§1: Iterative solution of equations

Only few equations have a "closed form" solution, e.g., polynomials of order ≤ 4. Most equations don't, e.g.

$$f(x) = x^5 - 4x - 2 = 0$$

We will study the existence of solutions to

$$f(x) = 0$$

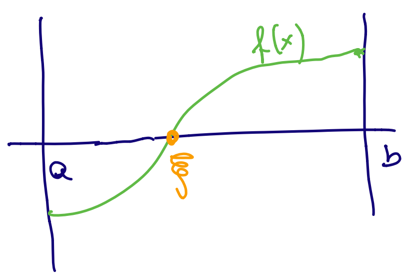
and techniques to compute solution ξ.

§1.2 Simple / Fixedpoint ITERATION

Consider $f : [a,b] \to \mathbb{R}$, $a < b$

<u>Thm:</u> f continuous, $f(a) \leq 0$, $f(b) \geq 0$ (or vice-versa)

$\Longrightarrow \exists\ \xi \in [a,b] : f(\xi) = 0$



<u>Proof:</u> Assume $f(a) < 0$, $f(b) > 0$

$\underset{\text{Intermed. value theorem}}{\Longrightarrow} \exists\ \xi \in (a,b) : f(\xi) = 0$ $\square$.

Alternative fixed point formulation: Find x with

$$g(x) = x \quad \Big[\text{equivalent to } \underbrace{f(x) + x}_{g(x)} = x\Big]$$

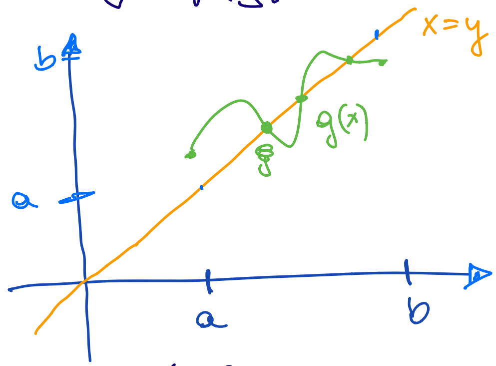
<u>Thm</u> (Brouwer's fixed point, holds in more general settings)

$g : [a,b] \to \mathbb{R}$ continuous, $g(x) \in [a,b]\ \forall x \in [a,b]$

$\Longrightarrow \exists\ \xi$ in $[a,b]$: $\xi = g(\xi)$.

Proof: $f(x) = x - g(x) \Longrightarrow f(a) \leq 0, \; f(b) \geq 0$

$\overset{\text{prev.}}{\underset{\text{Thm}}{\Longrightarrow}} \exists \, \xi : f(\xi) = 0 \Longrightarrow g(\xi) = \xi \quad \square.$



Example: $f(x) = e^x - 2x - 1, \quad x \in [1, 2]$

Solve $f(x) = 0$, has a root since $f(1) < 0$

$f(2) > 0$, f cont.

As fixed point equation

$g(x) = e^x - x - 1 \quad [f(x) = 0 \Longleftrightarrow g(x) = x]$

$g(x) = \ln(2x+1) \quad [\ln(2x+1) = x \Longleftrightarrow 2x + 1 = e^x \Longleftrightarrow f(x) = 0]$

$g(x) = \dfrac{e^x - 1}{2}$

SIMPLE / FIXEDPOINT ITERATION

$g : [a,b] \to \mathbb{R}$, $g(x) \in [a,b] \; \forall x$, g continuous

Given $x_0 \in [a,b]$, recursion

$$x_{k+1} = g(x_k) \qquad k = 0, 1, 2, \ldots \qquad (*)$$

iterates

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Upon convergence to ξ

$$\lim_{k \to \infty} x_{k+1} = \xi = g\left(\lim_{k \to \infty} x_k\right) = g(\xi)$$

