§ 2.1 Solving linear systems

We're interested in solving the linear system

$$Ax = b, \quad A \in \mathbb{R}^{n \times n}, \quad x, b \in \mathbb{R}^n, \quad n \in \mathbb{N}$$

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

Recall: $A^{-1} \in \mathbb{R}^{n \times n}$, $AA^{-1} = A^{-1}A = I = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$

inverse exists if $\det(A) \neq 0$

we call these matrices non-singular or regular or invertible

Cramer's rule:

$$x_i = \frac{\det(A_i^b)}{\det(A)}$$

matrix A with i-th column replaced by b

requires $(n+1)$ determinants, which is expensive (computationally)

Computing the determinant requires $\sim n!$ operations (i.e. additions, summations) floating point operations, "flops"

§ 2.2: Gaussian elimination

Example
$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & 5 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 16 \\ -3 \end{bmatrix}$$

Generate triangular system by adding multiples of rows to other rows (this does not change the solution).

add (-2) first row $\rightarrow$ to 2nd row

add 1st row $\rightarrow$ to 3rd row

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 6 & -3 \end{bmatrix} x = \begin{bmatrix} 6 \\ 4 \\ 3 \end{bmatrix}$$

this is identical to multiplying the system with

$$L_1 = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I + \mu_{21} E^{2,1}} \quad \text{and} \quad L_2 = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}}_{I + \mu_{31} E^{3,1}} \quad \text{from the left}$$

$$\mu_{21} = -2, \quad E^{2,1} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

... (−3) x second row $\longrightarrow$ add to 3rd row

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ -9 \end{bmatrix} \qquad (*)$$

upper triangular matrix

$$L_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

$(*)$ can be solved by "backwards substitution"

L_1, L_2, L_3 lower triangular matrices $\longrightarrow \quad x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

__Def:__ $L \in \mathbb{R}^{n \times n}$ lower triangular if $l_{ij} = 0$ for all $1 \le i < j \le n$:

$$\begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ & & & 0 \\ l_{n1} & \cdots & \cdots & l_{nn} \end{bmatrix}$$

unit lower triangular if additionally $l_{11} = \ldots = l_{nn} = 1$

Analogue for upper triangular, unit upper triangular matrices

<u>Thm:</u> (properties of lower triangular matrices; identical results hold for upper triangular matrices)

(i) products of lower triangular matrices are lower triangular

(ii) as above for <u>unit</u> lower triangular

(iii) lower triangular matrices are non-singular if $l_{11} \neq 0, \ldots, l_{nn} \neq 0$.

(iv) invertible lower triangular matrices have lower-triangular inverses

(v) same as (iv) for unit lower triangular.

<u>Proof of iv:</u> (rest is easy): Induction over matrix size n

<u>$n=2$:</u> $L = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$ $L^{-1} = \begin{pmatrix} d & e \\ f & g \end{pmatrix}$

$$LL^{-1} = I \Longrightarrow ae = 0, \; a \neq 0 \Longrightarrow e = 0 \; ✓$$

<u>$n \mapsto n+1$:</u>

$$L = \left[\begin{array}{c|c} L_1 & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline - r^T - & \alpha \end{array}\right] \in \mathbb{R}^{(n+1)\times(n+1)} \qquad L^{-1} = \left[\begin{array}{c|c} L_1' & \begin{matrix} | \\ c \\ | \end{matrix} \\ \hline - r'^T - & \mu \end{array}\right]$$

Since $LL^{-1} = I \in \mathbb{R}^{(n+1)\times(n+1)}$

$$\Longrightarrow L_1 L_1' = I \in \mathbb{R}^{n\times n}, \quad L_1 c = 0, \quad r^T L_1' + \alpha r'^T = 0$$

$$r^T c + \alpha\mu = 1$$

$\longrightarrow$ $L_1' = L_1^{-1}$ is lower triangular due to induction assumption

$$L_1 c = 0 \Longrightarrow c = 0 \Longrightarrow L^{-1} \text{ is lower triangular}$$

$\square$.