Def: $g : [a,b] \longrightarrow \mathbb{R}$, g continuous, $g(\xi) = \xi$

ξ is stable if fixed point iteration converges to ξ whenever x_0 is close to ξ.

ξ is unstable if no fixed point iteration sequence started close to ξ converges to the fixed point, unless $x_0 = \xi$

A fixed point can be neither stable nor unstable

Under the assumption that g is differentiable, the previous theorem tells us:

$$|g'(\xi)| < 1 \longrightarrow \text{stable fixed point}$$

$$|g'(\xi)| > 1 \longrightarrow \text{unstable fixed point}$$ (proof is in book)

Example: $g(x) = \frac{1}{2}(x^2 + c)$, $c \in \mathbb{R}$ fixed

$$g(x) = x \longrightarrow \xi_{1,2} = 1 \pm \sqrt{1-c} \qquad c \leq 1$$

$$\xi_1 = 1 - \sqrt{1-c}, \quad \xi_2 = 1 + \sqrt{1-c}$$

$$g'(x) = x \Longrightarrow |g'(\xi_2)| = |1 + \sqrt{1-c}| > 1$$

$$\Longrightarrow \underline{\xi_2 \text{ is unstable}} \text{ if } c < 1$$

$$|g'(\xi_1)| = |1 - \sqrt{1-c}|$$

for ξ_1 to be stable, I need $1 - \sqrt{1-c} > -1$

$$\Longleftrightarrow \sqrt{1-c} < 2$$

$$\Longleftrightarrow 1 - c < 4$$

$$\Longleftrightarrow \underline{c > -3}$$

ξ_1 is stable if $\underline{-3 < c < 1}$

Speed of convergence $\quad x_k \longrightarrow \xi$

$(x_k)_{k\geq 1}$ converges (at least) linearly if

$$\lim_{k\to\infty} \frac{|x_{k+1}-\xi|}{|x_k-\xi|} = \mu \in \mathbb{R}$$

(a more general definition that works around the problem that the denominator can be 0 is in the book)

if $\mu = 0 \longrightarrow$ superlinear convergence

if $\mu \in (0,1) \longrightarrow$ linear convergence, asymptotic rate $\rho = -\log_{10}\mu$ — \mu

\rho

if $\mu = 1$ sublinear convergence

(e.g. $x_k = \frac{1}{k} \longrightarrow \xi = 0$)

For fixed point iterations and differentiable g:

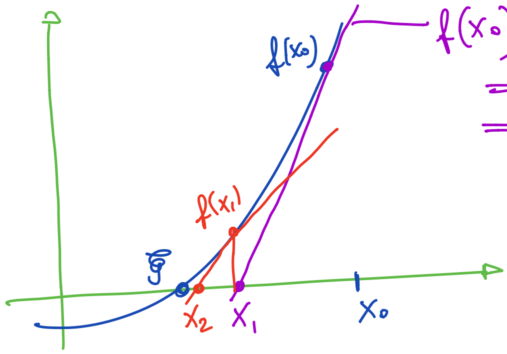
$$\lim_{k\to\infty} \frac{|x_{k+1}-\xi|}{|x_k-\xi|} = \lim_{k\to\infty} \frac{|g(x_k)-g(\xi)|}{|x_k-\xi|} \overset{\eta_k\in(x_k,\xi)}{=} \lim_{k\to\infty} |g'(\eta_k)| = |g'(\xi)|$$

§1.4 Newton's method

This is a method to solve $f(x)=0$ if f is differentiable.

$$\boxed{\text{Initial point } x_0 \in \mathbb{R} \qquad x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \quad k=0,1,2,\dots}$$

(assume $f'(x_k) \neq 0$)



$f(x_0) + (x - x_0) f'(x_0) = 0$

$\Longrightarrow x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

① linearization
② solve "linearization" $= 0$
③ iterate

Def: $x_k \to \xi$ as $k \to \infty$

If $\dfrac{|x_{k+1} - \xi|}{|x_k - \xi|^q} \xrightarrow{k \to \infty} \mu < \infty$

then $(x_k)_{k \geq 1}$ converges with order q.

In particular, $q = 2 \longrightarrow$ quadratic convergence.

How fast is that?

Assume $|x_k - \xi| \sim 10^{-1}$

$|x_{k+1} - \xi| \sim \mu (10^{-1})^2$

$|x_{k+2} - \xi| \sim \mu |x_{k-1} - \xi|^2 \sim \mu^3 10^{-4}$

$|x_{k+3} - \xi| \sim \mu^7 10^{-8}$

Thm 1.8: (Convergence of Newton's method)
f twice continuously differentiable on $I_\delta = [\xi - \delta, \xi + \delta]$, $\delta > 0$, $f(\xi) = 0$, $f''(\xi) \neq 0$. Suppose $\exists A > 0$ s.t.

$$\frac{|f''(x)|}{|f'(y)|} \leq A \quad \text{for all } x, y \in I_\delta$$

Then: If $|\xi - x_0| \leq h$, $h = \min(\delta, 1/A)$, then $(x_k)_{k \geq 1}$ defined by Newton's method converges quadratically to ξ.