coefficients

$$[h^\mu, h^\mu]_{ij} = 2\partial_i h^\mu \partial_j h^\mu - |\nabla h^\mu|^2 \delta_{ij}, \tag{1.17}$$

with δ_{ij} the Kronecker symbol. We may compute that

$$\operatorname{div}[h^\mu, h^\mu] = 2\Delta h^\mu \nabla h^\mu = -\frac{2}{\mathsf{c_d}}\mu \nabla h^\mu. \tag{1.18}$$

(Here the divergence is a vector with entries equal to the divergence of each row of $[h^\mu, h^\mu]$.) We thus see how this stress-energy tensor allows to give a weak meaning to the product $\mu\nabla h^\mu = \mu \nabla \mathsf{g} * \mu$, with $[h^\mu, h^\mu]$ well-defined in energy space and pointwise controlled by $|\nabla h^\mu|^2$, which can by the way serve to give a notion of weak solutions of the equation in the energy space (as in [De, LZ]). Note that in dimension 2, it is known since [De] that even though $[h^\mu, h^\mu]$ is nonlinear, it is stable under weak limits in energy space provided μ has a sign, but this fact *does not extend* to higher dimension.

Let us now present the short proof of (1.15) as it will be a model for the main proof. We focus on the dissipative case (the conservative one is an easy adaptation) and still the Coulomb case for simplicity. Let μ_1 and μ_2 be two solutions to (1.7) and $h_i = \mathsf{g} * \mu_i$ the associated potentials, which solve (1.16). Let us compute

$$\begin{aligned} \partial_t \int_{\mathbb{R}^\mathsf{d}} |\nabla(h_1 - h_2)|^2 &= 2\mathsf{c_d} \int_{\mathbb{R}^\mathsf{d}} (h_1 - h_2)\partial_t(\mu_1 - \mu_2) \\ &= 2\mathsf{c_d} \int_{\mathbb{R}^\mathsf{d}} (h_1 - h_2)\operatorname{div}(\mu_1 \nabla h_1 - \mu_2 \nabla h_2) \\ &= -2\mathsf{c_d} \int_{\mathbb{R}^\mathsf{d}} (\nabla h_1 - \nabla h_2)\cdot(\mu_1 \nabla h_1 - \mu_2 \nabla h_2) \\ &= -2\mathsf{c_d} \int_{\mathbb{R}^\mathsf{d}} |\nabla(h_1 - h_2)|^2 \mu_1 - 2\mathsf{c_d}\int_{\mathbb{R}^\mathsf{d}} \nabla h_2 \cdot \nabla(h_1 - h_2)(\mu_1 - \mu_2) \end{aligned} \tag{1.19}$$

In the right-hand side, we recognize from (1.18) the divergence of the stress-energy tensor $[h_1 - h_2, h_1 - h_2]$, hence

$$\partial_t \int_{\mathbb{R}^\mathsf{d}} |\nabla(h_1 - h_2)|^2 \le -2\mathsf{c_d} \int_{\mathbb{R}^\mathsf{d}} \nabla h_2 \cdot \operatorname{div}[h_1 - h_2, h_1 - h_2]$$

so if $\nabla^2 h_2$ is bounded, we may integrate by parts the right-hand side and bound it pointwise by

$$\|\nabla^2 h_2\|_{L^\infty} \int_{\mathbb{R}^\mathsf{d}} |[h_1 - h_2, h_1 - h_2]| \le 2\|\nabla^2 h_2\|_{L^\infty} \int_{\mathbb{R}^\mathsf{d}} |\nabla(h_1 - h_2)|^2,$$

and the claimed result follows by Gronwall's lemma.

In the Riesz case, the Riesz potential $h^\mu = \mathsf{g} * \mu$ is no longer the solution to a local equation, and to find a replacement to (1.16)–(1.18), we use an extension procedure as popularized by [CaffSi] in order to obtain a local integral in h^μ in the extended space $\mathbb{R}^{\mathsf{d}+1}$.

In the discrete case of the original ODE system, all the above integrals are singular and these singularities have to be removed. In place of the second term in the right-hand side of (1.19), we then have to control a term which by symmetry can be written in the form

$$\iint_{\mathbb{R}^\mathsf{d}\times\mathbb{R}^\mathsf{d}\setminus\triangle} (\nabla h^{\mu_1}(x) - \nabla h^{\mu_2}(y))\cdot \nabla \mathsf{g}(x-y)\, d(\mu_1 - \mu_2)(x)\, d(\mu_1 - \mu_2)(y) \tag{1.20}$$

where $\triangle$ denotes the diagonal, μ_1 is the limiting measure μ^t and μ_2 is the discrete empirical measure μ_N^t. Such terms are well known (see for instance [Scho2]), and create the main difficulty due to the singularity of g. The key is as above to reexpress it as a single integral in terms of a stress-energy tensor, but where this time we renormalize the singularity in (1.20) via an appropriate truncation of h^μ, and then to show that (1.20) can be controlled by the Coulomb (Riesz) distance itself, provided the limiting solution is regular enough. In carrying out all these steps, we depart significantly from the previous proof of [Du]. The idea of expressing the interaction energy as a local integral in h^μ and its renormalization procedure, were previously used in the study of Coulomb and Riesz energies in [RS, PS, LS1, LS2], but it was not clear how to adapt these ideas to control (1.20): in fact in [Du] this was dealt with by a ball-construction procedure inspired from the analysis of Ginzburg-Landau vortices, which led to the restriction $\mathsf{s} < 1$ and $\mathsf{d} \le 2$.

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2. Main proof

In all the paper, we will use the notation $\mathbf{1}_{(1.5)}$ to indicate a term which is only present in the logarithmic cases (1.5) and $\mathbf{1}_{\mathsf{s}<\mathsf{d}-1}$ for a term which is present only if $\mathsf{s} < \mathsf{d} - 1$.

Differentiating from formula (1.10), we have

Lemma 2.1. *If X_N^t is a solution of (1.1), then*

$$\partial_t F_N(X_N^t, \mu^t) = -2N^2 \int |\nabla h^{\mu_N^t - \mu^t}|^2 d\mu_N^t + N^2 \iint_{\mathbb{R}^\mathsf{d}\times\mathbb{R}^\mathsf{d}\setminus\triangle} \left(\nabla h^{\mu^t}(x) - \nabla h^{\mu^t}(y)\right) \cdot \nabla \mathsf{g}(x-y) d(\mu_N^t - \mu^t)(x) d(\mu_N^t - \mu^t)(y). \tag{2.1}$$

If X_N^t is a solution of (1.2), then

$$\partial_t F_N(X_N^t, \mu^t) = -N^2 \iint_{\mathbb{R}^\mathsf{d}\times\mathbb{R}^\mathsf{d}\setminus\triangle} \mathbb{J}\left(\nabla h^{\mu^t}(x) - \nabla h^{\mu^t}(y)\right) \cdot \nabla \mathsf{g}(x-y)(\mu_N^t - \mu^t)(x)(\mu_N^t - \mu^t)(y). \tag{2.2}$$

Proof. We note that if $\mathsf{s} \ge \mathsf{d} - 1$, $\nabla \mathsf{g}$ is not integrable near 0, so $\nabla \mathsf{g} * \mu$ should be understood in a distributional sense and $\mu \nabla(\mathsf{g} * \mu) = \mu \mathsf{g} * \nabla \mu$ as well, assuming that μ is regular enough. We may also check that this distributional definition is equivalent to defining

$$\nabla h^\mu(x) = P.V. \int_{\mathbb{R}^\mathsf{d}\setminus\{x\}} \nabla \mathsf{g}(x-y) d\mu(y)$$

where the principal value (which may be omitted for $\mathsf{s} < \mathsf{d} - 1$) is defined by

$$P.V. \int_{\mathbb{R}^\mathsf{d}\setminus\{x\}} := \lim_{r\to 0} \int_{\mathbb{R}^\mathsf{d}\setminus B(x,r)}.$$

We may now give the proof, which is as in [Du] (at least for the dissipative case) but we reproduce it here for the sake of completeness.

In the case (1.1), we have

$$\partial_t F_N(X_N^t,\mu^t) = N^2\partial_t \iint_{\mathbb{R}^d\times\mathbb{R}^d} \mathsf{g}(x-y)d\mu^t(x)d\mu^t(y)+\partial_t \sum_{i\neq j}\mathsf{g}(x_i^t-x_j^t)-2N\partial_t\sum_{i=1}^N\int_{\mathbb{R}^d}\mathsf{g}(x_i^t-y)d\mu^t(y)$$

$$= -2N^2\int_{\mathbb{R}^d}|\nabla h^{\mu^t}|^2(x)d\mu^t(x) - 2\sum_{i=1}^N\left|\sum_{j\neq i}\nabla\mathsf{g}(x_i^t-x_j^t)\right|^2$$

$$+2\sum_{i\neq j}\nabla h^{\mu^t}(x_i^t)\cdot\nabla\mathsf{g}(x_i^t-x_j^t)+2N\sum_{i=1}^N P.V.\int_{\mathbb{R}^d\setminus\{x_i^t\}}\nabla h^{\mu^t}(x)\cdot\nabla\mathsf{g}(x-x_i^t)d\mu^t(x).$$

We then recombine the terms to obtain

$$\begin{aligned}\partial_t F_N(X_N^t,\mu^t) &= -2N^2\int_{\mathbb{R}^d}\left|P.V.\int_{\mathbb{R}^d\setminus\{x\}}\nabla\mathsf{g}(x-y)d(\mu_N^t-\mu^t)(y)\right|^2 d\mu_N^t\\ &\quad -2N^2\int_{\mathbb{R}^d}|\nabla h^{\mu^t}|^2d\mu^t(x)+2N^2\int_{\mathbb{R}^d}|\nabla h^{\mu^t}|^2d\mu_N^t(x)\\ &\quad -2N^2\int_{\mathbb{R}^d}\nabla h^{\mu^t}(x)\cdot\int_{\mathbb{R}^d\setminus\{x\}}\nabla\mathsf{g}(x-y)d\mu_N^t(y)d\mu_N^t(x)\\ &\quad +2N^2\int_{\mathbb{R}^d}P.V.\int_{\mathbb{R}^d\setminus\{y\}}\nabla h^{\mu^t}(x)\cdot\nabla\mathsf{g}(x-y)d\mu^t(x)d\mu_N^t(y).\end{aligned}$$

We recognize that the last four terms can be recombined and symmetrized into

$$-N^2\iint_{\triangle^c}\left(\nabla h^{\mu^t}(x)-\nabla h^{\mu^t}(y)\right)\cdot\nabla\mathsf{g}(x-y)d(\mu_N^t-\mu^t)(x)d(\mu_N^t-\mu^t)(y)$$

which gives the desired formula.

