Power method for finding largest eigenvalue of $A \in \mathbb{R}^{n \times n}$

Start $x_0 \in \mathbb{R}^n$, compute

$$x_{k+1} = A x_k \qquad k = 0, 1, 2, \dots$$

Theorem: λ_1 is a simple eigenvalue of the symmetric matrix $A \in \mathbb{R}^{n \times n}$

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \dots \geq |\lambda_n|$$

Let $x_0 \in \mathbb{R}^n$ be a vector that is not orthogonal to the eigenspace of λ_1. Then the sequence $y_k := \frac{x_k}{\|x_k\|}$ with x_k defined in (*) converges towards a normalized eigenvector of A for λ_1.

Proof: $\eta_1, \dots, \eta_n$ orthonormal basis of $\mathbb{R}^n$ of eigenvectors of A, i.e.

$A\eta_i = \lambda_i \eta_i \quad i = 1, \dots, n$, Since $\eta_1, \dots, \eta_n$ are a basis,

$$x_0 = \sum_{i=1}^{n} \alpha_i \eta_i, \qquad \alpha_i \in \mathbb{R} \quad \text{and} \quad \alpha_1 = \eta_1^T x_0 \neq 0$$

$$x_k = A x_{k-1} = A^k x_0 = \sum_{i=1}^{n} \alpha_i A^k \eta_i = \sum_{i=1}^{n} \alpha_i \lambda_i^k \eta_i$$

$$= \alpha_1 \lambda_1^k \underbrace{\left(\eta_1 + \sum_{i=2}^{n} \frac{\alpha_i}{\alpha_1} \left(\frac{\lambda_i}{\lambda_1} \right)^k \eta_i \right)}_{z_k}$$

$$\longrightarrow \lim_{k \to \infty} z_k = \eta_1 \quad \text{since } |\lambda_i| < |\lambda_1|$$

$$y_k = \frac{x_k}{\|x_k\|} = \frac{\alpha_1 \lambda_1^k z_k}{|\alpha_1| |\lambda_1|^k \|z_k\|} = \pm \frac{z_k}{\|z_k\|} \longrightarrow \pm \eta_1$$

for $k \to \infty$. $\square$

Remarks: •) This method only allows to find the largest eigenvalue & eigenvector.
If I have the eigenvector η_1, know that

$$A\eta_1 = \lambda_1 \eta_1$$

$$\Longrightarrow \eta_1^T A \eta_1 = \lambda_1 \eta_1^T \eta_1$$

$$\Longrightarrow \boxed{\lambda_1 = \frac{\eta_1^T A \eta_1}{\eta_1^T \eta_1}} \quad \text{Rayleigh coefficient}$$

•) The speed of convergence depends on the gap between $|\lambda_1|$ and $|\lambda_2|$. If that gap is small,
e.g. $\frac{|\lambda_2|}{|\lambda_1|} = 0.9999 \longrightarrow$ slow convergence
If gap is larger, e.g. $\frac{|\lambda_2|}{|\lambda_1|} = 0.7 \rightarrow$ fast convergence.

Q4, HW3 : $A \in \mathbb{R}^{n \times n}$, λ is an eigenvalue of A^TA, $x \in \mathbb{R}^n$, $x \neq 0$
corr. eigenvector

(a) Show that $\|Ax\|_2^2 = \lambda \|x\|_2^2$ and thus $\lambda \geq 0$.

$$\underbrace{x^T A^T A x}_{\|Ax\|_2^2} = \underbrace{x^T \lambda x}_{\lambda \|x\|_2^2}$$

(b) $\|\cdot\|$ norm in $\mathbb{R}^n$

$$\|A\|_2 \leq \|A^TA\|^{\frac{1}{2}}$$

$$A^TAx = \lambda x \longrightarrow \|A^TAx\| = |\lambda| \, \|x\| = \lambda \|x\|$$
$$\leq \|A^TA\| \, \|x\|$$

$$\longrightarrow \lambda \leq \|A^TA\|$$

(c)

$$\|A\|_2 = \max_{1\le i\le n}\sqrt{\underbrace{\lambda_i(A^TA)}_{\text{eigenvalues of } A^TA.}} \le \|A^TA\|^{\frac{1}{2}}$$

$$\|A\|_2 \le \|A^TA\|_1^{\frac{1}{2}} \le \|A\|_1^{\frac{1}{2}}\|A^T\|_1^{\frac{1}{2}} = \|A\|_1^{\frac{1}{2}}\|A\|_\infty^{\frac{1}{2}}$$

$$\kappa_2(A) \le \left(\kappa_1(A)\,\kappa_\infty(A)\right)^{\frac{1}{2}}$$

$$\|A\|_2\|A^{-1}\|_2 \le \left(\|A\|_1\|A^{-1}\|_1\|A\|_\infty\|A^{-1}\|_\infty\right)^{\frac{1}{2}}$$

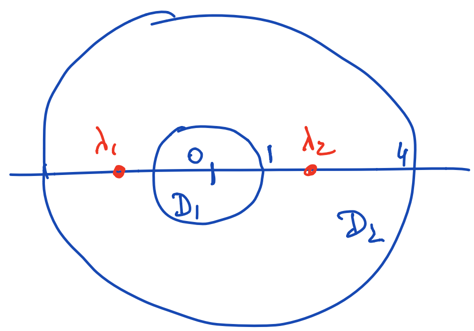
$$\|A^{-1}\|_2 \le \|A^{-1}\|_1^{\frac{1}{2}}\|A^{-1}\|_\infty^{\frac{1}{2}}$$

Gerschgorin:

$$A = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}$$

eigenvalues are roots of

$\lambda^2 - 4 \Rightarrow \lambda_{1,2} = \pm 2$



Can the number 6.125 be represented exactly in a computer?

$4 + 2 + \frac{1}{8}$ in binary: 110.001

$1/10$ cannot be represented: 0.000100100 1001...

Example when Newton does not converge quadratically for a C^2 function: At solution $f'(x^*) \neq 0$

Example: Solve $f(x) = x^3 = 0$, $f'(x) = 3x^2$ at

Solution $f'(x^*) = f'(0) = 0$.

Assumption: $\left|\dfrac{f''(x)}{f'(y)}\right| \le A \quad \forall\, x, y$ close to solution

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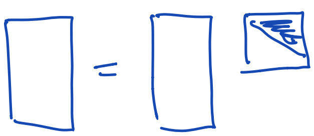
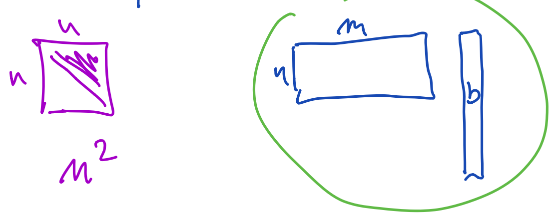
Least squares problem $Ax = b$ $\quad A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$

$A = \hat{Q}\hat{R}$



How many flops does it take to solve the least squares problem?

$$\hat{R}x = \hat{Q}^T b$$



n^2

$\left.\begin{array}{l} m \text{ mult} \\ m-1 \text{ addition} \end{array}\right\} \times n\text{-times}$

$$\underline{\underline{n(2m-1)}}$$

$n^2 + 2mn + \dots$