Power method: Finds the largest eigenvalue + eigenvector under some assumptions.

How do we find other eigenvalues?

§5.8 Inverse iteration

We'd like to find an eigenvalue of A that is close to ϑ

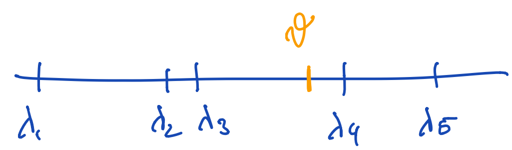
Consider $A - \vartheta I$, the eigenvalues are $\lambda_i(A) - \vartheta$, $i = 1, \dots, n$

eigenvalue of A

The matrix $(A - \vartheta I)^{-1}$ has the eigenvalues

$(\lambda_i(A) - \vartheta)^{-1}$ — this is the largest

for i where $\lambda_i(A)$ is closest to ϑ.



$Av = \lambda v$, $v \neq 0$, A^{-1} exists

$\longrightarrow$ $v = A^{-1}(\lambda v) = \lambda A^{-1} v \Rightarrow A^{-1} v = \frac{1}{\lambda} v$

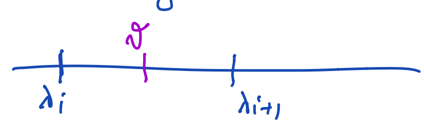
Power method with $(A - \vartheta I)^{-1}$: $x_0 \in \mathbb{R}^n$

$$x_{k+1} = (A - \vartheta I)^{-1} x_k \qquad k = 0, 1, 2, \dots$$

equivalent to $(A - \vartheta I) x_{k+1} = x_k$

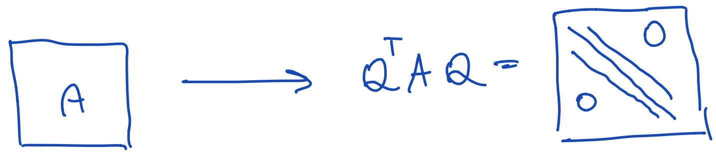
$\longrightarrow$ in every iteration of the inverse iteration, we have to solve a linear system.

Thm: $A \in \mathbb{R}^{n \times n}_{\text{symm.}}$, $y_{k+1} = \frac{x_k}{\|x_k\|}$ with x_k as defined in the inverse iteration, converges to the normalized eigenvector $\bar{y}$ corresponding to eigenvalue λ of A closest to ϑ if ϑ is not an eigenvalue and if x_0 is not orthogonal to $\bar{y}$.



Now, we aim at methods to compute **all** eigenvalues of a matrix.
We'll use 2 steps:

Step 1: Transform matrix A ($A^T = A$) to a tridiagonal matrix without changing the eigenvalues:



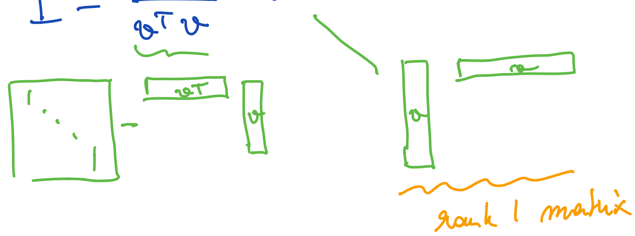
Step 2: Find eigenvalues of tridiagonal matrices iteratively (qr-algorithm)

§5.5 Householder's method for tridiagonalization

Goal: Reduce a matrix to tri-diagonal form using orthogonal transformations

Def: For $v \in \mathbb{R}^n$, $v \neq 0$ define

$$H = H(v) = I - \frac{2}{v^T v} v v^T \in \mathbb{R}^{n \times n}$$

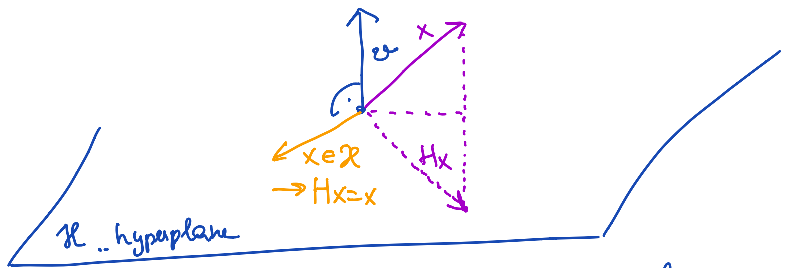


$$x \in \mathbb{R}^n, \quad Hx = x - \frac{2}{\|v\|_2^2} v(v^T x)$$

Hx, x, v are linearly dependent,

If $v^T x = 0 \Longrightarrow Hx = x$

$$v^T Hx = v^T x - \frac{2}{\cancel{\|v\|_2^2}} (\cancel{v^T v})(v^T x) = -v^T x$$