In the case (1.2) we have

$$\partial_t F_N(X_N^t,\mu^t) = N^2\partial_t\iint\mathsf{g}(x-y)d\mu^t(x)d\mu^t(y)+\partial_t\sum_{i\neq j}\mathsf{g}(x_i^t-x_j^t)-2N\partial_t\sum_{i=1}^N\int_{\mathbb{R}^d}\mathsf{g}(x_i^t-y)d\mu^t(y)$$

$$=2\sum_{i\neq j}\nabla h^{\mu^t}(x_i^t)\cdot\mathbb{J}\nabla\mathsf{g}(x_i^t-x_j^t)+2N\sum_{i=1}^N P.V.\int_{\mathbb{R}^d\setminus\{x_i^t\}}\mathbb{J}\nabla h^{\mu^t}(x)\cdot\nabla\mathsf{g}(x-x_i^t)d\mu^t(x)$$

We then rewrite this as

$$\begin{aligned}\partial_t F_N(X_N^t,\mu^t) &= 2N^2\int_{\mathbb{R}^d}\nabla h^{\mu^t}(x)\cdot\int_{\mathbb{R}^d\setminus\{x\}}\mathbb{J}\nabla\mathsf{g}(x-y)d\mu_N^t(y)d\mu_N^t(x)\\ &\quad +2N^2\int_{\mathbb{R}^d}P.V.\int_{\mathbb{R}^d\setminus\{y\}}\mathbb{J}\nabla h^{\mu^t}(x)\cdot\nabla\mathsf{g}(x-y)d\mu^t(x)d\mu_N^t(y).\end{aligned}$$

By antisymmetry of $\mathbb{J}$, we recognize that the right-hand side can be symmetrized into

$$-N^2\iint_{\triangle^c}\mathbb{J}\left(\nabla h^{\mu^t}(x)-\nabla h^{\mu^t}(y)\right)\cdot\nabla\mathsf{g}(x-y)d(\mu_N^t-\mu^t)(x)d(\mu_N^t-\mu^t)(y).$$

□

The main point is thus to control the last term in the right-hand side of (2.1) or (2.2) : we may reinterpret it via an appropriate stress-energy tensor, which we may show is itself pointwise controlled by the modulated energy density, once we have given a renormalized sense to all the quantities.

The crucial result we obtain this way is the following proposition.

Proposition 2.2. *Assume that μ is a probability density, with $\mu \in C^\sigma(\mathbb{R}^{\mathsf{d}})$ with $\sigma > \mathsf{s}-\mathsf{d}+1$ if $\mathsf{s} \geq \mathsf{d}-1$; respectively $\mu \in L^\infty(\mathbb{R}^{\mathsf{d}})$ or $\mu \in C^\sigma(\mathbb{R}^{\mathsf{d}})$ with $\sigma > 0$ if $\mathsf{s} < \mathsf{d}-1$. For any Lipschitz map $\psi : \mathbb{R}^{\mathsf{d}} \to \mathbb{R}^{\mathsf{d}}$, we have*

$$\left|\iint_{\triangle^c} (\psi(x)-\psi(y))\cdot \nabla \mathsf{g}(x-y) d(\sum_{i=1}^N \delta_{x_i} - N\mu)(x) d(\sum_{i=1}^N \delta_{x_i} - N\mu)(y)\right| \tag{2.3}$$
$$\leq C\|\nabla\psi\|_{L^\infty}\left(F_N(X_N,\mu) + (1+\|\mu\|_{L^\infty})N^{1+\frac{\mathsf{s}}{\mathsf{d}}} + N^{2-\frac{\mathsf{s}+1}{2}} + \left(\frac{N}{\mathsf{d}}\log N\right)\mathbf{1}_{(1.5)}\right)$$
$$+ C\min\left(\|\psi\|_{L^\infty}\|\mu\|_{L^\infty}N^{1+\frac{\mathsf{s}+1}{\mathsf{d}}} + \|\nabla\psi\|_{L^\infty}\|\mu\|_{L^\infty}N^{1+\frac{\mathsf{s}}{\mathsf{d}}}, \|\psi\|_{W^{1,\infty}}\|\mu\|_{C^\sigma}N^{1+\frac{\mathsf{s}+1-\sigma}{\mathsf{d}}}\right)$$
$$+ C\begin{cases}\|\nabla\psi\|_{L^\infty}(1+\|\mu\|_{C^\sigma})N^{2-\frac{1}{\mathsf{d}}} & \text{if } \mathsf{s}\geq \mathsf{d}-1\\ \|\nabla\psi\|_{L^\infty}(1+\|\mu\|_{L^\infty})N^{2-\frac{1}{\mathsf{d}}} & \text{if } \mathsf{s}< \mathsf{d}-1,\end{cases}$$

where C depends only on s, d.

The proof of this proposition will occupy Section 4. Here, all the additive terms should be considered as moderate perturbations of F_N, in particular what matters is that they are $o(N^2)$.

With this result at hand we immediately deduce from Lemma 2.1 that

$$\partial_t F_N(X_N^t,\mu^t) \leq C\|\nabla^2 h^{\mu^t}\|\Big[\left(F_N(X_N^t,\mu^t) + (1+\|\mu^t\|_{L^\infty})N^{1+\frac{\mathsf{s}}{\mathsf{d}}} + N^{\frac{3}{2}} + \left(\frac{N}{\mathsf{d}}\log N\right)\mathbf{1}_{(1.5)}\right)$$
$$+C\begin{cases}(1+\|\mu^t\|_{C^\sigma})N^{2-\frac{1}{\mathsf{d}}} + \left(\|\nabla h^{\mu^t}\|_{L^\infty} + \|\nabla^2 h^{\mu^t}\|_{L^\infty}\right)\|\mu\|_{C^\sigma}N^{1+\frac{\mathsf{s}+1-\sigma}{\mathsf{d}}}\Big] & \text{if } \mathsf{s}\geq\mathsf{d}-1\\ (1+\|\mu^t\|_{L^\infty})N^{2-\frac{1}{\mathsf{d}}} + \|\nabla h^{\mu^t}\|_{L^\infty}\|\mu^t\|_{L^\infty}N^{1+\frac{\mathsf{s}+1}{\mathsf{d}}} + \|\nabla^2 h^{\mu^t}\|_{L^\infty}\|\mu^t\|_{L^\infty})N^{1+\frac{\mathsf{s}}{\mathsf{d}}}\Big] & \text{if } \mathsf{s}<\mathsf{d}-1.\end{cases}$$

Since $\mathsf{s} < \mathsf{d}$ and $\sigma > \mathsf{s}-\mathsf{d}+1$, this implies by Gronwall's lemma and in view of (1.11) that for every $t \leq T$,

$$F_N(X_N^t,\mu^t) \leq \left(F_N(X_N^0,\mu^0) + C_1 N^\beta\right)e^{C_2 t} \quad \text{for some } \beta < 2.$$

In view Proposition 3.5 below, this proves the main theorem.

3. Formulation via the electric potential

3.1. **The extension representation for the fractional Laplacian.** In general, the kernel g is not the convolution kernel of a local operator, but rather of a fractional Laplacian. Here we use the *extension representation* popularized by [CaffSi]: by adding one space variable $y \in \mathbb{R}$ to the space $\mathbb{R}^{\mathsf{d}}$, the nonlocal operator can be transformed into a local operator of the form $-\mathrm{div}\,(|z|^\gamma \nabla \cdot)$.

In what follows, k will denote the dimension extension. We will take $\mathsf{k} = 0$ in the Coulomb cases for which g itself is the kernel of a local operator. In all other cases, we will take $\mathsf{k} = 1$. For now, points in the space $\mathbb{R}^{\mathsf{d}}$ will be denoted by x, and points in the extended space $\mathbb{R}^{\mathsf{d}+\mathsf{k}}$

by X, with $X = (x, z)$, $x \in \mathbb{R}^{\mathsf{d}}$, $z \in \mathbb{R}^{\mathsf{k}}$. We will often identify $\mathbb{R}^{\mathsf{d}} \times \{0\}$ and $\mathbb{R}^{\mathsf{d}}$ and thus $(x_i, 0)$ with x_i.

If γ is chosen such that

$$\mathsf{d} - 2 + \mathsf{k} + \gamma = \mathsf{s}, \tag{3.1}$$

then, given a probability measure μ on $\mathbb{R}^{\mathsf{d}}$, the g-potential generated by μ, defined in $\mathbb{R}^{\mathsf{d}}$ by

$$h^\mu(x) := \int_{\mathbb{R}^{\mathsf{d}}} \mathsf{g}(x - \tilde{x})\, d\mu(\tilde{x})$$

can be extended to a function h^μ on $\mathbb{R}^{\mathsf{d}+\mathsf{k}}$ defined by

$$\mathsf{h}^\mu(X) := \int_{\mathbb{R}^{\mathsf{d}}} \mathsf{g}(X - (\tilde{x}, 0))\, d\mu(\tilde{x}),$$

and this function satisfies

$$-\operatorname{div}(|z|^\gamma \nabla \mathsf{h}^\mu) = \mathsf{c}_{\mathsf{d},\mathsf{s}} \mu \delta_{\mathbb{R}^{\mathsf{d}}} \tag{3.2}$$

where by $\delta_{\mathbb{R}^{\mathsf{d}}}$ we mean the uniform measure on $\mathbb{R}^{\mathsf{d}} \times \{0\}$. The corresponding values of the constants $\mathsf{c}_{\mathsf{d},\mathsf{s}}$ are given in [PS, Section 1.2]. In particular, the potential g seen as a function of $\mathbb{R}^{\mathsf{d}+\mathsf{k}}$ satisfies

$$-\operatorname{div}(|z|^\gamma \nabla \mathsf{g}) = \mathsf{c}_{\mathsf{d},\mathsf{s}} \delta_0. \tag{3.3}$$

To summarize, we will take

- $\mathsf{k} = 0, \gamma = 0$ in the Coulomb cases. The reader only interested in the Coulomb cases may thus just ignore the k and the weight $|z|^\gamma$ in all the integrals.
- $\mathsf{k} = 1, \gamma = \mathsf{s} - \mathsf{d} + 2 - \mathsf{k}$ in the Riesz cases and in the one-dimensional logarithmic case (then we mean $\mathsf{s} = 0$). Note that our assumption $(\mathsf{d} - 2)_+ \le \mathsf{s} < \mathsf{d}$ implies that γ is always in $(-1, 1)$. We refer to [PS, Section 1.2] for more details.

