Plane rotations, Givens rotations (§ 5.3)

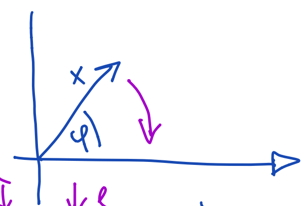
Besides reflections, rotations are orthogonal transformations $\rightarrow (AA^T = A^TA = I)$

In 2D, rotations around the origin look like:

$$R(\varphi) = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}, \quad c^2 + s^2 = 1$$

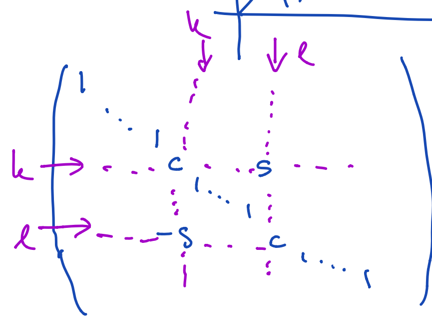
Properties: $R(\varphi)^T = R(\varphi)^{-1} = R(-\varphi)$

$R(\varphi)\, R(-\varphi) = I$



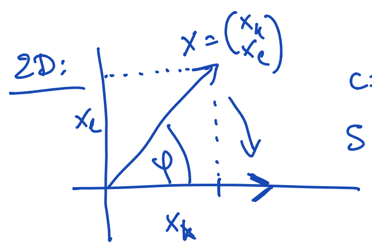
Plane rotation in $\mathbb{R}^n$

$R^{kl} =$



$$R^{kl} x = \begin{pmatrix} x_1 \\ \vdots \\ x_{k-1} \\ c x_k + s x_l \\ x_{k+1} \\ \vdots \\ -s x_k + c x_l \\ \vdots \\ x_n \end{pmatrix} \begin{matrix} \\ \\ \\ \leftarrow k \\ \\ \\ \leftarrow l \\ \\ \\ \end{matrix}$$

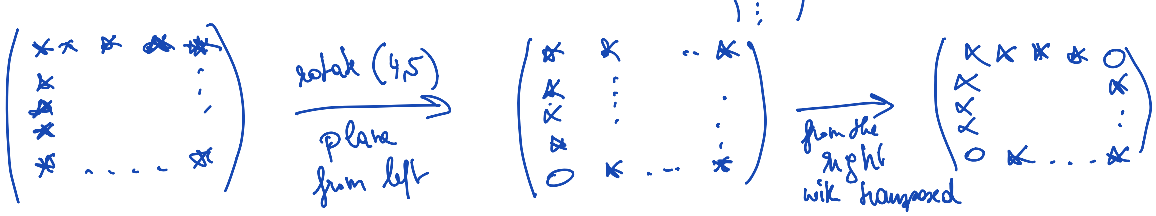
$r = \|x\| = \sqrt{x_k^2 + x_l^2}$

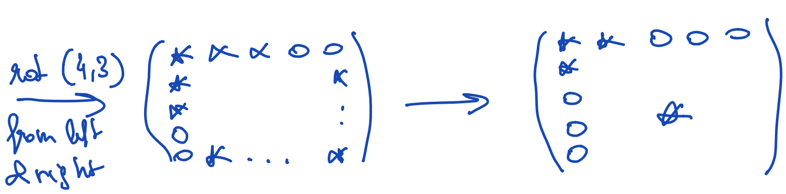


$c = \cos(\varphi) = \dfrac{x_k}{r}$

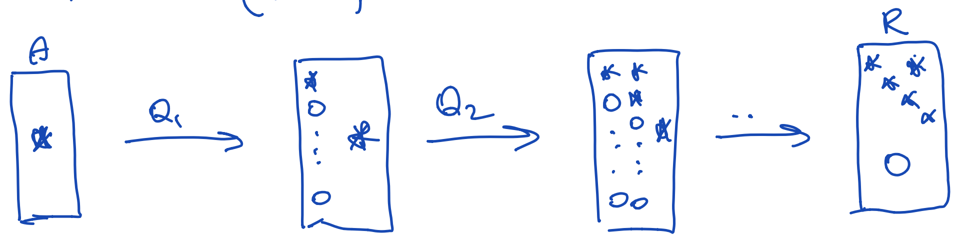
$s = \sin\varphi = \dfrac{x_l}{r}$

$$\longrightarrow R^{kl} x = \begin{pmatrix} \vdots \\ r \\ \vdots \\ 0 \\ \vdots \end{pmatrix}$$





Both, Householder & Givens can be used to compute QR factorization of $A \in \mathbb{R}^{m \times n}$ $(m \geq n)$:



$$\longrightarrow Q_n \cdots Q_2 \cdot Q_1 A = R \longrightarrow A = \underbrace{Q_1^T \cdot Q_2^T \cdots Q_n^T}_{\hat{Q}} R = \hat{Q}\hat{R}$$

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}$$ compute QR factorization with Givens:

$$R_1 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & c & s \\ & & -s & c \end{bmatrix} \qquad r = \sqrt{5}, \quad s = \frac{2}{\sqrt{5}}, \quad c = \frac{1}{\sqrt{5}} \quad : \quad R_1 A = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ \sqrt{5} & \frac{1}{\sqrt{5}} \\ 0 & -\frac{2}{\sqrt{5}} \end{pmatrix}$$

$$R_2 = \begin{bmatrix} 1 & & & \\ & c & s & \\ & -s & c & \\ & & & 1 \end{bmatrix}, \quad r = 3, \quad s = \frac{\sqrt{5}}{3}, \quad c = \frac{2}{3} \qquad R_2 \cdot \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ \sqrt{5} & \frac{1}{\sqrt{5}} \\ 0 & \frac{-2}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & * \\ 0 & * \\ 0 & * \end{bmatrix}$$

The QR algorithm for eigenvalues of tridiagonal matrices (§5.7)

$$A = \begin{bmatrix} \diagdown\diagdown\diagdown & 0 \\ 0 & \end{bmatrix}$$

$A \in \mathbb{R}^{n \times n}$ symmetric, tri-diagonal

The QR algorithm computes matrices $A^{(k)}$ $k = 0, 1, 2, \ldots$ starting from $A^{(0)} = A$:

for $k = 0, 1, 2, \ldots$

- compute QR decomposition of $A^{(k)}$, $A^{(k)} = QR$
- $A^{(k+1)} := RQ$

end

This algorithm converges to a diagonal matrix containing the eigenvalues of A.

First: Eigenvalues of $A^{(0)}, A^{(1)}, \ldots$ are the same:

$$A^{(k+1)} = RQ \qquad A^{(k)} = QR \longrightarrow R = Q^T A^{(k)}$$

$$= Q^T A^{(k)} Q$$

Since multiplication with an orthogonal matrix from left & right does not change the eigenvalues, $A^{(k)}$, $A^{(k+1)}$ have the same eigenvalues.