Last class:
* Newton-Cotes quadrature rules \((n=1, n=2)\) — replace integrand \(f(x)\) with \(p_n(x)\) and integrate \(p_n(x)\)

Today:
* error estimates
* composite formulae.

Trapezoid rule \((n=1)\)
\[
\int_a^b f(x) \, dx \approx \int_a^b p_1(x) \, dx = \frac{b-a}{2} \left[ f(a) + f(b) \right]
\]

Simpson’s rule \((n=2)\)
\[
\int_a^b f(x) \, dx \approx \int_a^b p_2(x) \, dx = \frac{b-a}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]
\]

Warning: Runge phenomenon can occur for large \(n\).
\[
\int_{-5}^{5} \frac{1}{1+x^2} \, dx \approx \int_{-5}^{5} \text{h(x)} \, dx \text{ error increases without bound for large } n.
\]

CURE: composite quadrature rules

**ERROR ESTIMATES**

We seek to study the error
\[
E_n(f) = \int_a^b f(x) \, dx - \sum_{k=0}^{n} w_k f(x_k)
\]
i.e., what is the size of the error that is being committed by integrating the Lagrange interpolant?

\[ |E_n(f)| = \left| \int_a^b f(x) \, dx - \sum_{k=0}^n w_k f(x_k) \right| \]

\[ = \left| \int_a^b f(x) \, dx - \int_a^b p_n(x) \, dx \right| \]

\[ = \left| \int_a^b (f - p_n)(x) \, dx \right| \]

\[ \leq \int_a^b |f(x) - p_n(x)| \, dx \]

\[ \leq \int_a^b \frac{M_{n+1}}{(n+1)!} |\Pi_{n+1}(x)| \, dx \]

\[ \leq \frac{M_{n+1}}{(n+1)!} \max_{x \in [a,b]} |f(x)| \]

Now, we can use this estimate to determine an upper bound on the error committed when using...
the trapezoid rule \((n=1)\)

\[
|E_1(f)| \leq \frac{M_2}{2} \int_a^b |f''(x)| \, dx
\]

\[
= \frac{M_2}{2} \int_a^b |(x-a)(x-b)| \, dx
\]

\[
= \frac{M_2}{2} \int_a^b (x-a)(b-x) \, dx
\]

\[
= \frac{(b-a)^3}{12} M_2
\]

For Simpson's rule \((n=2)\)

\[
|E_2(f)| \leq \frac{M_3}{6} \int_a^b |f(x-a)(x-(1/2)(a+b))(x-b)| \, dx
\]

\[
= \frac{(b-a)^5}{192} M_3 \quad \text{A considerable over estimate}
\]

\[\text{NOTE: } E_1(f) = 0 \quad \text{when } f \in \mathcal{S}_1, \quad \text{This is true too for } f \in \mathcal{S}_3\]

\[E_2(f) = 0 \quad \text{when } f \in \mathcal{S}_2, \quad \text{can be exact for polynomials}\]
of order \( n \)

\( n \) is even, Newton-Cotes is exact for polynomials of order \( n+1 \).

Error bound is a considerable over estimate of the actual error. It does not reflect the fact that \( E_2(f) = 0 \) when \( f \in \mathcal{P}_3 \).

Theorem: (improved) estimate for Simpson's rule:

\[ f : [a,b] \to \mathbb{R} \]

\( f^{(n)} \) exists and is continuous

Then:

\[
\int_a^b f(x) \, dx = \frac{b-a}{6} \left[ f(a) + 4f\left( \frac{a+b}{2} \right) + f(b) \right] - \frac{(b-a)^5}{2880} f^{(4)}(\xi) ; \quad \xi \in (a,b)
\]

\[
|E_2(f)| \leq \frac{(b-a)^5}{2880} M_4 \quad M_4 = \max_{x \in [a,b]} |f^{(4)}(x)|
\]
This error bound correctly shows that $E_2(f) = 0$ when $f$ is a polynomial of degree 3.