We now make a remark on the regularity of h^μ:

Lemma 3.1. *Assume μ is a probability density in $C^\theta(\mathbb{R}^{\mathsf{d}})$ for some $\theta > \mathsf{s} - \mathsf{d} + 2$, then we have*

$$\|\nabla h^\mu\|_{L^\infty(\mathbb{R}^{\mathsf{d}})} \le C \left(\|\mu\|_{C^{\theta-1}(\mathbb{R}^{\mathsf{d}})} + \|\mu\|_{L^1(\mathbb{R}^{\mathsf{d}})} \right), \tag{3.4}$$

and

$$\|\nabla^2 h^\mu\|_{L^\infty(\mathbb{R}^{\mathsf{d}})} \le C \left(\|\mu\|_{C^{\theta}(\mathbb{R}^{\mathsf{d}})} + \|\mu\|_{L^1(\mathbb{R}^{\mathsf{d}})} \right). \tag{3.5}$$

Proof. As is well known, g is (up to a constant) the kernel of $\Delta^{\frac{\mathsf{d}-\mathsf{s}}{2}}$, hence $h^\mu = \mathsf{c}_{\mathsf{d},\mathsf{s}} \Delta^{\frac{\mathsf{s}-\mathsf{d}}{2}} \mu$ and the relations follow (cf. also [Du, Lemma 2.5]). □

3.2. **Electring rewriting of the energy.** We briefly recall the procedure used in [RS, PS] for truncating the interaction or, equivalently, spreading out the point charges. It will also be crucial to use the variant introduced in [LSZ, LS2] where we let the truncation distance depend on the point.

For any $\eta \in (0, 1)$, we define

$$\mathsf{g}_\eta := \min(\mathsf{g}, \mathsf{g}(\eta)), \quad \mathsf{f}_\eta := \mathsf{g} - \mathsf{g}_\eta \tag{3.6}$$

and

$$\delta_0^{(\eta)} := -\frac{1}{\mathsf{c}_{\mathsf{d},\mathsf{s}}} \operatorname{div}(|z|^\gamma \nabla \mathsf{g}_\eta), \tag{3.7}$$

which is a positive measure supported on $\partial B(0,\eta)$.

Remark 3.2. *This nonsmooth truncation of* g_η *can be replaced with no change by a smooth one such that*

$$\mathsf{g}_\eta(x)=\mathsf{g}(x) \text{ for } |x|\ge\eta, \quad \mathsf{g}_\eta(x)=cst \text{ for } |x|\le \eta-\varepsilon \qquad \varepsilon<\frac{1}{2}\eta$$

and this way $\delta_0^{(\eta)}$ *gets replaced by a probability measure with a regular density supported in* $B(0,\eta)\backslash B(0,\eta-\varepsilon)$. *We make this modification whenever the integrals against the singular measures may not be well-defined.*

We will also let

$$\mathsf{f}_{\alpha,\eta}:=\mathsf{f}_\alpha-\mathsf{f}_\eta=\mathsf{g}_\eta-\mathsf{g}_\alpha, \tag{3.8}$$

and we observe that $\mathsf{f}_{\alpha,\eta}$ has the sign of $\alpha-\eta$, vanishes outside $B(0,\max(\alpha,\eta))$, and satisfies

$$\mathsf{g}*(\delta_x^{(\eta)}-\delta_x^{(\alpha)})=\mathsf{f}_{\alpha,\eta}(\cdot-x) \tag{3.9}$$

and

$$-\operatorname{div}(|z|^\gamma\nabla \mathsf{f}_{\alpha,\eta})=\mathsf{c}_{\mathsf{d},\mathsf{s}}(\delta_0^{(\eta)}-\delta_0^{(\alpha)}). \tag{3.10}$$

For any configuration $X_N=(x_1,\dots,x_N)$, we define for any i the minimal distance

$$\mathsf{r}_i=\min\left(\frac{1}{4}\min_{j\neq i}|x_i-x_j|, N^{-\frac{1}{\mathsf{d}}}\right). \tag{3.11}$$

For any $\vec{\eta}=(\eta_1,\dots,\eta_N)\in\mathbb{R}^N$ and measure μ, we define the electric potential

$$H_N^\mu[X_N]=\int_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}}\mathsf{g}(x-y)d\left(\sum_{i=1}^N\delta_{x_i}-N\mu\delta_{\mathbb{R}^\mathsf{d}}\right)(y) \tag{3.12}$$

and the truncated potential

$$H_{N,\vec{\eta}}^\mu[X_N]=\int_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}}\mathsf{g}(x-y)d\left(\sum_{i=1}^N\delta_{x_i}^{(\eta_i)}-N\mu\delta_{\mathbb{R}^\mathsf{d}}\right)(y), \tag{3.13}$$

where we will quickly drop the dependence in X_N. We note that

$$H_{N,\vec{\eta}}^\mu[X_N]=H_N^\mu[X_N]-\sum_{i=1}^N\mathsf{f}_\eta(x-x_i). \tag{3.14}$$

These functions are viewed in the extended space $\mathbb{R}^{\mathsf{d}+\mathsf{k}}$ as described in the previous subsection, and solve

$$-\operatorname{div}(|z|^\gamma\nabla H_N^\mu)=\mathsf{c}_{\mathsf{d},\mathsf{s}}\left(\sum_{i=1}^N\delta_{x_i}-N\mu\delta_{\mathbb{R}^\mathsf{d}}\right) \quad \text{in } \mathbb{R}^{\mathsf{d}+\mathsf{k}}, \tag{3.15}$$

and

$$-\operatorname{div}(|z|^\gamma\nabla H_{N,\vec{\eta}}^\mu)=\mathsf{c}_{\mathsf{d},\mathsf{s}}\left(\sum_{i=1}^N\delta_{x_i}^{(\eta_i)}-N\mu\delta_{\mathbb{R}^\mathsf{d}}\right) \quad \text{in } \mathbb{R}^{\mathsf{d}+\mathsf{k}}. \tag{3.16}$$

The following proposition shows how to express F_N in terms of the truncated electric fields $\nabla H^\mu_{N,\vec\eta}$. In addition, we show that the quantities

$$\int_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}} |z|^\gamma |\nabla H^\mu_{N,\vec\eta}|^2 - \mathsf{c}_{\mathsf{d},\mathsf{s}} \sum_{i=1}^N \mathsf{g}(\eta_i)$$

converge almost monotonically (i.e. up to a small error) to F_N, while the discrepancy between the two can serve to control the energy of close pairs of points.

Proposition 3.3. *Let μ be a bounded probability density on $\mathbb{R}^{\mathsf{d}}$ and X_N be in $(\mathbb{R}^{\mathsf{d}})^N$. We may re-write $F_N(X_N,\mu)$ as*

$$F_N(X_N,\mu) := \frac{1}{\mathsf{c}_{\mathsf{d},\mathsf{s}}} \lim_{\eta\to 0} \left(\int_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}} |z|^\gamma |\nabla H^\mu_{N,\vec\eta}|^2 - \mathsf{c}_{\mathsf{d},\mathsf{s}} \sum_{i=1}^N \mathsf{g}(\eta_i) \right), \tag{3.17}$$

and we have the bound

$$\sum_{i\neq j} (\mathsf{g}(x_i-x_j) - \mathsf{g}(\eta_i))_+ \le F_N(X_N,\mu) - \left(\frac{1}{\mathsf{c}_{\mathsf{d},\mathsf{s}}} \int_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}} |z|^\gamma |\nabla H^\mu_{N,\vec\eta}|^2 - \sum_{i=1}^N \mathsf{g}(\eta_i) \right) + CN\|\mu\|_{L^\infty} \sum_{i=1}^N \eta_i^{\mathsf{d}-\mathsf{s}}, \tag{3.18}$$

for some C depending only on d and s.

The proof, which is an adaptation and improvement of [PS, LS2], is postponed to Section 5.

What makes our main proof work is the ability to find some choice of truncation $\vec\eta$ such that $\int_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}} |z|^\gamma |\nabla H^\mu_{N,\vec\eta}|^2$ (*without* the renormalizing term $-\mathsf{c}_{\mathsf{d},\mathsf{s}} \sum_{i=1}^N \mathsf{g}(\eta_i)$) is controlled by $F_N(X_N,\mu)$ *and* the balls $B(x_i,\eta_i)$ are disjoint. In view of (3.18) the former could easily be achieved by taking the η_i's large enough, say $\eta_i = N^{-1/\mathsf{d}}$, but the balls would not necessarily be disjoint. Instead the choice of $\eta_i = \mathsf{r}_i$ where r_i are the minimal distances as in (3.11) allows to fulfill both requirements, as seen in the following

Corollary 3.4. *We have*

$$\sum_{i=1}^N \mathsf{g}(\mathsf{r}_i) \le C\left(F_N(X_N,\mu) + (1+\|\mu\|_{L^\infty}) N^{1+\frac{\mathsf{s}}{\mathsf{d}}} + \left(\frac{N}{\mathsf{d}} \log N\right) \mathbf{1}_{(1.5)} \right) \tag{3.19}$$

and

$$\int_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}} |z|^\gamma |\nabla H^\mu_{N,\vec{\mathsf{r}}}|^2 \le C\left(F_N(X_N,\mu) + (1+\|\mu\|_{L^\infty}) N^{1+\frac{\mathsf{s}}{\mathsf{d}}} + \left(\frac{N}{\mathsf{d}} \log N\right) \mathbf{1}_{(1.5)} \right) \tag{3.20}$$

for some C depending only on s,d.

Proof. Let us choose $\eta_i = N^{-1/\mathsf{d}}$ for all i in (3.18) and observe that for each i, by definition (3.11) there exists $j\neq i$ such that $(\mathsf{g}(|x_i-x_j|) - \mathsf{g}(N^{-1/\mathsf{d}}))_+ = \mathsf{g}(4\mathsf{r}_i) - \mathsf{g}(N^{-1/\mathsf{d}})$. We may thus write that

$$\sum_{i=1}^N (\mathsf{g}(4\mathsf{r}_i) - \mathsf{g}(N^{-\frac{1}{\mathsf{d}}})) \le F_N(X_N,\mu) - \frac{1}{\mathsf{c}_{\mathsf{d},\mathsf{s}}} \int_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}} |z|^\gamma |\nabla H_{N,\vec\eta}|^2 + N\mathsf{g}(N^{-\frac{1}{\mathsf{d}}}) + O(N\|\mu\|_{L^\infty}) N^{\frac{\mathsf{s}}{\mathsf{d}}}. \tag{3.21}$$

from which (3.19) follows after rearranging and using the definition of g.

