# §5 Eigenvalues & eigenvectors of symmetric matrices

$$A \in \mathbb{R}^{n\times n}, \quad x \in \mathbb{R}^n, \lambda \in \mathbb{C}, x \neq 0$$

eigenvalue

$$Ax = \lambda x$$

eigenvector

We'll try to find ways to compute $x, \lambda$. That's a non-linear problem since both $\lambda, x$ are unknown. If $\lambda$ is known, then it's linear since $x$ solves $(A-\lambda I)x = 0$.

<u>Recall:</u> $A \in \mathbb{R}^{n\times n}$ symmetric, then:

- There exists $n$ linearly independent eigenvectors $x_i \in \mathbb{R}^n$ with eigenvalues $\lambda_i \in \mathbb{R}$.
- $\lambda \longrightarrow \det(A-\lambda I)$ is a polynomial of degree $n$, and the eigenvalues $\lambda_i$ are its roots (characteristic polynomial)
- Eigenvectors corresponding to distinct eigenvalues are orthogonal, i.e. $\lambda_i \neq \lambda_j \longrightarrow x_i^T x_j = 0$
- $\lambda_i$ has multiplicity $m \geq 1$, then there exists a basis of orthogonal eigenvectors corresponding to $\lambda_i$
- $A$ and $B := Q^T A Q$ with $Q$ orthogonal have the same eigenvalues.
- There exists an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of $A$.

# §5.4 Gerschgorin theorems

They allow estimation of regions where the eigenvalues lie. This does not require $A$ to be symmetric, they hold for any $A \in \mathbb{C}^{n\times n}$.

**Def:** Gerschgorin discs $D_i$ $i=1,\dots,n$ are defined as

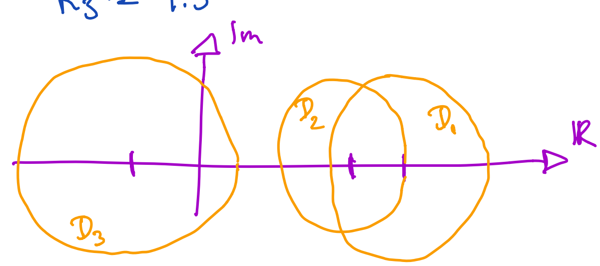
$$D_i = \left\{ z \in \mathbb{C} \,\middle|\, |z - a_{ii}| \le R_i \right\} \text{ with}$$

$$R_i = \sum_{\substack{j=1 \\ j \ne i}}^{n} |a_{ij}|$$

**Ex:** $B = \begin{bmatrix} 3 & 1 & -0.5 \\ 1 & 2 & 0 \\ 1 & 0.5 & -1 \end{bmatrix}$ $\quad R_1 = 1.5$, $R_2 = 1$, $R_3 = 1.5$ $\quad D_1 = \left\{ z \in \mathbb{C} \,/\, |z-3| \le 1.5 \right\}$

**Theorem** (Gershgorin's 1st theorem)

All eigenvalues of $A \in \mathbb{C}^{n\times n}$ lie in $D = \bigcup_{i=1}^{n} D_i$.




**Proof:** $\lambda \in \mathbb{C}$, $x \ne 0$ $x \in \mathbb{C}^n$

$$Ax = \lambda x \longrightarrow \sum_{j=1}^{n} a_{ij} x_j = \lambda x_i \quad i = 1,\dots,n$$

Let $k$ be such that $|x_k| \ge |x_i|$ for all $i$, i.e. $k$ is the index with the largest entry in absolute value.

$$\underline{|\lambda - a_{kk}|\,|x_k|} = |\lambda x_k - a_{kk} x_k| = \left| \sum_{j=1}^{n} a_{kj} x_j - a_{kk} x_k \right|$$

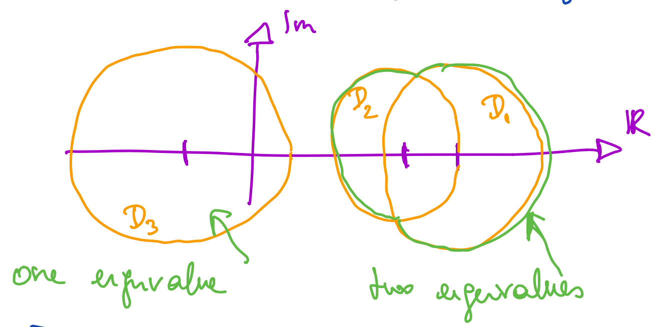
$$= \left| \sum_{\substack{j=1 \\ j \ne k}}^{n} a_{kj} x_j \right| \le \sum_{\substack{j=1 \\ j \ne k}}^{n} |a_{kj}| \underbrace{|x_j|}_{\le |x_k|}$$

$$\le \underline{R_k |x_k|}$$

$$\xrightarrow[|x_k| \ne 0]{\text{divide throug}} |\lambda - a_{kk}| \le R_k \rightarrow \lambda \in D_k$$

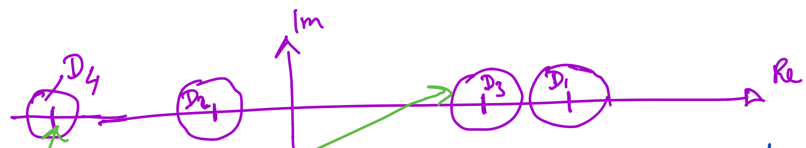
$\square$

**Theorem** (Gerschgorin's 2nd theorem). Let the $D_i$'s be divided into disjoint sets $D^{(p)}, D^{(q)}$ with $p$ and $q = n-p$ discs. Then the union of $D^{(p)}$ contains $p$ eigenvalues and the union of $D^{(q)}$ contains $q$ eigenvalues. In particular, disjoint discs contain exactly one eigenvalue.




**Example:**

$$A = \begin{bmatrix} 4 & 0.2 & -0.1 & 0.1 \\ 0.2 & -1 & -0.1 & 0.05 \\ -0.1 & -0.1 & 3 & 0.1 \\ 0.1 & 0.05 & 0.1 & -3 \end{bmatrix} \qquad \begin{aligned} R_1 &= 0.4 \\ R_2 &= 0.35 \\ R_3 &= 0.3 \\ R_4 &= 0.25 \end{aligned}$$




Since $A$ is symmetric and the discs are disjoint, we know that

$$\lambda_4 \in [-3.25, -2.75]$$
$$\lambda_3 \in [2.7, 3.3]$$
$$\vdots$$

## Power method for computing eigenvectors

Simple idea: Start with a vector $x_0 \in \mathbb{R}^n$ and iterate

$$x_{k+1} = A x_k \qquad k = 0, 1, 2, \dots$$

If a simple $\lambda$ is strictly larger in absolute value than the other eigenvalues, it will start to dominate in $x_k$, i.e. $x_k$ will "converge" to eigenvector of $\lambda$.

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \dots \geq |\lambda_n|.$$