**Thm 1.8** (Convergence of Newton's method)

$f$ twice continuously diff'able on $I_\delta = [\xi - \delta, \xi + \delta]$, $\delta > 0$, $f(\xi) = 0$, $f''(\xi) \neq 0$. Suppose $\exists A > 0$ s.t.

$$\frac{|f''(x)|}{|f'(y)|} \leq A \quad \text{for all } x, y \in I_\delta$$

**Then:** If $|\xi - x_0| \leq h$, $h = \min(\delta, \frac{1}{A})$, then $x_k$, $k = 0, 1, 2, \dots$ defined by Newton's method converges quadratically to $\xi$.

**Proof:** Suppose $|\xi - x_k| \leq h$. Taylor expansion:

$$0 = f(\xi) = f(x_k) + (\xi - x_k) f'(x_k) + \frac{(\xi - x_k)^2}{2} f''(\eta_k)$$

$$\eta_k \in (\xi, x_k)$$

divide by $f'(x_k)$

$$0 = \underbrace{\frac{f(x_k)}{f'(x_k)} + \xi - x_k}_{= \xi - x_{k+1}} + \frac{(\xi - x_k)^2}{2} \underbrace{\frac{f''(\eta_k)}{f'(x_k)}}_{|\cdot| \leq A}$$

$= -x_{k+1}$

$|\xi - x_k| \leq \frac{1}{A}$

$$\Longrightarrow |\xi - x_{k+1}| = \frac{(\xi - x_k)^2}{2|f'(x_k)|} |f''(\eta_k)| \leq \frac{|\xi - x_k|}{2} \dots$$

$$\leq 2^{-k-1} |\xi - x_0|$$

$$\Longrightarrow x_k \longrightarrow \xi \text{ as } k \to \infty$$

$$\eta_k \longrightarrow \xi \text{ as } k \to \infty$$

**and** $$\frac{|\xi - x_{k+1}|}{|\xi - x_k|^2} \xrightarrow{k \to \infty} \frac{|f''(\xi)|}{2|f'(\xi)|} = \eta$$

$\Longrightarrow$ quadr. convergence $\square$

**Remarks:** 

- requires $C^2$ (twice cont' diff'able)
- requires $f'(\xi) \neq 0$
- only converges if started "close enough" to the solution $(|x_0 - \xi| \le h)$

Depending on the initialization:

- convergence to $\xi$, with $x_k \neq \xi$ for all $k$
- converges to $\xi$ in finite number of steps, i.e. $x_k = \xi$ for $k \geq k_0$
- diverge to $\pm\infty$
- no convergence (cycling)

## §1.5 The secant method

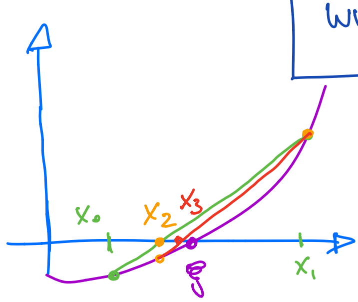
Newton's method requires computation of $f'$, which could be expensive or not available. In these cases, one could replace

$$f'(x_k) \approx \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$$

This results in:

$$\boxed{x_{k+1} = x_k - f(x_k)\,\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} \qquad k = 1, 2, 3, \ldots}$$

with starting values $x_0, x_1$



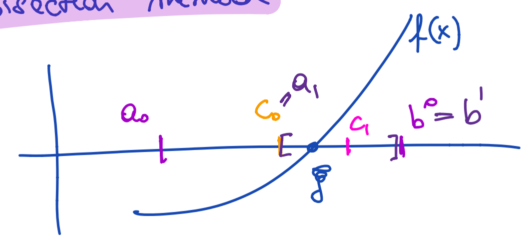
**Thm 1.10:** $f$ cont' differentiable on $I=[\xi-h,\xi+h]$, $h>0$
$f(\xi)=0$, $f'(\xi)\neq 0$

**Then:** If $x_0, x_1$ are sufficiently close to $\xi$, the sequence generated by the secant method converges at least linearly.

**Proof:** (book)

This method is cheaper as it does not require computing $f'(x_n)$.

## §1.6 Bisection method




Iterative method that halves intervals recursively:

1.) Initialize $(a_0, b_0)$ s.t. $f(a_0), f(b_0)$ have different sign, $k=0$

2.) Compute $c_k = \frac{a_k+b_k}{2}$ and set

$$(a_{k+1}, b_{k+1}) = \begin{cases} (a_k, c_k) & \text{if } f(c_k)f(b_k) > 0 \\ (c_k, b_k) & \text{if } \quad \text{—}\,\text{"}\,\text{—} < 0 \end{cases}$$

Sequence $c_k$ converges to $\xi$ with rate $\rho = \log_{10} 2$

Since $|c_k - \xi| \le 2^{-k-1}|b_0 - a_0|$

→ •) Only continuity on $f$ needed

•) robust method, if the interval contains more than one root, the result depends on $[a_0, b_0]$