Let us next choose $\eta_i = \mathsf{r}_i$ in (3.18). Using that $\mathsf{r}_i \le N^{-1/\mathsf{d}}$, this yields

$$0 \le F_N(X_N, \mu) - \frac{1}{\mathsf{c}_{\mathsf{d},\mathsf{s}}} \int_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}} |z|^\gamma |\nabla H_{N,\vec{\mathsf{r}}}|^2 + \sum_{i=1}^N \mathsf{g}(\mathsf{r}_i) + O(N\|\mu\|_{L^\infty}) N^{\frac{\mathsf{s}}{\mathsf{d}}}.$$

Combining with (3.19), (3.20) follows. ☐

3.3. **Coerciveness of the modulated energy.** Here we prove that the modulated energy does metrize the convergence of μ_N^t to μ^t.

Proposition 3.5. *We have, for B_R the ball of radius R in $\mathbb{R}^{\mathsf{d}}$ centered at 0, for every $1 < p < \frac{2\mathsf{d}}{\mathsf{s}+\mathsf{d}}$,*

$$\|\nabla H_N^\mu[X_N]\|_{L^p(B_R, L^2(\mathbb{R}^{\mathsf{k}}, |z|^\gamma dy))} \le C_p N^{\frac{1}{p}} + C_R \left(F_N(X_N, \mu) + \left(\frac{N}{\mathsf{d}} \log N\right) \mathbf{1}_{(1.5)} + (1 + \|\mu\|_{L^\infty}) N^{1+\frac{\mathsf{s}}{\mathsf{d}}} \right)^{\frac{1}{2}} \tag{3.22}$$

In particular, if $\frac{1}{N^2} F_N(X_N, \mu) \to 0$ as $N \to \infty$, we have that

$$\frac{1}{N} \sum_{i=1}^N \delta_{x_i} \rightharpoonup \mu \quad \text{in the weak sense.} \tag{3.23}$$

Proof. In the Coulomb cases where $\mathsf{k} = 0$, we have

$$\int_{\mathbb{R}^{\mathsf{d}}} |\nabla \mathsf{f}_{\mathsf{r}_i}|^p \le C_p \qquad \forall p < \frac{\mathsf{d}}{\mathsf{d}-1},$$

while in the cases where $\mathsf{k} = 1$, we may write instead that

$$\int_{B_R} \left(\int_{\mathbb{R}^{\mathsf{k}}} |z|^\gamma |\nabla \mathsf{f}_{\mathsf{r}_i}|^2 \right)^{\frac{p}{2}} \le C_p \quad \text{for} \quad p < \frac{2\mathsf{d}}{\mathsf{s}+\mathsf{d}}.$$

Using (3.14) and the fact that the balls $B(x_i, \mathsf{r}_i)$ are disjoint, we may thus write

$$\int_{B_R} \left(\int_{\mathbb{R}^{\mathsf{k}}} |z|^\gamma |\nabla H_N^\mu|^2 \right)^{\frac{p}{2}} \le C \int_{B_R} \left(\int_{\mathbb{R}^{\mathsf{k}}} |z|^\gamma |\nabla H_{N,\vec{\mathsf{r}}}^\mu|^2 \right)^{\frac{p}{2}} + N C_p.$$

Combining with (3.20) and Hölder's inequality, it follows that

$$\int_{B_R} \left(\int_{\mathbb{R}^{\mathsf{k}}} |z|^\gamma |\nabla H_N^\mu|^2 \right)^{\frac{p}{2}} \le N C_p + C_R \left(F_N(X_N, \mu) + C(1 + \|\mu\|_{L^\infty}) N^{1+\frac{\mathsf{s}}{\mathsf{d}}} + \left(\frac{N}{\mathsf{d}} \log N \right) \mathbf{1}_{(1.5)} \right)^{\frac{p}{2}},$$

which implies the result. Since

$$\frac{1}{N} \sum_{i=1}^N \delta_{x_i} - \mu = -\frac{1}{N \mathsf{c}_{\mathsf{d},\mathsf{s}}} \mathrm{div}\, (|z|^\gamma \nabla H_N^\mu),$$

testing the left-hand side against any smooth test-function yields a result which is $o(1)$. ☐

Remark 3.6. *In a density formulation aiming at proving propagation of chaos, arguing exactly as in [RS, Lemma 8.4] for instance, we may deduce from this result and the main theorem the convergence of the k-marginal densities in the dual of (some weighted) $W^{1,p}$ space with p large enough, with rate k/N times the right-hand side of (3.22).*

4. Proof of Proposition 2.2

4.1. Stress-energy tensor.

Definition 4.1. For any functions h, f in $\mathbb{R}^{\mathsf{d}+\mathsf{k}}$ such that $\int_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}} |z|^\gamma |\nabla h|^2$ and $\int_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}} |z|^\gamma |\nabla f|^2$ are finite, we define the stress tensor $[h, f]$ as the $(\mathsf{d}+\mathsf{k}) \times (\mathsf{d}+\mathsf{k})$ tensor

$$[h, f] = |z|^\gamma (\partial_i h \partial_j f + \partial_i h \partial_j f) - |z|^\gamma \nabla h \cdot \nabla f \delta_{ij} \tag{4.1}$$

where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise.

We note that

Lemma 4.2. *If h and f are regular enough, we have*

$$\mathrm{div}\,[h, f] = \mathrm{div}\,(|z|^\gamma \nabla h)\nabla f + \mathrm{div}\,(|z|^\gamma \nabla f)\nabla h - \nabla |z|^\gamma \nabla h \cdot \nabla f \tag{4.2}$$

where $\mathrm{div}\, T$ *here denotes the vector with components* $\sum_i \partial_i T_{ij}$, *with* j *ranging from* 1 *to* $\mathsf{d}+\mathsf{k}$.

Proof. This is a direct computation. Below, all sums range from 1 to $\mathsf{d}+\mathsf{k}$.

$$\begin{aligned}
&\sum_i \partial_i [h, f]_{ij} \\
&= \sum_i [\partial_i(|z|^\gamma \partial_i h)\partial_j f + \partial_i(|z|^\gamma \partial_i f)\partial_j h + |z|^\gamma \partial_{ij} h \partial_i f + |z|^\gamma \partial_{ij} f \partial_i h] - \partial_j \left(|z|^\gamma \sum_i \partial_i h \partial_i f\right) \\
&\qquad = \mathrm{div}\,(|z|^\gamma \nabla h) + \mathrm{div}\,(|z|^\gamma \nabla f) - \nabla h \cdot \nabla f \partial_j |z|^\gamma.
\end{aligned}$$

□

In view of (4.2), we have

Lemma 4.3. *Let $\psi : \mathbb{R}^{\mathsf{d}} \to \mathbb{R}^{\mathsf{d}}$ be Lipschitz, and if $\mathsf{k} = 1$ let $\hat\psi$ be an extension of it to a map from $\mathbb{R}^{\mathsf{d}+\mathsf{k}}$ to $\mathbb{R}^{\mathsf{d}+\mathsf{k}}$, whose last component identically vanishes, which tends to 0 as $|y| \to \infty$ and has the same pointwise and Lipschitz bounds as ψ. For any measures μ, ν on $\mathbb{R}^{\mathsf{d}+\mathsf{k}}$, if $-\mathrm{div}\,(|z|^\gamma \nabla \mathsf{h}^\mu) = \mathsf{c}_{\mathsf{d},\mathsf{s}}\mu$ and $-\mathrm{div}\,(|z|^\gamma \nabla \mathsf{h}^\nu) = \mathsf{c}_{\mathsf{d},\mathsf{s}}\nu$, and assuming that $\int_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}} |z|^\gamma |\nabla \mathsf{h}^\mu|^2$ and $\int_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}} |z|^\gamma |\nabla \mathsf{h}^\nu|^2$ are finite and the left-hand side in (4.3) is well-defined, we have*

$$\iint_{\mathbb{R}^{\mathsf{d}+\mathsf{k}} \times \mathbb{R}^{\mathsf{d}+\mathsf{k}}} (\hat\psi(x) - \hat\psi(y)) \cdot \nabla \mathsf{g}(x-y) d\mu(x) d\nu(y) = \frac{1}{\mathsf{c}_{\mathsf{d},\mathsf{s}}} \int_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}} \nabla \hat\psi(x) : [\mathsf{h}^\mu, \mathsf{h}^\nu]. \tag{4.3}$$

Proof. If μ is smooth enough then we may use (4.2) to write

$$\begin{aligned}
\iint_{\mathbb{R}^{\mathsf{d}+\mathsf{k}} \times \mathbb{R}^{\mathsf{d}+\mathsf{k}}} (\hat\psi(x) - \hat\psi(y)) \cdot \nabla \mathsf{g}(x-y) d\mu(x) d\nu(y) &= \int_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}} \hat\psi \cdot (\nabla \mathsf{h}^\mu d\nu + \nabla \mathsf{h}^\nu d\mu) \\
&= -\frac{1}{\mathsf{c}_{\mathsf{d},\mathsf{s}}} \int_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}} \hat\psi \cdot \mathrm{div}\,[\mathsf{h}^\mu, \mathsf{h}^\nu]
\end{aligned}$$

since the last component of $\hat\psi$ vanishes identically. Integrating by parts, we obtain

$$\iint_{\mathbb{R}^{\mathsf{d}+\mathsf{k}} \times \mathbb{R}^{\mathsf{d}+\mathsf{k}}} (\hat\psi(x) - \hat\psi(y)) \cdot \nabla \mathsf{g}(x-y) d\mu(x) d\nu(y) = \frac{1}{\mathsf{c}_{\mathsf{d},\mathsf{s}}} \int_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}} \nabla \hat\psi : [\mathsf{h}^\mu, \mathsf{h}^\nu].$$

By density, we may extend this relation to all measures μ, ν such that both sides of (4.3) make sense. □

4.2. **Proof of Proposition 2.2.** We now proceed to the proof. Given the Lipschitz map $\psi:\mathbb{R}^\mathsf{d}\to\mathbb{R}^\mathsf{d}$, we choose an extension $\hat\psi$ to $\mathbb{R}^{\mathsf{d}+\mathsf{k}}$ which satisfies the same conditions as in Lemma 4.3.

