Last class:

* Lagrange interpolation
* Hermite interpolation (finish)
* Numerical integration (start)

\[ x_0, x_1, \ldots, x_n \in \mathbb{R} \quad \text{(distinct)} \]
\[ y_0, y_1, \ldots, y_n \in \mathbb{R} \]
\[ z_0, z_1, \ldots \in \mathbb{R} \]

Find \( P_n \in \mathbb{P}_n \) such that
\[
P_n(x_i) = y_i, \quad i = 0, \ldots, n
\]
\[
P_n'(x_i) = 2^i
\]

\[ P_n(x) = \sum_{k=0}^{n} H_k(x) x_k + K_k(x) \xi_k \]

is the Hermite interpolation polynomial.

What properties of \( H_k \) and \( K_k \) do we expect?

\[
\begin{align*}
H_k(x_i) &= \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases} \\
H_k'(x_i) &= 0 \\
K_k(x_i) &= \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases} \\
K_k'(x_i) &= 0
\end{align*}
\]
\[ H_n(x) = (L_n(x))^2 \left(1 - 2L_n'(x_n)(x-x_n)\right) \in \mathcal{S}_{2n+1} \]
\[ K_n(x) = (L_n(x))^2 (x-x_n) \in \mathcal{S}_{2n+1} \]

Theorem: \( n \geq 0 \), \( f : [a,b] \rightarrow \mathbb{R}, f^{(2n+2)} \) is continuous, then the Hermite interpolant of \( f \) satisfies (unique)

\[ |f(x) - P_{2n+1}(x)| \leq \frac{M_{2n+2}}{(2n+2)!} \left(\left|\frac{f^{(2n+2)}(x)}{(2n+2)!}\right| \right]^2 \]

\[ \frac{M_{2n+2}}{(2n+2)!} \]

Proof: see Suli

Example: For \( n=1 \), construct a cubic polynomial \( P_3 \) such that

\[ P_3(0) = 0 \quad P_3(1) = 1 \quad P_3'(0) = 0 \quad P_3'(1) = 0 \]
\[ x_0 = 0 \quad x_1 = 1 \]
\[
\begin{align*}
p_3(x) &= H_0(x) y_0 + K_0(x) x_0 + H_1(x) y_1 + K_1(x) x_1 \\
&= K_0(x) + H_1(x) \\
L_0(x) &= 1 - x \quad L_1(x) = x \quad \text{for } x_0 = 0 \quad x_1 = 1 \\
K_0(x) &= \left[ L_0(x) \right]^2 (x - x_0) = (1 - x)^2 x \\
H_1(x) &= \left[ L_1(x) \right]^2 (1 - 2 L_1(x)(x - x_0)) = x^2 (3 - 2x) \\
\Rightarrow \quad p_3(x) &= -x^3 + x^2 + x
\]

Using the interpolant (e.g. Lagrange) to compute approximate derivatives to \( f \).

**How accurately does \( p_n(x) \) approximate \( f'(x) \)?**

Recall **Numerical differentiation**

\[ f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \cdot \delta_{n+1}(x) \]

We can try to differentiate the above, except... it's not clear that \( f(x) \) is continuous, or even differentiable.
If $P_n$ is Lagrange interpolant of $f$ on $[a,b]$, we have

$$|f''(x) - P_n'(x)| \leq \frac{(b-a)^n}{n!} M_{n+1}$$

where $M_{n+1} = \max_{x \in [a,b]} |f^{(n+1)}(x)|$.

If $\lim_{n \to \infty} \frac{(b-a)^n}{n!} M_{n+1} = 0$, then $P_n \to f'$ uniformly on $[a,b]$.

**Numerical Integration**

The definite integrals
\[ \int e^x \, dx \quad \text{and} \quad \int \cos(x) \, dx \]
are easy to evaluate analytically.

Unfortunately, most integrals cannot be evaluated analytically (with for example a table of integrals).
For example:
\[
\int e^x \, dx \quad \text{and} \quad \int \cos(x^2) \, dx
\]

Now, for general integrals:
\[
\int_a^b f(x) \, dx
\]

since polynomials are easy to integrate, the idea is to replace \( f \) with a polynomial and integrate the polynomial exactly.

Newton-Cotes Formulae

Replace \( f(x) \) with its Lagrange interpolant of degree \( n \).

\[
\int_a^b f(x) \, dx \approx \int_a^b \tilde{P}_n(x) \, dx \quad \text{where the}
\]

interpolation points, \( x_i = a + \frac{(b-a) \cdot i}{n} \quad i = 0, 1, 2, \ldots, n \),

are equidistant.

Recall \( \tilde{P}_n(x) = \sum_{i=0}^{n} L_i(x) \cdot f(x_i) \); \( L_i(x) = \frac{n!}{(x-x_i)(x-x_0)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)} \).
\[
\int_{a}^{b} f(x) \, dx \approx \int_{a}^{b} \sum_{n=0}^{m} L_n(x) f(x_n) \, dx
\]

\[
= \sum_{n=0}^{m} f(x_n) \int_{a}^{b} L_n(x) \, dx
\]

can be easily precomputed, exactly.

\[
= \sum_{n=0}^{m} \omega_n f(x_n)
\]

With above choices, we obtain a Newton-Cotes formula of order \( n \).