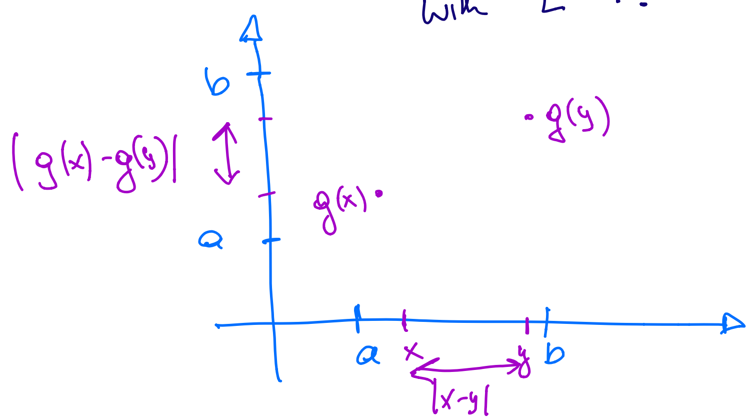
When does (*) converge?

<u>Def:</u> (contraction) $g:[a,b] \to \mathbb{R}$ is a contraction if

$$|g(x)-g(y)| \leq L\,|x-y| \quad \text{for all } x,y \in [a,b]$$

Lipschitz condition

with $L<1$.



<u>Thm:</u> $g:[a,b] \to \mathbb{R}$, $g(x) \in [a,b]$ for all $x \in [a,b]$
continuous, contraction.

$\Longrightarrow$ $\exists!$ fixed point ξ with $g(\xi)=\xi$ and the sequence (*) converges to ξ for any starting point x_0.

<u>Proof:</u> Existence ✓

<u>Uniqueness:</u> ξ, η two fixed points

$$|\xi-\eta| = |g(\xi)-g(\eta)| \leq L\,|\xi-\eta|, \quad L<1$$

$$\Longrightarrow \xi = \eta \quad ✓$$

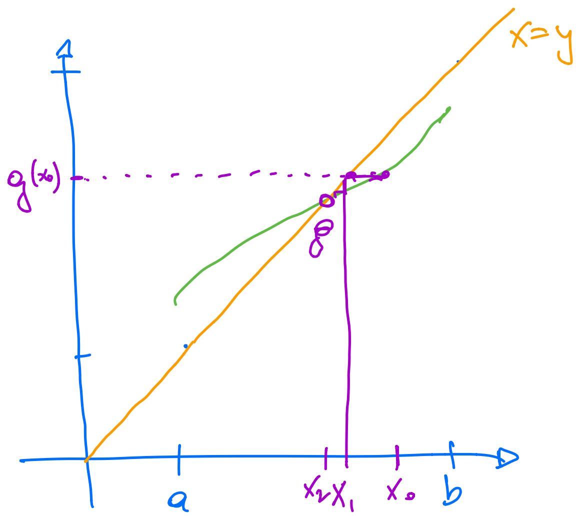
<u>Convergence:</u> $x_0 \in [a,b]$

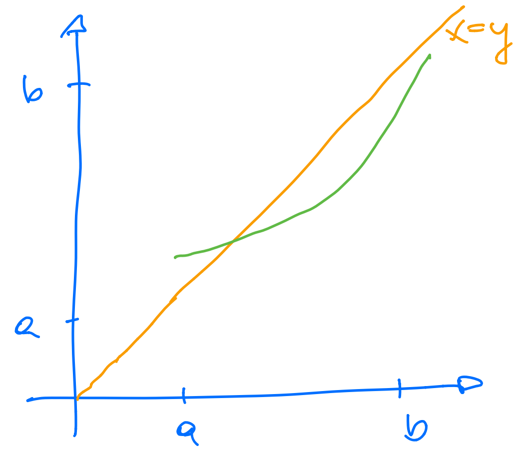
$$|x_k - \xi| = |g(x_{k-1}) - g(\xi)| \leq L\,|x_{k-1}-\xi| \quad k \geq 1$$

$$\Longrightarrow |x_k-\xi| \leq L^k\,|x_0-\xi| \quad k \geq 1$$

$$\Longrightarrow |x_k-\xi| \to 0 \text{ as } L^k \to 0 \text{ for } k \to \infty$$

(as $L<1$) $\square$.


x=y
g(x₀)
δ
a
x₂ x₁
x₀
b


x=y
b
a
a
b