Step 1: *renormalizing the quantity and expressing it with the stress-energy tensor.* Clearly,

$$\begin{aligned}(4.4)\quad &\iint_{\triangle^c}(\psi(x)-\psi(y))\cdot\nabla\mathsf{g}(x-y)d(\sum_{i=1}^N\delta_{x_i}-N\mu)(x)d(\sum_{i=1}^N\delta_{x_i}-N\mu)(y)\\
&=\lim_{\eta\to0}\Big[\iint_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}\times\mathbb{R}^{\mathsf{d}+\mathsf{k}}}\Big(\hat\psi(x)-\hat\psi(y)\Big)\cdot\nabla\mathsf{g}(x-y)d(\sum_{i=1}^N\delta_{x_i}^{(\eta)}-N\mu\delta_{\mathbb{R}^\mathsf{d}})(x)d(\sum_{i=1}^N\delta_{x_i}^{(\eta)}-N\mu\delta_{\mathbb{R}^\mathsf{d}})(y)\\
&\qquad-\sum_{i=1}^N\iint_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}\times\mathbb{R}^{\mathsf{d}+\mathsf{k}}}\Big(\hat\psi(x)-\hat\psi(y)\Big)\cdot\nabla\mathsf{g}(x-y)d\delta_{x_i}^{(\eta)}(x)d\delta_{x_i}^{(\eta)}(y)\Big].\end{aligned}$$

Applying Lemma 4.3, in view of (3.13), (3.16) we find that

$$\begin{aligned}(4.5)\quad &\iint_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}\times\mathbb{R}^{\mathsf{d}+\mathsf{k}}}\Big(\hat\psi(x)-\hat\psi(y)\Big)\cdot\nabla\mathsf{g}(x-y)d(\sum_{i=1}^N\delta_{x_i}^{(\eta)}-N\mu\delta_{\mathbb{R}^\mathsf{d}})(x)d(\sum_{i=1}^N\delta_{x_i}^{(\eta)}-N\mu\delta_{\mathbb{R}^\mathsf{d}})(y)\\
&=\frac{1}{\mathsf{c}_{\mathsf{d},\mathsf{s}}}\int_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}}\nabla\hat\psi:[H^\mu_{N,\vec\eta},H^\mu_{N,\vec\eta}].\end{aligned}$$

Step 2: *analysis of the diagonal terms.* The main point is to understand how they vary with η_i. Let $\vec\alpha$ be such that $\alpha_i\ge\eta_i$ for every i.

We may write that

$$\begin{aligned}(4.6)\quad &\iint_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}\times\mathbb{R}^{\mathsf{d}+\mathsf{k}}}\Big(\hat\psi(x)-\hat\psi(y)\Big)\cdot\nabla\mathsf{g}(x-y)d\delta_{x_i}^{(\eta_i)}(x)d\delta_{x_i}^{(\eta_i)}(y)\\
&\quad-\iint_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}\times\mathbb{R}^{\mathsf{d}+\mathsf{k}}}\Big(\hat\psi(x)-\hat\psi(y)\Big)\cdot\nabla\mathsf{g}(x-y)d\delta_{x_i}^{(\alpha_i)}(x)d\delta_{x_i}^{(\alpha_i)}(y)\\
&=\iint_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}\times\mathbb{R}^{\mathsf{d}+\mathsf{k}}}(\hat\psi(x)-\hat\psi(y))\cdot\nabla\mathsf{g}(x-y)d(\delta_{x_i}^{(\eta_i)}-\delta_{x_i}^{(\alpha_i)})(x)d(\delta_{x_i}^{(\eta_i)}-\delta_{x_i}^{(\alpha_i)})(y)\\
&\quad+2\iint_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}\times\mathbb{R}^{\mathsf{d}+\mathsf{k}}}(\hat\psi(x)-\hat\psi(y))\cdot\nabla\mathsf{g}(x-y)d\delta_{x_i}^{(\alpha_i)}(x)d(\delta_{x_i}^{(\eta_i)}-\delta_{x_i}^{(\alpha_i)})(y).\end{aligned}$$

We claim that

$$(4.7)\quad \iint_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}\times\mathbb{R}^{\mathsf{d}+\mathsf{k}}}(\hat\psi(x)-\hat\psi(y))\cdot\nabla\mathsf{g}(x-y)d\delta_{x_i}^{(\alpha_i)}(x)d(\delta_{x_i}^{(\eta_i)}-\delta_{x_i}^{(\alpha_i)})(y)=0.$$

Assuming this, inserting it to (4.6) and using (3.9) and Lemma 4.3, we conclude that

$$\begin{aligned}(4.8)\quad &\iint_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}\times\mathbb{R}^{\mathsf{d}+\mathsf{k}}}\Big(\hat\psi(x)-\hat\psi(y)\Big)\cdot\nabla\mathsf{g}(x-y)d\delta_{x_i}^{(\eta)}(x)d\delta_{x_i}^{(\eta)}(y)\\
&\quad-\iint_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}\times\mathbb{R}^{\mathsf{d}+\mathsf{k}}}\Big(\hat\psi(x)-\hat\psi(y)\Big)\cdot\nabla\mathsf{g}(x-y)d\delta_{x_i}^{(\alpha_i)}(x)d\delta_{x_i}^{(\alpha_i)}(y)\\
&=\frac{1}{\mathsf{c}_{\mathsf{d},\mathsf{s}}}\int_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}}\nabla\hat\psi:[\mathsf{f}_{\alpha_i,\eta_i}(\cdot-x_i),\mathsf{f}_{\alpha_i,\eta_i}(\cdot-x_i)].\end{aligned}$$

Step 3: *proof of* (4.7). Let us write the quantity in (4.7) as

$$2\iint_{\mathbb{R}^{\mathsf{d+k}}\times\mathbb{R}^{\mathsf{d+k}}}(\hat\psi(x)-\hat\psi(y))\cdot\nabla \mathsf{g}(x-y)d\delta_{x_i}^{(2\alpha_i)}(x)d(\delta_{x_i}^{(\eta_i)}-\delta_{x_i}^{(\alpha_i)})(y)$$
$$+2\iint_{\mathbb{R}^{\mathsf{d+k}}\times\mathbb{R}^{\mathsf{d+k}}}(\hat\psi(x)-\hat\psi(y))\cdot\nabla \mathsf{g}(x-y)d\left(\delta_{x_i}^{(\alpha_i)}-\delta_{x_i}^{(2\alpha_i)}\right)(x)d\left(\delta_{x_i}^{(\eta_i)}-\delta_{x_i}^{(\alpha_i)}\right)(y). \tag{4.9}$$

In view of (3.7), $\nabla \mathsf{g}*\delta_{x_i}^{(2\alpha_i)}=\nabla \mathsf{g}_{2\alpha_i}(\cdot-x_i)$ and in view of (3.9), $\nabla\mathsf{g}*(\delta_{x_i}^{(\eta_i)}-\delta_{x_i}^{(\alpha_i)})=\nabla \mathsf{f}_{\alpha_i,\eta_i}(x-x_i)$. We may thus rewrite the first term in (4.9) as

$$2P.V.\int_{\mathbb{R}^{\mathsf{d+k}}}\hat\psi\cdot\nabla\mathsf{f}_{\alpha_i,\eta_i}(\cdot-x_i)d\delta_{x_i}^{(2\alpha_i)}+2P.V.\int_{\mathbb{R}^{\mathsf{d+k}}}\hat\psi\cdot\nabla\mathsf{g}_{2\alpha_i}(\cdot-x_i)d\left(\delta_{x_i}^{(\eta_i)}-\delta_{x_i}^{(\alpha_i)}\right).$$

But $\nabla\mathsf{f}_{\alpha_i,\eta_i}(\cdot-x_i)$ is supported in $B(x_i,\alpha_i)$ while $\delta_{x_i}^{(2\alpha_i)}$ is supported on $\partial B(x_i,2\alpha_i)$, and in the same way $\nabla\mathsf{g}_{\alpha_i}(\cdot-x_i)$ vanishes in $B(x_i,2\alpha_i)$ where $\delta_{x_i}^{(\eta_i)}-\delta_{x_i}^{(\alpha_i)}$ is supported, so we conclude that the first term in (4.9) is zero. The second term in (4.9) is equal by (3.9) and Lemma 4.3 to

$$\frac{1}{\mathsf{c_{d,s}}}\int_{\mathbb{R}^{\mathsf{d+k}}}\nabla\hat\psi:[\mathsf{f}_{2\alpha_i,\alpha_i}(\cdot-x_i),\mathsf{f}_{\alpha_i,\eta_i}(\cdot-x_i)]$$

and it is zero, since $\mathsf{f}_{2\alpha_i,\alpha_i}$ and $\mathsf{f}_{\alpha_i,\eta_i}$ have disjoint supports. This finishes the proof of (4.7).

Step 4: *combining* (4.5) *and* (4.8). The following lemma allows to recombine the terms obtained at different values of η_i while making only a small error.

Lemma 4.4. *Assume that $\mu\in C^\sigma(\mathbb{R}^{\mathsf{d}})$ with $\sigma>\mathsf{s}-\mathsf{d}+1$ if $\mathsf{s}\ge\mathsf{d}-1$. Assume $\mu\in L^\infty(\mathbb{R}^{\mathsf{d}})$ or $\mu\in C^\sigma(\mathbb{R}^{\mathsf{d}})$ with $\sigma>0$ if $\mathsf{s}<\mathsf{d}-1$. If for each i we have $\eta_i<\alpha_i\le\mathsf{r}_i$, then*

$$\int_{\mathbb{R}^{\mathsf{d+k}}}\nabla\hat\psi:[H^\mu_{N,\vec\eta},H^\mu_{N,\vec\eta}]=\int_{\mathbb{R}^{\mathsf{d+k}}}\nabla\hat\psi:[H^\mu_{N,\vec\alpha},H^\mu_{N,\vec\alpha}]+\sum_{i=1}^N\int_{\mathbb{R}^{\mathsf{d+k}}}\nabla\hat\psi:[\mathsf{f}_{\alpha_i,\eta_i}(\cdot-x_i),\mathsf{f}_{\alpha_i,\eta_i}(\cdot-x_i)]+\mathcal{E}$$

with

$$|\mathcal{E}|\le C\|\nabla\psi\|_{L^\infty}\left(N^{2-\frac{\mathsf{s}+1}{2}}+F_N(X_N,\mu)+\left(\frac{N}{\mathsf{d}}\log N\right)\mathbf{1}_{(1.5)}+(1+\|\mu\|_{L^\infty})N^{1+\frac{\mathsf{s}}{\mathsf{d}}}\right) \tag{4.10}$$
$$+CN\min\left(\|\psi\|_{L^\infty}\|\mu\|_{L^\infty}\sum_{i=1}^N\alpha_i^{\mathsf{d-s}-1}+\|\nabla\psi\|_{L^\infty}\|\mu\|_{L^\infty}\sum_{i=1}^N\alpha_i^{\mathsf{d-s}},\|\psi\|_{W^{1,\infty}}\|\mu\|_{C^\sigma}\sum_{i=1}^N\alpha_i^{\mathsf{d-s}+\sigma-1}\right)$$
$$+CN\begin{cases}\|\nabla\psi\|_{L^\infty}(1+\|\mu\|_{C^\sigma})\sum_{i=1}^N\alpha_i & \text{if } \mathsf{s}\ge\mathsf{d}-1\\ \|\nabla\psi\|_{L^\infty}(1+\|\mu\|_{L^\infty})\sum_{i=1}^N\alpha_i & \text{if } \mathsf{s}<\mathsf{d}-1\end{cases}$$

where C depends only on s,d.