**Composite formulae.**

Subdivide the interval $[a, b]$ into smaller, equal-sized subintervals, and use Newton-Cotes on each subinterval. For example, select an $m > 2$, and divide $[a, b]$ into $m$ subintervals.

\[
\int_a^b f(x) \, dx = \sum_{i=1}^{m} \int_{x_{i-1}}^{x_i} f(x) \, dx
\]

where $x_i = a + \frac{i}{m} (b-a), \quad i = 0, 1, \ldots, m; \quad h = \frac{b-a}{m}$

\[x_0 = a \quad x_1 \quad x_2 \quad x_3 \quad \cdots \quad x_m = b\]

On each subinterval, use the trapezoidal rule.

\[
\int_{x_{i-1}}^{x_i} f(x) \, dx = \frac{1}{2} h \left[ f(x_{i-1}) + f(x_i) \right]
\]
The composite trapezoidal rule of integration is given by
\[
\int_a^b f(x) \, dx \approx \sum_{i=1}^{m} \frac{h}{2} [f(x_i) + f(x_{i-1})]
\]
where \(h = \frac{b-a}{m}\). This is an approximation of the definite integral of a function \(f(x)\) from \(a\) to \(b\) using \(m\) trapezoids.

The error estimate for the composite trapezoidal rule is
\[
E_1(f) = \int_a^b f(x) \, dx - \sum_{i=1}^{m} \frac{h}{2} [f(x_i) + f(x_{i-1})]
\]
\[
= \sum_{i=1}^{m} \left\{ \int_{x_{i-1}}^{x_i} f(x) \, dx - \frac{h}{2} [f(x_i) + f(x_{i-1})] \right\}
\]
Recall: 
\[
|E_1(f)| \leq \frac{h^3}{12} \max_{x \in [x_{i-1}, x_i]} |f^{(3)}(x)|
\]
Take absolute value of both sides
\[
|E_1(f)| \leq \sum_{i=1}^{m} \frac{h^3}{12} \max_{x \in [x_{i-1}, x_i]} |f^{(3)}(x)|
\]
$$\begin{align*}
&= \frac{1}{12} h^3 \sum_{i=1}^{m} \max_{x \in [x_{i-1}, x_i]} |f^{(3)}(x)| \\
&= \frac{1}{12} h^3 m \mu_2 \quad \mu_2 = \max_{x \in [a,b]} |f^{(3)}(x)| \\
&= \frac{1}{12} \left( \frac{b-a}{m} \right)^3 m \mu_2 \\
&= \frac{1}{12} \left( \frac{b-a}{2m} \right)^3 m \mu_2
\end{align*}$$

**Composite Simpson's Rule**

Let us suppose that $[a,b]$ now is divided into $2m$ intervals with the points

$$x_i = a + i \frac{b-a}{2m}, \quad i = 0, 1, 2, \ldots, 2m$$

$$h = \frac{b-a}{2m}$$

and Simpson's rule is applied to each interval $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, m$. 

\[ x_0 = a \quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad \ldots \quad x_{2m} = b \]
Simpson's rule on each subinterval.

\[ \int_{a}^{b} f(x) \, dx = \sum_{i=1}^{m} \int_{x_{i-2}}^{x_{i}} f(x) \, dx \]

\[ = \sum_{i=1}^{m} \frac{2h}{b} \left[ f(x_{i-2}) + 4f(x_{i-1}) + f(x_{i}) \right] \]

\[ = \frac{h}{3} \left[ f(x_{0}) + 4f(x_{1}) + 2f(x_{2}) + 4f(x_{3}) + \ldots + 2f(x_{2m-2}) + 4f(x_{2m-1}) + f(x_{2m}) \right] \]

Error estimate:

\[ |E_2(f)| \leq \frac{(b-a)^5}{2880m^4} M_y \quad \text{where } M_y = \max_{x \in [a, b]} |f^{(4)}(x)| \]

Remark: Composite formulae are more accurate than the Newton-Cotes formulae.

As long as \( f \) is sufficiently smooth, the error can be made arbitrarily small by choosing enough subintervals.