Orthogonal polynomials on $[0,1]$, $w(x) \equiv 1$

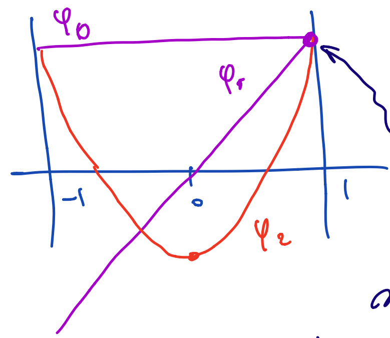
$$\langle \varphi, \psi \rangle = \int_0^1 \varphi(x)\psi(x)\,dx$$

$$\varphi_0(x) = 1$$
$$\varphi_1(x) = x - \frac{1}{2}$$
$$\varphi_2(x) = x^2 - x + \frac{1}{6}$$
$$\varphi_3(x) = x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20}$$
$$\vdots$$

To find an orthogonal family of polynomials on $[a,b]$ w.r.to $w(x) \equiv 1$ we can use a linear transformation $x \longmapsto (b-a)x + a$, then the resulting polynomials are orthogonal on $[a,b]$.

For $[a,b] = [-1,1]$ these polynomials are called Legendre polynomials:

$$\varphi_0 = 1$$
$$\varphi_1 = x$$
$$\varphi_2 = \frac{3}{2}x^2 - \frac{1}{2}$$
$$\varphi_3 = \frac{5}{2}x^3 - \frac{3}{2}x$$
$$\vdots$$



These are scaled such that $\varphi_i(1) = 1$; one could scale them such that $\|\varphi_i\| = \left(\int_{-1}^{1} \varphi_i^2(x)\,dx \right)^{\frac{1}{2}} = 1$ (then they would be orthonormal)

Thm: Given $f : [a,b] \to \mathbb{R}$, there exists a unique polynomial $p_n \in P_n$ such that

$$\|f - p_n\|_2 = \min_{q_n \in P_n} \|f - q_n\|_2$$

Proof: $\varphi_0, \dots, \varphi_n$ family of orthogonal polynomials, and we normalize them $\psi_j = \dfrac{\varphi_j}{\|\varphi_j\|}$ $j = 0, \dots, n$

Every $q_n \in P_n$ is of the form $q_n(x) = \beta_0 \psi_0(x) + \dots + \beta_n \psi_n(x)$
$\beta_i \in \mathbb{R}$

Goal: choose β_i such that q_n minimizes $\|f-q_n\|_2$ over all $q_n \in P_n$

Define $E(\beta_0,\dots,\beta_n) = \|f-q_n\|_2^2 = \langle f-q_n, f-q_n\rangle$

$$= \langle f,f\rangle - 2\langle f,q_n\rangle + \langle q_n,q_n\rangle$$

$$= \|f\|_2^2 - 2\sum_{i=0}^{n} \beta_i \langle f,\psi_i\rangle + \underbrace{\sum_{j=0}^{n}\sum_{k=0}^{n} \beta_j\beta_k \langle \psi_j,\psi_k\rangle}_{\sum_{j=0}^{n} \beta_j^2}$$

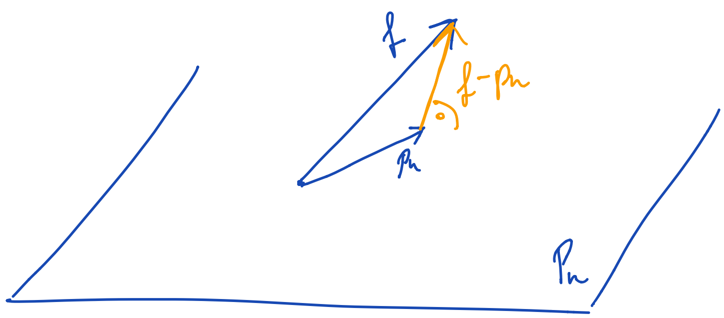
$$= \sum_{j=0}^{n} \{\beta_j - \langle f,\psi_j\rangle\}^2 + \underbrace{\|f\|_2^2 - \sum_{j=0}^{n} |\langle f,\psi_j\rangle|^2}_{\text{indep. of } \beta_i}$$

Minimum will be attained for

$$\beta_j^* = \langle f,\psi_j\rangle \quad j=0,\dots,n$$

$\longrightarrow$ $p_n(x) = \beta_0^*\psi_0(x) + \dots + \beta_n^*\psi_n(x)$ is the unique minimizer. $\square$