Assuming this, and combining (4.4), (4.5) and (4.8) we find that for any $\alpha_i\le\mathsf{r}_i$,

$$\iint_{\triangle^c}(\psi(x)-\psi(y))\cdot\nabla\mathsf{g}(x-y)d(\sum_{i=1}^N\delta_{x_i}-N\mu)(x)d(\sum_{i=1}^N\delta_{x_i}-N\mu)(y)=\frac{1}{\mathsf{c_{d,s}}}\int_{\mathbb{R}^{\mathsf{d+k}}}\nabla\hat\psi:[H^\mu_{N,\vec\alpha},H^\mu_{N,\vec\alpha}]$$
$$-\sum_{i=1}^N\iint_{\mathbb{R}^{\mathsf{d+k}}\times\mathbb{R}^{\mathsf{d+k}}}\left(\hat\psi(x)-\hat\psi(y)\right)\cdot\nabla\mathsf{g}(x-y)d\delta_{x_i}^{(\alpha_i)}(x)d\delta_{x_i}^{(\alpha_i)}(y)+O(\mathcal{E})$$

where $\mathcal{E}$ is as in (4.10). Using the Lipschitz character of ψ and the expression of g, we find that the second term on the right-hand side can be bounded by [1]

$$C\|\nabla\psi\|_{L^\infty}\sum_{i=1}^N\iint_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}\times\mathbb{R}^{\mathsf{d}+\mathsf{k}}}\mathsf{g}(x-y)d\delta_{x_i}^{(\alpha_i)}(x)d\delta_{x_i}^{(\alpha_i)}(y)$$
$$=C\|\nabla\psi\|_{L^\infty}\sum_{i=1}^N\int_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}}\mathsf{g}_{\alpha_i}(\cdot-x_i)d\delta_{x_i}^{(\alpha_i)}=C\|\nabla\psi\|_{L^\infty}\sum_{i=1}^N\mathsf{g}(\alpha_i)$$

where we have used (3.7). Choosing finally $\alpha_i=\mathsf{r}_i\le N^{-1/\mathsf{d}}$, bounding pointwise $[H^\mu_{N,\vec{\mathsf{r}}},H^\mu_{N,\vec{\mathsf{r}}}]$ by $2|z|^\gamma|\nabla H^\mu_{N,\vec{\mathsf{r}}}|^2$ and using (3.20), while using (3.19) to bound $\sum_{i=1}^N\mathsf{g}(\mathsf{r}_i)$, we conclude the proof of Proposition 2.2.

Proof of Lemma 4.4. First, we observe from (3.14) that $[H^\mu_{N,\vec{\alpha}},H^\mu_{N,\vec{\alpha}}]$ and $[H^\mu_{N,\vec{\eta}},H^\mu_{N,\vec{\eta}}]$ only differ in the balls $B(x_i,\alpha_i)$ which are disjoint since $\alpha_i\le\mathsf{r}_i$, and that in each $B(x_i,\alpha_i)$ we have

$$H^\mu_{N,\vec{\eta}}=H^\mu_{N,\vec{\alpha}}+\mathsf{f}_{\alpha_i,\eta_i}(\cdot-x_i).$$

We thus deduce that

$$\int_{B(x_i,\alpha_i)}\nabla\hat{\psi}:\left([H^\mu_{N,\vec{\eta}},H^\mu_{N,\vec{\eta}}]-[H^\mu_{N,\vec{\alpha}},H^\mu_{N,\vec{\alpha}}]\right)\tag{4.11}$$
$$=\int_{B(x_i,\alpha_i)}\nabla\hat{\psi}:\left([\mathsf{f}_{\alpha_i,\eta_i},\mathsf{f}_{\alpha_i,\eta_i}](\cdot-x_i)+2[\mathsf{f}_{\alpha_i,\eta_i}(\cdot-x_i),H^\mu_{N,\vec{\alpha}}]\right)$$
$$=\int_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}}\nabla\hat{\psi}:\left([\mathsf{f}_{\alpha_i,\eta_i},\mathsf{f}_{\alpha_i,\eta_i}](\cdot-x_i)+2[\mathsf{f}_{\alpha_i,\eta_i}(\cdot-x_i),H^\mu_{N,\vec{\alpha}}]\right).$$

There only remains to control the second part of the right-hand side. By Lemma 4.3, we have

$$\int_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}}\nabla\hat{\psi}:[\mathsf{f}_{\alpha_i,\eta_i}(\cdot-x_i),H^\mu_{N,\vec{\alpha}}]$$
$$=\mathsf{c}_{\mathsf{d},\mathsf{s}}\iint_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}\times\mathbb{R}^{\mathsf{d}+\mathsf{k}}}(\hat{\psi}(x)-\hat{\psi}(y))\cdot\nabla\mathsf{g}(x-y)d\left(\sum_{j=1}^N\delta_{x_j}^{(\alpha_j)}-N\mu\delta_{\mathbb{R}^{\mathsf{d}}}\right)(x)d\left(\delta_{x_i}^{(\eta_i)}-\delta_{x_i}^{(\alpha_i)}\right)(y).$$

In view of (4.7), we just need to bound the sum over i of

(4.12)

$$\mathsf{c}_{\mathsf{d},\mathsf{s}}\iint_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}\times\mathbb{R}^{\mathsf{d}+\mathsf{k}}}(\hat{\psi}(x)-\hat{\psi}(y))\cdot\nabla\mathsf{g}(x-y)d\left(\sum_{j:j\neq i}\delta_{x_j}^{(\alpha_j)}-N\mu\delta_{\mathbb{R}^{\mathsf{d}}}\right)(x)d\left(\delta_{x_i}^{(\eta_i)}-\delta_{x_i}^{(\alpha_i)}\right)(y)$$
$$=\mathsf{c}_{\mathsf{d},\mathsf{s}}\int_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}}\hat{\psi}\cdot\nabla\mathsf{f}_{\alpha_i,\eta_i}(\cdot-x_i)d\left(\sum_{j:j\neq i}\delta_{x_j}^{(\alpha_j)}-N\mu\delta_{\mathbb{R}^{\mathsf{d}}}\right)$$
$$+\mathsf{c}_{\mathsf{d},\mathsf{s}}\int_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}}\hat{\psi}\cdot\left(\sum_{j:j\neq i}\nabla\mathsf{g}_{\alpha_j}(x-x_j)-N\nabla\mathsf{h}^\mu\right)d\left(\delta_{x_i}^{(\eta_i)}-\delta_{x_i}^{(\alpha_i)}\right),$$

where we used (3.9).

[1] In the case (1.5) we bound instead $|x-y||\nabla\mathsf{g}(x-y)|$ by 1, which yields an even better control.

Step 1: first term in (4.12). Since $\mathsf{f}_{\alpha_i,\eta_i}(\cdot - x_i)$ is supported in $B(x_i,\alpha_i)$, $\delta_{x_j}^{(\alpha_j)}$ in $B(x_j,\alpha_j)$ and the balls are disjoint, one type of terms vanishes and there remains

$$-N\mathsf{c}_{\mathsf{d},\mathsf{s}}\int_{\mathbb{R}^\mathsf{d}}\psi\cdot\nabla\mathsf{f}_{\alpha_i,\eta_i}(\cdot - x_i)d\mu.$$

Thanks to the explicit form of $\mathsf{f}_{\alpha,\eta}$ we have

$$\mathsf{f}_{\alpha_i,\eta_i}(\cdot - x_i) = \begin{cases}\mathsf{g}(x-x_i)-\mathsf{g}(\alpha_i) & \text{for } \ \eta<|x-x_i|\le\alpha_i\\ \mathsf{g}(\eta_i)-\mathsf{g}(\alpha_i) & \text{for } \ |x-x_i|\le\eta_i\end{cases}$$

and

$$\nabla\mathsf{f}_{\alpha_i,\eta_i}(\cdot - x_i) = \nabla\mathsf{g}(x-x_i)\mathbf{1}_{\eta_i\le|x-x_i|\le\alpha_i}.$$

It follows that

$$\int_{\mathbb{R}^\mathsf{d}}|\mathsf{f}_\alpha|\le C\alpha^{\mathsf{d}-\mathsf{s}},\qquad \int_{\mathbb{R}^\mathsf{d}}|\mathsf{f}_{\alpha_i,\eta_i}|\le C\alpha_i^{\mathsf{d}-\mathsf{s}},\qquad \int_{\mathbb{R}^\mathsf{d}}|\nabla\mathsf{f}_{\alpha_i,\eta_i}|\le C\alpha_i^{\mathsf{d}-\mathsf{s}-1}. \tag{4.13}$$

Indeed, it suffices to observe that

$$\int_{B(0,\eta)}\mathsf{f}_\alpha = C\int_0^\alpha(\mathsf{g}(r)-\mathsf{g}(\alpha))r^{\mathsf{d}-1}\,dr = -\frac{C}{\mathsf{d}}\int_0^r\mathsf{g}'(r)r^\mathsf{d}\,dr, \tag{4.14}$$

with an integration by parts.

We may always write

$$\begin{aligned}\left|\int_{\mathbb{R}^\mathsf{d}}\psi\cdot\nabla\mathsf{f}_{\alpha_i,\eta_i}(\cdot - x_i)(\mu-\mu(x_i))\right| &\le C\|\psi\|_{L^\infty}\|\mu\|_{C^{0,1}}\int_{\eta_i}^{\alpha_i}\frac{r^\mathsf{d}}{r^{\mathsf{s}+1}}dr\\ &\le C\|\psi\|_{L^\infty}\|\mu\|_{C^{0,1}}\alpha_i^{\mathsf{d}-\mathsf{s}},\end{aligned} \tag{4.15}$$

and, integrating by parts and using (4.13),

$$\left|\int_{\mathbb{R}^\mathsf{d}}\psi\cdot\nabla\mathsf{f}_{\alpha_i,\eta_i}(\cdot - x_i)\mu(x_i)\right|\le\|\mu\|_{L^\infty}\|\nabla\psi\|_{L^\infty}\int_{\mathbb{R}^\mathsf{d}}|\mathsf{f}_{\alpha_i,\eta_i}|\le\|\mu\|_{L^\infty}\|\nabla\psi\|_{L^\infty}\alpha_i^{\mathsf{d}-\mathsf{s}}. \tag{4.16}$$

Alternatively, we may use the simpler bound derived from (4.13),

$$\left|\int_{\mathbb{R}^\mathsf{d}}\psi\cdot\nabla\mathsf{f}_{\alpha_i,\eta_i}(\cdot - x_i)d\mu\right|\le C\|\psi\|_{L^\infty}\|\mu\|_{L^\infty}\alpha_i^{\mathsf{d}-\mathsf{s}-1}. \tag{4.17}$$

