Iterative solution of equations

Only few equations have a "closed form" solution, e.g., polynomials of order \( \leq 4 \). Most equations don't, e.g.,

\[ f(x) = x^5 - 4x - 2 = 0 \]

We will study the existence of solutions to

\[ f(x) = 0 \]

and techniques to compute solutions \( \phi \).

§1.2 Simple Fixedpoint Iteration

Consider \( f : [a,b] \rightarrow \mathbb{R}, \ a < b \)

Thm: \( f \) continuous, \( f(a) \leq 0, f(b) \geq 0 \) (or vice versa)

\[ \Rightarrow \exists \ \phi \in [a,b] : f(\phi) = 0 \]

Proof: Assume \( f(a) < 0, f(b) > 0 \)

\[ \Rightarrow \exists \ \phi \in (a,b) : \text{Intermediate Value Theorem} \quad f(\phi) = 0 \]

Alternative fixed point formulation: Find \( x \) with

\[ q(x) = x \quad [\text{equivalent to } f(x) + x = x] \]

Thm (Brouwer's fixed point, holds in more general settings)

\[ q : [a,b] \rightarrow \mathbb{R} \text{ continuous, } q(x) \in [a,b] \quad \forall x \in [a,b] \]

\[ \Rightarrow \exists \ \phi \text{ in } [a,b] : \phi = q(\phi). \]
Proof: \( f(x) = x - g(x) \Rightarrow f(a) \leq 0, f(b) \geq 0 \)

\[ \text{Thm} \]

Then \( \exists \ g: f(g) = 0 \Rightarrow g(g) = x \ \square \)

**Example:**

\[ f(x) = e^x - 2x - 1, \ x \in [1,2] \]

Solve \( f(x) = 0 \), has a root since \( f(1) < 0 \)
\( f(2) > 0 \), + cont.

As fixed point equation
\[ g(x) = e^x - x - 1 \quad [f(x) = 0 \Leftrightarrow g(x) = x] \]
\[ g(x) = \ln(2x + 1) \quad [\ln(2x + 1) = x \Leftrightarrow 2x + 1 = e^x \Rightarrow f(x) = 0] \]
\[ g(x) = \frac{e^x - 1}{2} \]

**SIMPLE/FIXED POINT ITERATION**

\( g: [a,b] \rightarrow \mathbb{R}, g(x) \in [a,b] \ \forall x, g \text{ continuous} \)

Given \( x_0 \in [a,b] \), recursion
\[ x_{k+1} = g(x_k), k = 0,1,2,\ldots \]

Upon convergence to \( \xi \)
\[ \lim_{k \to \infty} x_{k+1} = \xi = g(\lim_{k \to \infty} x_k) = g(\xi) \]
When does (*) converge?

**Def:** (contraction) \( g : [a, b] \rightarrow \mathbb{R} \) is a contraction if
\[
|g(x) - g(y)| \leq L |x - y| \text{ for all } x, y \in [a, b]
\]
with \( L < 1 \).

**Thm:** \( g : [a, b] \rightarrow \mathbb{R}, g(x) \in [a, b] \) for all \( x \in [a, b] \) continuous, contraction.

\[ \Rightarrow \exists ! \text{ fixed point } \bar{g} \text{ with } g(\bar{g}) = \bar{g} \] and the sequence (*) converges to \( \bar{g} \) for any starting point \( x_0 \).

**Proof:**

Existence \( \checkmark \)

Uniqueness: \( \bar{g}, \bar{g}' \) two fixed points

\[
|\bar{g} - \bar{g}'| = |g(\bar{g}) - g(\bar{g}')| \leq L |\bar{g} - \bar{g}'|, \ L < 1
\]

\[ \Rightarrow \bar{g} = \bar{g}' \checkmark \]

Convergence: \( x_0 \in [a, b] \)

\[
|x_k - \bar{g}| \leq |g(x_{k-1}) - g(\bar{g})| \leq L |x_{k-1} - \bar{g}|, \ k \geq 1
\]

\[ \Rightarrow |x_k - \bar{g}| \leq L^k |x_0 - \bar{g}|, \ k \geq 1
\]

\[ \Rightarrow |x_k - \bar{g}| \rightarrow 0 \text{ as } L^k \rightarrow 0 \text{ for } k \rightarrow \infty \] (as \( L < 1 \)) \( \square \).