Eigenvalues & eigenvectors of symmetric matrices

\[ A \in \mathbb{R}^{n \times n}, \quad x \in \mathbb{R}^n, \quad \lambda \in \mathbb{C}, \quad x \neq 0 \]

\[ Ax = \lambda x \]

We'll try to find ways to compute \( x, \lambda \). That's a non-linear problem since both \( \lambda, x \) are unknown. If \( \lambda \) is known, then it's linear since \( x \) solves \((A - \lambda I)x = 0\).

Recall: \( A \in \mathbb{R}^{n \times n} \) symmetric, then:

- There exists \( n \) linearly independent eigenvectors \( x_i \in \mathbb{R}^n \) with eigenvalue \( \lambda_i \in \mathbb{R}^n \).
- \( \lambda \rightarrow \det(A - \lambda I) \) is a polynomial of degree \( n \) and the eigenvalues \( \lambda_i \) are its roots (characteristic polynomial).
- Eigenvectors corresponding to distinct eigenvalues are orthogonal, i.e., \( x_i \perp x_j \Rightarrow x_i^T x_j = 0 \).
- \( \lambda_i \) has multiplicity \( m \geq 1 \), then there exists a basis of orthogonal eigenvectors corresponding to \( \lambda_i \).
- \( A \) and \( B : = Q^T AQ \) with \( Q \) orthogonal have the same eigenvalues.
- There exists an orthonormal basis of \( \mathbb{R}^n \) consisting of eigenvectors of \( A \).

§5.4 Gerschgorin Theorems

They allow estimation of regions where the eigenvalues lie. This does not require \( A \) to be symmetric, they hold for any \( A \in \mathbb{C}^{n \times n} \).
Def: Gerschgorin discs $D_i$ $i = 1, \ldots, n$ are defined as
$$D_i = \{ z \in \mathbb{C} \mid |z - a_{ii}| \leq R_i \}$$
with
$$R_i = \sum_{j=1}^{n} |a_{ij}|$$

Ex: $B = \begin{bmatrix} 3 & 1 & -0.5 \\ 1 & 2 & 0 \\ 1 & 0.5 & -1 \end{bmatrix}$
$R_1 = 1.5$  $D_1 = \{ z \in \mathbb{C} \mid |z - 3| \leq 1.5 \}$
$R_2 = 1.4$  $D_2 = \{ z \in \mathbb{C} \mid |z - 2| \leq 1.4 \}$
$R_3 = 1.4$  $D_3 = \{ z \in \mathbb{C} \mid |z - 0.5| \leq 1.4 \}$

Theorem (Gershgorin's 1st theorem)
All eigenvalues of $A \in \mathbb{C}^{n \times n}$ lie in $D = \bigcup_{i=1}^{n} D_i$.

Proof: $\lambda \in \mathbb{C}, x \neq 0 \times \in \mathbb{C}^n$
$$Ax = \lambda x \Rightarrow \sum_{j=1}^{n} a_{ij} x_j = \lambda x_i \quad i = 1, \ldots, n$$

Let $k$ be such that $|x_k| \geq |x_i|$ for all $i$, i.e. $k$ is the index with the largest entry in absolute value.
$$|\lambda - a_{kk}| |x_k| = |\lambda x_k - a_{kk} x_k| = \left| \sum_{j=1}^{n} a_{kj} x_j - a_{kk} x_k \right|$$
$$= \left| \sum_{j=1}^{n} a_{kj} x_j - a_{kk} x_k \right| \leq \sum_{j=1}^{n} |a_{kj}| |x_j| \leq \sum_{j=1}^{n} |a_{kj}| x_j$$
$$\leq R_k |x_k|$$
$$\therefore \quad |x_k| \to 0 \quad |\lambda - a_{kk}| \leq R_k \Rightarrow \lambda \in D_k$$

Theorem (Gershgorin's 2nd theorem). Let the $D_i$’s be divided into disjoint sets $D^{(p)}, D^{(q)}$ with $p$ and $q = n - p$ discs.
Then the union of $D^{(p)}$ contains $p$ eigenvalues and the
union of \( \mathbb{D}(a) \) contains \( q \) eigenvalues. In particular, disjoint discs contain exactly one eigenvalue.

Example:

\[
A = \begin{bmatrix}
4 & 0.2 & -0.1 & 0.1 \\
0.2 & -1 & -0.1 & 0.05 \\
-0.1 & -0.1 & 3 & 0.1 \\
-0.1 & 0.05 & 0.1 & -3 \\
\end{bmatrix}
\]

\( R_1 = 0.4 \)

\( R_2 = 0.35 \)

\( R_3 = 0.3 \)

\( R_4 = 0.25 \)

Since \( A \) is symmetric and the discs are disjoint, we know that:

\( \lambda_1 \in [-3.25, -2.75] \)

\( \lambda_3 \in [2.7, 3.5] \)

Power method for computing eigenvectors

Simple idea: Start with a vector \( x_0 \in \mathbb{R}^n \) and iterate

\[ x_{k+1} = A x_k \quad k = 0, 1, 2, \ldots \]

If a simple \( \lambda \) is strictly larger in absolute value than the other eigenvalues, it will start to dominate in \( x_k \), i.e. \( x_k \) will "converge" to eigenvector of \( \lambda \).

\[ |\lambda_1| > |\lambda_2| \geq |\lambda_3| \ldots \geq |\lambda_q| \]