Thm: $p_n \in P_n$ is the best fit polynomial for $f:[a,b]\to\mathbb{R}$ if and only if $f-p_n$ is orthogonal to every $q_n \in P_n$, i.e. $\langle f-p_n, q_n\rangle = 0$ for all $q_n \in P_n$



Practical computation of p_n for given f:

$\varphi_0,\dots,\varphi_n$ orthogonal, $\psi_j = \dfrac{\varphi_j}{\|\varphi_j\|_2}$, $\beta_j = \langle f,\psi_j\rangle$

$$\rightarrow p_n(x) = \beta_0 \varphi_0(x) + \dots + \beta_n \varphi_n(x)$$

$$= \beta_0 \frac{\varphi_0(x)}{\|\varphi_0(x)\|} + \dots + \beta_n \frac{\varphi_n(x)}{\|\varphi_n(x)\|}$$

$$= \gamma_0 \varphi_0(x) + \dots + \gamma_n \varphi_n(x)$$

$$\gamma_j = \frac{\langle f, \varphi_j \rangle}{\|\varphi_j\|} = \frac{\langle f, \varphi_j \rangle}{\langle \varphi_j, \varphi_j \rangle} \qquad j = 0, \dots, n$$

<u>Example</u>: Best fit polynomial in P_2 of $f : x \rightarrow e^x$ over $[0,1]$ with $w(x) \equiv 1$.

$\varphi_0(x) = 1$

$\varphi_1(x) = x - \frac{1}{2}$

$\varphi_2(x) = x^2 - x + \frac{1}{6}$

$$\underline{\gamma_0} = \frac{\langle f, \varphi_0 \rangle}{\langle \varphi_0, \varphi_0 \rangle} = \frac{\int_0^1 e^x \cdot 1 \, dx}{\int_0^1 1 \cdot 1 \, dx} = \underline{\underline{e - 1}}$$

$$\gamma_1 = \frac{\langle f, \varphi_1 \rangle}{\langle \varphi_1, \varphi_1 \rangle} = \frac{\int_0^1 e^x (x - \frac{1}{2}) dx}{\int_0^1 (x - \frac{1}{2})^2 dx} = 18 - 6e$$

$$\gamma_2 = 210e - 570$$

$$P_2(x) = (e-1) + (18 - 6e)\left(x - \frac{1}{2}\right) + (210e - 570)\left(x^2 - x + \frac{1}{6}\right)$$

§10 Numerical integration / quadrature

$f : [a,b] \rightarrow \mathbb{R}$ continuous & diff'able

$$\int_a^b w(x) f(x) dx, \quad w(x) > 0$$

Newton-Cotes allowed to compute integrals exactly for polynomials up to degree n; we fixed nodes $x_0, \dots, x_n$ as being uniform. In Gauss quadrature, we allow those points to change and hope to find more accurate rules

$$\int_a^b w(x) f(x)\,dx \simeq \sum_{i=0}^{n} w_i\, f(x_i)$$

weights   quadrature nodes

We assume to-be-determined $x_0, \dots, x_n$ and instead of Lagrange interpolation let us try Hermite interpolation

$$p_{2n+1}(x) = \sum_{k=0}^{n} H_k(x) f(x_k) + \sum_{k=0}^{n} K_k(x) f'(x_k)$$

Hermite interpolation of f using nodes $x_0, \dots, x_n$

$$H_k(x_j) = \begin{cases} 1 & \text{if } k=j \\ 0 & \text{else} \end{cases} \qquad H_k'(x_j) = 0$$

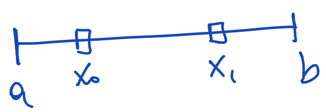
$$K_k(x_j) = 0, \qquad K_k'(x_j) = \begin{cases} 1 & k=j \\ 0 & \text{else} \end{cases}$$

$$\int_a^b w(x) f(x)\,dx \approx \int_a^b w(x)\, p_{2n+1}(x)\,dx =$$

$$= \sum_{k=0}^{n} f(x_k) \underbrace{\int_a^b w(x) H_k(x)\,dx}_{W_k} + \sum_{k=0}^{n} f'(x_k) \underbrace{\int_a^b w(x) K_k(x)\,dx}_{V_k}$$

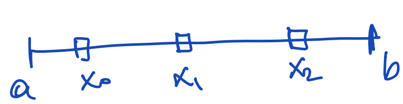
We don't want to involve $f'(x_k)$,
so can we find quadrature points such that $V_k = 0$ $k = 0, \dots, n$?

$n = 1$



Gauss points integrate polynomials up to degree $2n+1$ exactly!

$n = 2$



(Compared to Newton-Cotes, where it is n (or $n+1$))