\( \omega_n \): quadrature weights

\( x_n \): quadrature points
For $n=1$, Trapezoid rule

$$\int_a^b f(x) \, dx \approx \int_a^b P_1(x) \, dx$$

$x_i = a + \frac{(b-a)}{n} \cdot i \Rightarrow x_0 = a, \ x_1 = b$

$$P_1(x) = L_0(x) f(a) + L_1(x) f(b)$$

$$= \frac{x-b}{a-b} f(a) + \frac{x-a}{b-a} f(b)$$

$$= \frac{1}{b-a} \left\{ (b-x)f(a) + (x-a)f(b) \right\}$$

$$\int_a^b P_1(x) \, dx = \frac{1}{b-a} \left\{ \int_a^b f(a) \int_a^b (b-x) \, dx + \int_a^b f(b) \int_a^b (x-a) \, dx \right\}$$

$$= \frac{b-a}{2} \left[ f(a) + f(b) \right]$$
\[ x_0 = a \quad x_1 = b \quad (w_0, x_0) \]
\[ w_0 = \frac{b-a}{2} \quad w_1 = \frac{b-a}{2} \quad (w_0, x_1) \]

For \( n = 2 \): Simpson's Rule

\[ x_i = a + \frac{(b-a)i}{n} \Rightarrow x_0 = a, \quad x_1 = \frac{b-a}{2}, \quad x_2 = b \]

\[
\int_a^b f(x) \, dx = \int_a^b f(x_i) \, L_0 + f(x_i) \, L_1 + f(x_i) \, L_2 \, dx
\]

\[
W_0 = \int_a^b L_0(x) \, dx = \int_a^b \frac{(x-x_i)(x-x_0)}{(x_i-x_1)(x_0-x_2)} \, dx
\]

evaluate the integral with the change of variables

\[ n = \frac{b-a}{2} \, t + \frac{b+a}{2} \]

\[
= \int_{-1}^1 \frac{t(t-1)}{2} \frac{b-a}{2} \, dt = \frac{b-a}{6} = W_0 = W_2
\]

by symmetry.
Remark: you could use the change of variables, or, more simply, expand the product

$$\frac{1}{(x-1)(x-2)} \{ x^2 - (x_1 + x_2) x + x_1 x_2 \}$$

and integrate each term separately.

You can also compute $w_1 = \frac{4}{b} (b-a)$

$$\Rightarrow \int_a^b f(x) \, dx = \int_a^b \phi(x) \, dx = \frac{b-a}{b} \left[ f(a) + 4 f\left(\frac{a+b}{2}\right) + f(b) \right]$$

Since the weights are independent of $f$, they can be precomputed in advance.

**ERROR ESTIMATES**

We seek to study the error

$$E_n(f) = \int_a^b f(x) \, dx - \sum_{k=0}^{n-1} w_k f(x_k)$$
i.e., what is the size of the error that is being committed by integrating the Lagrange interpolant?

\[ |E_n(f)| = \left| \int_a^b f(x) \, dx - \sum_{k=0}^{n} w_k f(x_k) \right| \]

\[ = \left| \int_a^b f(x) \, dx - \int_a^b P_n(x) \, dx \right| \]

\[ = \left| \int_a^b (f - P_n)(x) \, dx \right| \]

\[ \leq \int_a^b |f(x) - P_n(x)| \, dx \]

RECALL:

\[ |f(x) - P_n(x)| \leq \frac{M_{n+1}}{(n+1)!} |\Pi_{n+1}(x)| \]

\[ M_{n+1} = \max_{x \in [a,b]} |f^{(n+1)}(x)| \]

Now, we can use this estimate to determine an upper bound on the error committed when using
the trapezoidal rule \( (n=1) \)

\[
|E_1(f)| \leq \frac{M_2}{2} \int_a^b |T_n(x)| \, dx
\]

\[
= \frac{M_2}{2} \int_a^b |(x-a)(x-b)| \, dx
\]

\[
= \frac{M_2}{2} \int_a^b (x-a)(b-x) \, dx
\]

\[
= \frac{(b-a)^3}{12} M_2
\]

For Simpson's rule \( (n=2) \)

\[
|E_2(f)| \leq \frac{M_3}{b} \int_a^b |(x-a)(x-(\frac{a+b}{2}))(x-b)| \, dx
\]

\[
= \frac{(b-a)^5}{192} M_3
\]

**NOTE:**

\( E_1(f) = 0 \) when \( f \in S_1 \)

\( E_2(f) = 0 \) when \( f \in S_3 \)

When \( n \) odd, Newton-Cotes is exact for polynomials.
of order $n$

$n$ is even, Newton-Cotes is exact for polynomials of order $n+1$.

Next class: composite quadrature rules.