A standard interpolation argument yields that $\|g\|_{(C^\sigma)^*}\le\|g\|^\sigma_{(C^1)^*}\|g\|^{1-\sigma}_{(C^0)^*}$ so interpolating between (4.15)–(4.16) and (4.17), we obtain

$$\left|\int_{\mathbb{R}^\mathsf{d}}\psi\cdot\nabla\mathsf{f}_{\alpha_i,\eta_i}(\cdot - x_i)d\mu\right|\le C\|\psi\|^{1-\sigma}_{L^\infty}\|\psi\|^\sigma_{W^{1,\infty}}\|\mu\|_{C^\sigma}\alpha_i^{\mathsf{d}-\mathsf{s}+\sigma-1}.$$

We conclude that the sum over i of the first terms in (4.12) is bounded by both

$$C\|\psi\|^{1-\sigma}_{L^\infty}\|\psi\|^\sigma_{W^{1,\infty}}\|\mu\|_{C^\sigma}\sum_i\alpha_i^{\mathsf{d}-\mathsf{s}+\sigma-1}\quad\text{and}\quad C\|\psi\|_{L^\infty}\|\mu\|_{L^\infty}\sum_i\alpha_i^{\mathsf{d}-\mathsf{s}-1}. \tag{4.18}$$

Step 2: second term in (4.12). We may rewrite the integral as

$$
\begin{aligned}
(4.19)\quad & -N\int_{\mathbb{R}^{d+k}} \hat{\psi}(x_i)\cdot\nabla_{\mathbb{R}^d}\mathsf{h}^\mu d\left(\delta_{x_i}^{(\eta_i)}-\delta_{x_i}^{(\alpha_i)}\right)\\
&+\sum_{j:j\neq i}\int_{\mathbb{R}^{d+k}}\hat{\psi}(x_j)\cdot\nabla \mathsf{g}_{\alpha_j}(x-x_j)d\left(\delta_{x_i}^{(\eta_i)}-\delta_{x_i}^{(\alpha_i)}\right)\\
&+\sum_{j:j\neq i}\int_{\mathbb{R}^{d+k}}(\hat{\psi}-\hat{\psi}(x_j))\cdot\nabla \mathsf{g}_{\alpha_j}(x-x_j)d\left(\delta_{x_i}^{(\eta_i)}-\delta_{x_i}^{(\alpha_i)}\right)\\
&+O\left(N\|\nabla\psi\|_{L^\infty}\int_{\mathbb{R}^{d+k}}|x-x_i|\,|\nabla_{\mathbb{R}^d}\mathsf{h}^\mu|\,d\left(\delta_{x_i}^{(\eta_i)}+\delta_{x_i}^{(\alpha_i)}\right)\right),
\end{aligned}
$$

where we used that the last component of $\hat{\psi}$ vanishes, so that only the derivatives along the $\mathbb{R}^\mathsf{d}$ directions appear.

Substep 2.1: first term of (4.19). We may write that $\delta_{x_i}^{(\eta_i)}-\delta_{x_i}^{(\alpha_i)}=-\frac{1}{\mathsf{c}_{\mathsf{d},\mathsf{s}}}\mathrm{div}\,(|z|^\gamma\nabla \mathsf{f}_{\alpha_i,\eta_i}(\cdot-x_i))$ and integrate by parts twice to get

$$\frac{1}{\mathsf{c}_{\mathsf{d},\mathsf{s}}}\int_{\mathbb{R}^{d+k}}|z|^\gamma\nabla\left(\hat{\psi}(x_i)\cdot\nabla_{\mathbb{R}^d}\mathsf{h}^\mu\right)\cdot\nabla \mathsf{f}_{\alpha_i,\eta_i}(\cdot-x_i)=\int_{\mathbb{R}^d}(\psi(x_i)\cdot\nabla\mu)\mathsf{f}_{\alpha_i,\eta_i}.$$

Here, we used that $-\mathrm{div}\,(|z|^\gamma\nabla\mathsf{h}^\mu)=\mathsf{c}_{\mathsf{d},\mathsf{s}}\mu\delta_{\mathbb{R}^d}$ and took the $\hat{\psi}(x_i)\cdot\nabla_{\mathbb{R}^d}$ of this relation. In view of (4.13), this is then bounded by

$$C\|\psi\|_{L^\infty}\|\nabla\mu\|_{L^\infty}\alpha_i^{\mathsf{d}-\mathsf{s}}.$$

Alternatively, we may integrate by parts in $\mathbb{R}^\mathsf{d}$ to bound it by

$$C\|\mu\|_{L^\infty}\left(\|\psi\|_{L^\infty}\int_{\mathbb{R}^d}|\nabla \mathsf{f}_{\alpha_i,\eta_i}|+\|\nabla\psi\|_{L^\infty}\alpha_i^{\mathsf{d}-\mathsf{s}}\right).$$

Interpolating as above, we conclude with (4.13) that the sum over i of these terms is bounded by both

$$
\begin{aligned}
(4.20)\quad & C\|\mu\|_{C^\sigma}\left(\|\psi\|_{L^\infty}^\sigma\|\nabla\psi\|_{L^\infty}^{1-\sigma}\alpha_i^{\mathsf{d}-\mathsf{s}}+\|\psi\|_{L^\infty}\sum_i\alpha_i^{\mathsf{d}-\mathsf{s}+\sigma-1}\right)\\
&\text{and } \|\mu\|_{L^\infty}\left(\|\psi\|_{L^\infty}\sum_i\alpha_i^{\mathsf{d}-\mathsf{s}-1}+\|\nabla\psi\|_{L^\infty}\sum_i\alpha_i^{\mathsf{d}-\mathsf{s}}\right).
\end{aligned}
$$

Substep 2.2: second term of (4.19). Arguing in the same way as for the first term, using that $-\mathrm{div}\,(|z|^\gamma\nabla\mathsf{g}_{\alpha_j}(\cdot-x_j))=\mathsf{c}_{\mathsf{d},\mathsf{s}}\delta_{x_j}^{(\alpha_j)}$ and the disjointness of the balls, we find that this term vanishes.

Substep 2.3: third term in (4.19). We separate the sum into two pieces and bound this term by

$$
\begin{aligned}
(4.21)\quad & \sum_{j\neq i,|x_i-x_j|\geq N^{-\frac{1}{\mathsf{d}}}}\int_{\mathbb{R}^{d+k}}(\hat{\psi}-\hat{\psi}(x_j))\cdot\nabla\mathsf{g}_{\alpha_j}(\cdot-x_j)d\left(\delta_{x_i}^{(\eta_i)}-\delta_{x_i}^{(\alpha_i)}\right)\\
&+\|\nabla\psi\|_{L^\infty}\sum_{j\neq i,|x_i-x_j|\leq N^{-\frac{1}{\mathsf{d}}}}\int_{\mathbb{R}^{d+k}}|x-x_j||\nabla\mathsf{g}_{\alpha_j}(x-x_j)|d\left(\delta_{x_i}^{(\eta_i)}+\delta_{x_i}^{(\alpha_i)}\right)(x)
\end{aligned}
$$

For the first term of (4.21), we may use that $|x_i - x_j| \ge N^{-\frac{1}{\mathsf{d}}}$ to write

$$\|\nabla \left(\hat\psi - \hat\psi(x_j)\right) \cdot \nabla \mathsf{g}_{\alpha_j}(x-x_j)\|_{L^\infty(B(x_i,\mathsf{r}_i))} \le C\|\nabla\psi\|_{L^\infty} N^{-\frac{\mathsf{s}+1}{\mathsf{d}}}$$

and we may thus bound it by

$$CN\|\nabla\psi\|_{L^\infty} N^{-\frac{\mathsf{s}+1}{\mathsf{d}}}.$$

Since $|\nabla \mathsf{g}_\alpha| \le |\nabla \mathsf{g}|$, we may bound the second term in (4.21) by

$$\|\nabla\psi\|_{L^\infty} \sum_{j\neq i, |x_i-x_j|\le N^{-\frac{1}{\mathsf{d}}}} |x_i - x_j|^{-\mathsf{s}}. \tag{4.22}$$

To bound this, let us choose $\eta_i = 2N^{-\frac{1}{\mathsf{d}}}$ in (3.18) to obtain that

$$\sum_{i\neq j} \left(\mathsf{g}(x_i-x_j) - \mathsf{g}(2N^{-\frac{1}{\mathsf{d}}})\right)_+ \le F_N(X_N,\mu) + N\mathsf{g}(2N^{-\frac{1}{\mathsf{d}}}) + C\|\mu\|_{L^\infty} N^{1+\frac{\mathsf{s}}{\mathsf{d}}}.$$

In the cases (1.4), it follows that

$$\sum_{i\neq j, |x_i-x_j|\le N^{-\frac{1}{\mathsf{d}}}} \mathsf{g}(x_i-x_j) \le F_N(X_N,\mu) + C(1+\|\mu\|_{L^\infty}) N^{1+\frac{\mathsf{s}}{\mathsf{d}}}.$$

In the cases (1.5), it follows that

$$\sum_{i\neq j, |x_i-x_j|\le N^{-\frac{1}{\mathsf{d}}}} \log 2 \le F_N(X_N,\mu) + \frac{N}{\mathsf{d}}\log N + C\|\mu\|_{L^\infty} N^{1+\frac{\mathsf{s}}{\mathsf{d}}},$$

and this suffices to bound (4.22) as well. We conclude in all cases that the sum over i of the third terms in (4.19) is bounded by

$$C\|\nabla\psi\|_{L^\infty}\left(N^{2-\frac{\mathsf{s}+1}{\mathsf{d}}} + F_N(X_N,\mu) + (1+\|\mu\|_{L^\infty})N^{1+\frac{\mathsf{s}}{\mathsf{d}}} + \left(\frac{N}{\mathsf{d}}\log N\right)\mathbf{1}_{(1.5)}\right). \tag{4.23}$$

Substep 2.4: fourth term in (4.19). We may bound it by $O\left(\|\nabla\psi\|_{L^\infty}\alpha_i\|\nabla_{\mathbb{R}^\mathsf{d}} \mathsf{h}^\mu\|_{L^\infty(\mathbb{R}^{\mathsf{d}+\mathsf{k}})}\right)$. But since $h^\mu = \mathsf{g} * \mu$, it is straightforward to check that $\|\nabla_{\mathbb{R}^\mathsf{d}} \mathsf{h}^\mu\|_{L^\infty(\mathbb{R}^{\mathsf{d}+\mathsf{k}})} \le \|\nabla h^\mu\|_{L^\infty(\mathbb{R}^\mathsf{d})}$. Using (3.4), we conclude the sum of these terms is bounded by

$$\begin{cases} C\sum_i \alpha_i \|\nabla\psi\|_{L^\infty}(1+\|\mu\|_{L^\infty}) & \text{if } \mathsf{s} < \mathsf{d}-1 \\ C\sum_i \alpha_i \|\nabla\psi\|_{L^\infty}(1+\|\mu\|_{C^\sigma}) & \text{if } \mathsf{s} \ge \mathsf{d}-1 \text{ and } \sigma > \mathsf{s}-\mathsf{d}+1. \end{cases} \tag{4.24}$$

