§9 Polynomial approximation in the 2-norm

We'll introduce an inner product on P_n; this allows us to define angles between polynomials, and in particular, when polynomials are orthogonal.

Inner product on a linear space V over $\mathbb{R}$ is a map $\langle \cdot, \cdot \rangle \longrightarrow \mathbb{R}$ that satisfies for all $f, g, h \in V$, $\lambda \in \mathbb{R}$

- $\langle f+g, h \rangle = \langle f, h \rangle + \langle g, h \rangle$
- $\langle \lambda f, g \rangle = \lambda \langle f, g \rangle$
- $\langle f, g \rangle = \langle g, f \rangle$
- $\langle f, f \rangle > 0$ if $f \neq 0$

We say that f, g are <u>orthogonal</u> if $\langle f, g \rangle = 0$.

Inner product induce norms:

$$\|f\| := \sqrt{\langle f, f \rangle}$$

<u>Examples:</u> 1.) $V = \mathbb{R}^n$ with $\langle x, y \rangle = x^T y = \sum_{i=1}^{n} x_i y_i$

induced norm: Euclidean (or 2-norm)

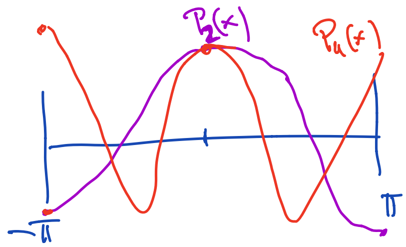
Note that the 2-norm is induced by an inner product, but the $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are not induced by an inner product.

2.) The space of continuous functions on $[a, b]$ is a linear space with inner product

$$\langle f, g \rangle = \int_a^b f(x) g(x) \, dx$$

As example for orthogonal functions in this inner product, consider $[-\pi, \pi]$, $P_{2k}(x) = \cos(kx)$, $P_{2k+1}(x) = \sin(kx)$ $k = 0, 1, \ldots$ These are orthogonal since:

$$\int_{-\pi}^{\pi} P_l(x) P_k(x) \, dx = \begin{cases} 0 & \text{if } l \neq k \\ \langle P_k, P_k \rangle > 0 & \text{if } l = k \end{cases}$$



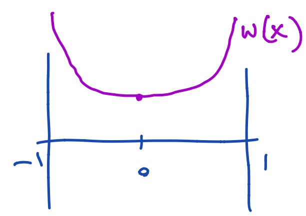
3.) We can also consider a weighted inner product between functions:

$$\langle f,g\rangle = \int_a^b w(x)\, f(x)\, g(x)\, dx \quad \text{with}$$

$$w: (a,b) \to \mathbb{R}, \quad w \geq 0$$

For $w(x) \equiv 1$ this reduces to Example 2. Other choices for w are:

$$w(x) = \frac{1}{\sqrt{1-x^2}} \quad \text{on } [-1,1]$$



the introduction of weights is also important on unbounded intervals, e.g. $\mathbb{R}_{\geq 0}$

Best approximation in 2-norm

Given $f: [a,b] \to \mathbb{R}$, find $p_n \in P_n$ such that

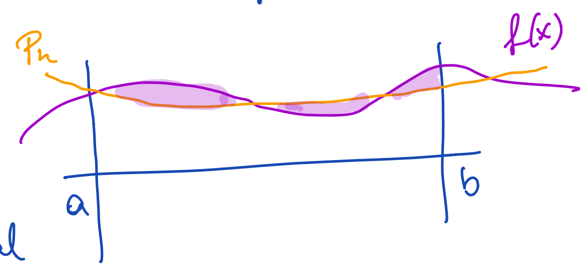
$$\|f - p_n\|_2 = \inf_{q_n \in P_n} \|f - q_n\|_2, \quad \text{where } \|\cdot\|_2 \text{ is induced}$$

by (weighted) inner product between functions on $[a,b]$.

Find p_n such that

$$\int_a^b w(x)(f-p_n)^2(x)\,dx \text{ is minimal}$$

$$\| \quad \|f-p_n\|^2 = \langle f-p_n, f-p_n\rangle$$



How can we compute the best 2-norm approximation?

Example: $f: [0,1] \to \mathbb{R}$, $p_n(x) = c_0 + c_1 x + \dots + c_n x^n$, $c_i \in \mathbb{R}$

$w(x) = 1$

$$\min_{p_n \in P_n} \|f - p_n\|_2^2 = \left(\int_0^1 (f(x) - p_n(x))^2\right)^{\cancel{1/2}}$$

$$\Leftrightarrow \min_{c_0, c_1, \dots c_n} \int_0^1 (f(x) - p_n(x))^2 \, dx$$

replace $p_n(x)$ by $c_0 + c_1 x + \dots + c_n x^n$, minimize over coefficients $c_0, \dots, c_n$

$$\text{Solve} \quad M \cdot \begin{bmatrix} c_0 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} b_0 \\ \vdots \\ b_n \end{bmatrix}$$

computed from above $b_j = \langle f, x^j \rangle$

with $M \in \mathbb{R}^{(n+1)\times(n+1)}$ is the Hilbert matrix, which is poorly conditioned — so it's prone to error and we would like to avoid solving systems with it.

Orthogonal polynomials

We try to find a basis in P_n that is better than the monomial basis $\{1, x, x^2, \dots\}$ when we write $p_n = \sum_{i=0}^{n} c_i \varphi_i(x)$

orthogonal polynom.

Def: Given a weight function $w(x)$, $w > 0$, φ_j are orthogonal on (a,b) with respect to $w(x)$ if φ_j has degree j and

$$\langle \varphi_j, \varphi_k \rangle = \int_a^b w(x) \varphi_j(x) \varphi_k(x) \, dx = \begin{cases} 0 & \text{if } j \neq k \\ \neq 0 & \text{else} \end{cases}$$

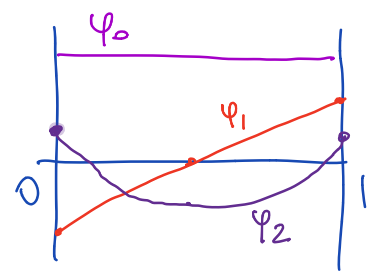
Example: $\{\varphi_0, \varphi_1, \varphi_2\}$ on $[0,1]$ w.r.to $w(x) \equiv 1$

$\varphi_0 \equiv 1$

$\varphi_1(x) = x - c_0 \varphi_0(x)$ with $c_0 \in \mathbb{R}$ such that

$$0 = \langle \varphi_0, \varphi_1 \rangle = \langle 1, x - c_0 1 \rangle = \int_0^1 x \cdot 1 \, dx - c_0 \int_0^1 1 \cdot 1 = \frac{1}{2} - c_0$$

$\Longrightarrow \varphi_1(x) = x - \frac{1}{2}$



$\varphi_2(x) = x^2 - d_1 \varphi_1(x) - d_0 \varphi_0(x)$

$$0 = \langle \varphi_2, \varphi_0 \rangle = \frac{1}{3} - d_0$$

$$0 = \langle \varphi_2, \varphi_1 \rangle \Longrightarrow d_1 = 1$$

$$\varphi_2(x) = x^2 - 1\,\varphi_1(x) - \frac{1}{3}\varphi_0(x) = x^2 - \left(x - \frac{1}{2}\right) - \frac{1}{3} = x^2 - x + \frac{1}{6}$$