Elimination process

$$\underbrace{L_N \cdots L_2 L_1}_{N = \frac{n(n-1)}{2}} A = U$$

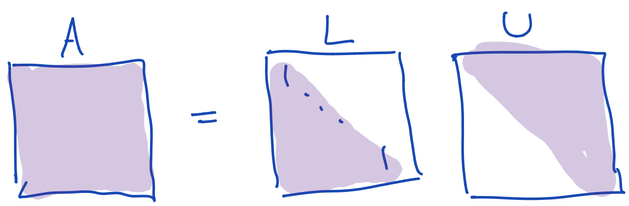
upper triangular matrix

$$L_j = I + \mu_{rs} E^{rs}, \quad \mu_{rs} \in \mathbb{R}$$

$$E^{rs} = \begin{array}{cc} r \rightarrow & \begin{bmatrix} 0 & \cdots & 0 \\ 1 & \ddots & \\ & & 0 \end{bmatrix} \\ & \uparrow s \end{array}$$

$$A = \underbrace{L_1^{-1} \cdot \ldots \cdot L_N^{-1}}_{L \text{ unit lower triangular}} U$$

upper triangular



"LU-decomposition of A"

How can L, U be used to solve $Ax = b$?

$$Ax = L\underbrace{Ux}_{y} = b$$

① Solve $Ly = b$ with lower triangular matrix L (forward substitution)

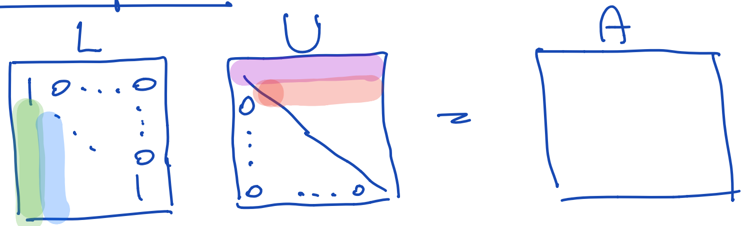
② Solve $Ux = y$ with upper triangular matrix U (backward substitution)

In MATLAB notation: $y = L\backslash b$, $x = U\backslash y$

$$\Longrightarrow \boxed{x = U\backslash(L\backslash b)}$$

A\b in MATLAB $\Longrightarrow$ it computes L, U, and then does $x = U \backslash (L \backslash b)$

<u>Direct computation:</u>



$u_{1j} = a_{1j}$ $\quad j = 1, 2, 3, \dots n$

$l_{11} = 1, \; l_{i1} = \dfrac{a_{i1}}{u_{11}}$ $\quad i = 2, \dots, n$

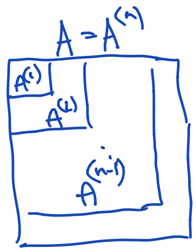
$u_{2j} = a_{2j} - l_{21} u_{12}$ $\quad j = 2, \dots, n$

$l_{j2} = \dfrac{1}{u_{22}} \left(a_{j2} - l_{21} u_{1j} \right)$ $\quad j = 1, 2, \dots$

$\vdots$

Next question: For what $A \in \mathbb{R}^{n \times n}$ does an LU decomposition exist?

<u>Def:</u> $A \in \mathbb{R}^{n \times n}$, the leading principle submatrices $A^{(k)}$ coincide with A on the "upper left"



<u>Thm:</u> $A \in \mathbb{R}^{n \times n}$, every leading principle submatrix $A^{(k)}$ is non-singular $k = 1, \dots, n-1$. Then $A = LU$ exists with a lower unit triangular matrix L and an upper triang. U.

Proof: Induction over matrix size k:

$\underline{k=2}$ $\quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a \neq 0$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \ell & 1 \end{pmatrix} \begin{pmatrix} u & v \\ 0 & \eta \end{pmatrix}$$

$$u = a, \quad v = b, \quad \ell u = c, \quad \ell v + \eta = d$$

$$\Longrightarrow \ell = \frac{c}{u}, \qquad u = a \neq 0$$

$\boxed{k \mapsto k+1}$

Assume $A \in \mathbb{R}^{(k+1)\times(k+1)}$ for which all principle leading submatrices of order k or lower are invertible, and thus they have an LU-decomposition.

$$A = \left[\begin{array}{c|c} A^{(k)} & b \\ \hline - \; c^T \; - & d \end{array}\right]$$

$A^{(k)}$ is non-singular, $A^{(k)} = L^{(k)} U^{(k)}$.

$b, c \in \mathbb{R}^k$, column vectors

$$A = \left[\begin{array}{c|c} L^{(k)} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline m^T & 1 \end{array}\right] \left[\begin{array}{c|c} U^{(k)} & v \\ \hline 0 \cdots 0 & \eta \end{array}\right]$$

$\xrightarrow[\text{multiplication}]{\text{block}}$ $A^{(k)} = L^{(k)} U^{(k)}$, $\quad L^{(k)} v = b$, $\quad \underbrace{m^T U^{(k)} = c^T}_{U^{(k)^T} m = c}$

$m^T v + \eta = d$

exists.

$$v = \left(L^{(k)}\right)^{-1} b$$

$$A^{(k)} = L^{(k)} U^{(k)}, \quad \overset{\neq 0}{\det(A^{(k)})} = \underbrace{\det(L^{(k)})}_{=1} \det(U^{k})$$

$\Longrightarrow U^{(k)}$ is invertible, and so is $(U^{(k)})^T$

$\Longrightarrow m = \left[(U^{(k)})^T\right]^{-1} c$ $\qquad \square$

§2.4: Pivoting

What do we do if $u_{ii} = 0$ (or very small)

Example: $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix}$ no LU decomposition exists, but exchanging 1^{st} & 2^{nd} row

$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 2 & 2 \end{bmatrix}$ now LU-decomposition exists!

Pivoting exchanges rows to ensure we don't find a zero (or something very small) in diagonal.

Def: (Permutation matrices): $P \in \mathbb{R}^{n \times n}$ which only contains zeros and ones and each column and row contains exactly one non-zero.

Examples: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \ldots$

Properties:
- product of perm. matrices are again perm. matrices
- det is ± 1
- product of "interchange" matrices (matrices that interchange two rows)
- inverse are again perm. matrices.