Combining the bounds (4.18), (4.20), (4.23), (4.24) concludes the proof of the lemma. □

5. Proof of Proposition 3.3

We drop the superscripts μ. First, $\int_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}} |z|^\gamma |\nabla H_{N,\vec\eta}|^2$ is a convergent integral and

(5.1)

$$\int_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}} |z|^\gamma|\nabla H_{N,\vec\eta}|^2 = \mathsf{c}_{\mathsf{d},\mathsf{s}} \iint_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}\times\mathbb{R}^{\mathsf{d}+\mathsf{k}}} \mathsf{g}(x-y)d\left(\sum_{i=1}^N \delta_{x_i}^{(\eta_i)} - N\mu\delta_{\mathbb{R}^\mathsf{d}}\right)(x)d\left(\sum_{i=1}^N \delta_{x_i}^{(\eta_i)} - N\mu\delta_{\mathbb{R}^\mathsf{d}}\right)(y).$$

Indeed, we may choose R large enough so that all the points of X_N are contained in the ball $B_R = B(0,R)$ in $\mathbb{R}^{\mathsf{d}+\mathsf{k}}$. By Green's formula and (3.16), we have

$$\int_{B_R} |z|^\gamma|\nabla H_{N,\vec\eta}|^2 = \int_{\partial B_R} |z|^\gamma H_{N,\vec\eta}\frac{\partial H_N}{\partial n} + \mathsf{c}_{\mathsf{d},\mathsf{s}}\int_{B_R} H_{N,\vec\eta}\, d\left(\sum_{i=1}^N \delta_{x_i}^{(\eta_i)} - N\mu\delta_{\mathbb{R}^\mathsf{d}}\right).$$

Since $\int d(\sum_i \delta_{x_i} - N\mu) = 0$, the function $H_{N,\vec\eta}$ decreases like $1/|x|^{\mathsf{s}+1}$ and $\nabla H_{N,\vec\eta}$ like $1/|x|^{\mathsf{s}+2}$ as $|x| \to \infty$, hence the boundary integral tends to 0 as $R \to \infty$, and we may write

$$\int_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}} |z|^\gamma |\nabla H_{N,\vec\eta}|^2 = \mathsf{c}_{\mathsf{d},\mathsf{s}} \int_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}} H_{N,\vec\eta}\, d\left(\sum_{i=1}^N \delta_{x_i}^{(\eta_i)} - N\mu\delta_{\mathbb{R}^\mathsf{d}}\right)$$

and thus by (3.16), (5.1) holds. We may next write that

$$\lim_{\eta\to 0} \Big[\iint_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}\times\mathbb{R}^{\mathsf{d}+\mathsf{k}}} \mathsf{g}(x-y) d\left(\sum_{i=1}^N \delta_{x_i}^{(\eta_i)} - N\mu\delta_{\mathbb{R}^\mathsf{d}}\right)(x) d\left(\sum_{i=1}^N \delta_{x_i}^{(\eta_i)} - N\mu\delta_{\mathbb{R}^\mathsf{d}}\right)(y) - \sum_{i=1}^N \mathsf{g}(\eta_i)\Big]$$
$$= -\iint_{\triangle^c} \mathsf{g}(x-y) d\left(\sum_{i=1}^N \delta_{x_i} - N\mu\delta_{\mathbb{R}^\mathsf{d}}\right)(x) d\left(\sum_{i=1}^N \delta_{x_i} - N\mu\delta_{\mathbb{R}^\mathsf{d}}\right)(y)$$

and we deduce in view of (5.1) that (3.17) holds.

We next turn to the proof of (3.18), adapted from [PS]. We have

$$-\operatorname{div}(|z|^\gamma \nabla H_{N,\vec\alpha}) = \mathsf{c}_{\mathsf{d},\mathsf{s}}\Big(\sum_{i=1}^N \delta_{x_i}^{(\alpha_i)} - \mu\delta_{\mathbb{R}^\mathsf{d}}\Big), \tag{5.2}$$

and in view of (3.13), $\nabla H_{N,\vec\eta} = \nabla H_{N,\vec\alpha} + \sum_{i=1}^N \nabla \mathsf{f}_{\alpha_i,\eta_i}(\cdot - x_i)$ thus

$$\int_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}} |z|^\gamma |\nabla H_{N,\vec\eta}|^2 = \int_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}} |z|^\gamma |\nabla H_{N,\vec\alpha}|^2 + \sum_{i,j} \int_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}} |z|^\gamma \nabla \mathsf{f}_{\alpha_i,\eta_i}(x-x_i)\cdot \nabla \mathsf{f}_{\alpha_i,\eta_i}(x-x_j)$$
$$+ 2\sum_{i=1}^N \int_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}} |z|^\gamma \nabla \mathsf{f}_{\alpha_i,\eta_i}(x-x_i)\cdot \nabla H_{N,\vec\alpha}.$$

Using (3.10), we first write

$$\sum_{i,j} \int_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}} |z|^\gamma \nabla \mathsf{f}_{\alpha_i,\eta_i}(x-x_i)\cdot \nabla \mathsf{f}_{\alpha_i,\eta_i}(x-x_j)$$
$$= -\sum_{i,j} \int_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}} \mathsf{f}_{\alpha_i,\eta_i}(x-x_i) \operatorname{div}(|z|^\gamma \nabla \mathsf{f}_{\alpha_i,\eta_i}(x-x_j)) = \mathsf{c}_{\mathsf{d},\mathsf{s}} \sum_{i,j} \int_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}} \mathsf{f}_{\alpha_i,\eta_i}(x-x_i) d(\delta_{x_j}^{(\eta_i)} - \delta_{x_j}^{(\alpha_i)}).$$

Next, using (5.2), we write

$$2\sum_{i=1}^N \int_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}} |z|^\gamma \nabla \mathsf{f}_{\alpha_i,\eta_i}(x-x_i)\cdot \nabla H_{N,\vec\alpha} = -2\sum_{i=1}^N \int_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}} \mathsf{f}_{\alpha_i,\eta_i}(x-x_i) \operatorname{div}(|z|^\gamma \nabla H_{N,\vec\alpha})$$
$$= 2\mathsf{c}_{\mathsf{d},\mathsf{s}} \sum_{i=1}^N \int_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}} \mathsf{f}_{\alpha_i,\eta_i}(x-x_i)\, d\Big(\sum_{j=1}^N \delta_{x_j}^{(\alpha_i)} - \mu\delta_{\mathbb{R}^\mathsf{d}}\Big).$$

These last two equations add up to give a right-hand side equal to

$$\sum_{i\neq j} \mathsf{c}_{\mathsf{d},\mathsf{s}} \int_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}} \mathsf{f}_{\alpha_i,\eta_i}(x-x_i) d(\delta_{x_j}^{(\alpha_i)} + \delta_{x_j}^{(\eta_j)}) - 2\mathsf{c}_{\mathsf{d},\mathsf{s}} \sum_{i=1}^N \int \mathsf{f}_{\alpha_i,\eta_i}(x-x_i) d\mu\delta_{\mathbb{R}^\mathsf{d}} \tag{5.3}$$
$$+ N\mathsf{c}_{\mathsf{d},\mathsf{s}} \int_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}} \mathsf{f}_{\alpha_i,\eta_i} d(\delta_0^{(\alpha_i)} + \delta_0^{(\eta_i)}).$$

We then note that $\int \mathsf{f}_{\alpha_i,\eta_i}\, d(\delta_0^{(\alpha_i)}+\delta_0^{(\eta_i)}) = -\int \mathsf{f}_{\eta_i} d\delta_0^{(\alpha_i)} = -(\mathsf{g}(\alpha_i)-\mathsf{g}(\eta_i))$ by definition of f_η and the fact that $\delta_0^{(\alpha)}$ is a measure supported on $\partial B(0,\alpha)$ and of mass 1. Secondly, by (4.13) we may bound $\int_{\mathbb{R}^\mathsf{d}} \mathsf{f}_{\alpha_i,\eta_i}(x-x_i)\mu\delta_{\mathbb{R}^\mathsf{d}}$ by $C\|\mu\|_{L^\infty}\alpha_i^{\mathsf{d}-\mathsf{s}}$.

Thirdly, we observe that since $\mathsf{f}_{\alpha_i,\eta_i}\le 0$, the first term in (5.3) is nonpositive and we may bound it above by

$$\sum_{i\neq j}\mathsf{c}_{\mathsf{d},\mathsf{s}}\int_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}} \mathsf{f}_{\alpha_i,\eta_i} d\delta_{x_j}^{(\alpha_j)} \le \sum_{i\neq j}\mathsf{c}_{\mathsf{d},\mathsf{s}}\int_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}} \left(\mathsf{g}_{\eta_i}(x-x_i)-\mathsf{g}_{\alpha_i}(x-x_i)\right) d\delta_{x_j}^{(\alpha_j)}$$
$$\le \sum_{i\neq j}\mathsf{c}_{\mathsf{d},\mathsf{s}}\int_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}} \left(\mathsf{g}(\eta_i)-\mathsf{g}_{\alpha_i}(|x_i-x_j|+\alpha_j)\right)_-$$

where we used the fact that g_α is radial decreasing. Combining the previous relations yields

$$-CN\|\mu\|_{L^\infty}\sum_{i=1}^N \eta_i^{\mathsf{d}-\mathsf{s}} + \mathsf{c}_{\mathsf{d},\mathsf{s}}\sum_{i\neq j}\left(\mathsf{g}_{\alpha_i}(|x_i-x_j|+\alpha_j)-\mathsf{g}(\eta_i)\right)_+$$
$$\le \left(\int_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}} |z|^\gamma|\nabla H_{N,\vec{\alpha}}|^2 - \mathsf{c}_{\mathsf{d},\mathsf{s}}\sum_{i=1}^N \mathsf{g}(\alpha_i)\right) - \left(\int_{\mathbb{R}^{\mathsf{d}+\mathsf{k}}} |z|^\gamma|\nabla H_{N,\vec{\eta}}|^2 - \mathsf{c}_{\mathsf{d},\mathsf{s}}\sum_{i=1}^N \mathsf{g}(\eta_i)\right)$$

and letting all $\alpha_i\to 0$ finishes the proof in view of (3.17).

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