<u>Properties:</u> H is symmetric and orthogonal.

<u>Proof:</u> $H = I - \frac{2}{v^Tv} vv^T$

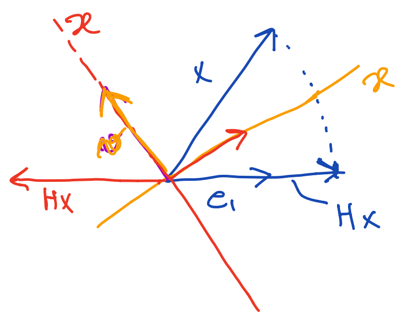
$$H^T = I^T - \frac{2}{v^Tv}(vv^T)^T = I - \frac{2}{v^Tv} vv^T = H$$

✓ symm.

<u>orthog:</u> $H^TH = HH^T = H^2 = \left(I - \frac{2}{v^Tv} vv^T\right)\left(I - \frac{2}{v^Tv} vv^T\right)$

$$= I - \frac{4}{v^Tv} vv^T + \frac{4}{(v^Tv)^{\cancel{2}}} v\cancel{(v^Tv)}v^T = I$$

<u>Lemma 5.4:</u> For any vector $x \in \mathbb{R}^n$ there exists $H(v)$ such that $Hx = \alpha e_1$, with e_1 is the first unit vector, $\alpha \in \mathbb{R}$



since H is orthogonal, $\|x\|_2 = \|Hx\|_2$

because:

$\|x\|_2^2 = x^Tx = (x^TH^T)(Hx) = \|Hx\|_2^2$

Thus: $|\alpha| = \|x\|_2$

<u>Proof:</u>

Need to find v s.t. $H(v)$ does the job

$v = x + ce_1$ and find c properly

$v^Tx = x^Tx + c\beta$ $\quad \beta = e_1^Tx$ (first component of x)

$v^Tv = x^Tx + 2c\beta + c^2$

$$Hx = x - \frac{2}{v^Tv} v(v^Tx) = \frac{(c^2 - x^Tx)x - 2c(x^Tx + c\beta)e_1}{x^Tx + 2c\beta + c^2}$$

For Hx to be a multiple of e_1, we need that

$c^2 = x^Tx$, and

we need that $x^T x + 2c\beta + c^2 \neq 0$. Since

$$x^T x + 2c\beta + c^2 \geq (\beta + c)^2 \neq 0 \text{ if } \beta + c \neq 0$$

Thus:

$$c = \begin{cases} \operatorname{sign}(\beta)\sqrt{x^T x} & \beta \neq 0 \\ \sqrt{x^T x} & \beta = 0 \end{cases} \qquad (*)$$

$$\longrightarrow Hx = -ce_1 \quad \text{as desired.} \qquad \square$$

So: Given x we choose $v = x + ce_1$ with c given in $(*)$.

$$\longrightarrow Hx = \alpha e_1$$

Example: $x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \sqrt{14}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+\sqrt{14} \\ 2 \\ 3 \end{pmatrix}$

$$H \overset{\in \mathbb{R}^{3\times 3}}{=} I - \frac{2}{v^T v}\begin{pmatrix} 1+\sqrt{14} \\ 2 \\ 3 \end{pmatrix}\left(1+\sqrt{14}, 2, 3\right) = \begin{pmatrix} \quad & \quad & \quad \end{pmatrix}$$

$$H\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -\sqrt{14} \\ 0 \\ 0 \end{pmatrix}$$

$$H \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} = \begin{bmatrix} * & \\ * & \\ 0 & * \\ \vdots & \\ 0 & \end{bmatrix}$$